FIXED FUZZY POINT THEOREMS FOR FUZZY MAPPINGS ON COMPLETE METRIC SPACES

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The aim of this paper is to present fixed fuzzy point results of fuzzy mappings in complete metric spaces. As an application, coincidence and common fixed fuzzy points of a hybrid fuzzy pair of mappings are obtained. An example is presented to support the result proved herein. Our results generalize and extend various results in the existing literature.

Keywords: Fixed fuzzy point, fuzzy mapping, fuzzy set, approximate quantity;

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1. Introduction

The Banach contraction principle appeared in explicit form in Banach's thesis [5] in 1922 where it was used to establish the existence of a solution for an integral equation. Since then, it has become a very popular tool in solving existence problems in many branches of mathematics. Extensions of this principle were obtained either by generalizing the domain of mappings or by extending the contractive condition on the mappings (see for example [1, 2, 4, 6, 7, 8, 10, 13, 14, 16, 18, 19])s. Nadler [20] proved multivalued version of Banach contraction principle. In mathematical modeling of the real world problems, there are many inconveniences including the complexity of models and imprecision in differentiating the events exactly in real situations. Advances in computer science industry developed and modified many areas of research. There is still a major shortcoming of computers to deal with the uncertain and imprecise situations. To deal with this uncertainty Zadeh [23] in 1965, initiated the concept of fuzzy sets. Since then, many authors have employed this concept extensively in topology and analysis to develop this theory further and obtained several interesting applications. Now it is well recognized theory to handle uncertainties arising in various real life situations. Heilpern [12] introduced fuzzy mappings on a metric space and proved a fixed point theorem for fuzzy contraction mappings as a generalization of Nadler's theorem [20]. For more results on fuzzy mappings we refer to [9, 21, 22].

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The aim of this paper is to present a new fixed point theorem for fuzzy mappings in complete metric spaces. As an application of the result presented herein, a new fixed point result of multivalued mappings is obtained. Also, coincidence and common fixed fuzzy point result of a hybrid fuzzy pair of mappings is derived. An example is given to show that the result proved in this paper is a proper generalization of comparable results in the existing recent literature.

2. **Preliminaries**

Let us first recall some basic definitions and known results needed in the sequel.

Let X be a space of points with generic elements of X denoted by x and $I = [0,1]$. A fuzzy set A in X is characterized by a membership function $A : X \to I$ such that each element in X is associated with a real number $A(x) \in [0,1]$. Let I^X be the collection of all fuzzy subsets of X. Let (X,d) be a metric space and A a fuzzy set in X. If $\alpha \in (0,1]$, then $\alpha-$ level set A_α of A is defined as:

$$A_\alpha = \{x : A(x) \geq \alpha\}.$$

For $\alpha = 0$, we have $A_0 = \overline{\{x \in X : A(x) > 0\}}$, where $\overline{B}$ denotes the closure of a set B in (X,d). The set A_α is a crisp approximation of the fuzzy set A. A fuzzy set A is said to be an approximate quantity if and only if for each $\alpha \in [0,1]$, A_α is compact, and convex subset of a metric linear space (X,d) with

$$\sup_{x \in X} A(x) = 1.$$

Let $W(X)$ be the family of all approximate quantities. A fuzzy set A is said to be more accurate than the fuzzy set B, denoted by $A \subset B$ if and only if $A(x) \leq B(x)$ for each x in X. It is obvious that if $0 < \alpha \leq \beta \leq 1$, then $A_\beta \subseteq A_\alpha$. Corresponding to each $\alpha \in [0,1]$ and $x \in X$, the fuzzy point x_α of X is a fuzzy set $x_\alpha : X \to [0,1]$ given by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

For $\alpha = 1$, we have following indicator function of $\{x\}$,

$$x_1(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

Define $W_\alpha(X) = \{A \in I^X : A_\alpha \text{ is nonempty and compact}\}$. For $A, B \in W_\alpha(X)$ and $\alpha \in [0,1]$, let

$$\begin{aligned} p_\alpha(A,B) &= \inf_{x \in A_\alpha, y \in B_\alpha} d(x,y), \delta_\alpha(A,B) = \sup_{x \in A_\alpha, y \in B_\alpha} d(x,y), \\ D_\alpha(A,B) &= \max\{\sup_{x \in A_\alpha} d(x, B_\alpha), \sup_{y \in B_\alpha} d(y, A_\alpha)\}, \\ D(A,B) &= \sup_\alpha D_\alpha(A,B). \end{aligned}$$

Note that p_α is nondecreasing mapping of α and D a metric on $W_\alpha(X)$. Let Y be an arbitrary subset in (X,d). A mapping $F : Y \to W_\alpha(X)$ is called a fuzzy mapping over the set Y, that is, a fuzzy set $F_y \in W_\alpha(X)$ for each y in Y. As, a fuzzy set F_y in X is characterized by a membership function $F_y : X \to [0,1]$, so $F_y(x)$ is a membership of x in F_y. Thus a fuzzy mapping F over Y is a fuzzy subset of $Y \times X$ having membership function $F_y(x) = F(y,x)$.

In a more general sense than that given in [12], a mapping $F\colon X \to I^X$ is a fuzzy mapping over X ([21]). Notice that $\alpha-$level set of fuzzy mapping F over X is given by

$$(F_x)_\alpha = \{y \in X : F_x(y) \geq \alpha\}.$$

Definition 2.1 ([9]). *A fuzzy point x_α in X is called a fixed fuzzy point of fuzzy mapping F if $x_\alpha \subset F_x$ that is $(F_x)x \geq \alpha$ or $x \in (F_x)_\alpha$. That is, the fixed degree of x in Fx is at least α. If $\{x\} \subset F_x$, then x is a fixed point of a fuzzy mapping F.*

Recently Ali and Abbas [3] gave the following definitions followed by a couple of results about fixed fuzzy points and common fixed fuzzy points of fuzzy mappings.

Definition 2.2 ([3]). *Let $F\colon X \to W_\alpha(X)$ be a fuzzy mapping and $g : X \to X$ a self mapping on X. A fuzzy point x_α in X is called:*

- **(a):** coincidence fuzzy point of hybrid fuzzy pair (g, F) if $(gx)_\alpha \subset F_x$, that is $(F_x)gx \geq \alpha$ or $gx \in (F_x)_\alpha$. That is, the fixed degree of gx in F_x is at least α.
- **(b):** common fixed fuzzy point of the hybrid fuzzy pair (g, F) if $x_\alpha = (gx)_\alpha \subset F_x$, that is $x = gx \in (F_x)_\alpha$ (the fixed degree of x and gx in F_x is same and is at least α).

We denote $C_\alpha(g, F)$ and $F_\alpha(g, F)$ by the set of all coincidence fuzzy point and set of all common fixed fuzzy point of the hybrid fuzzy pair (g, F), respectively.

Definition 2.3 ([3]). *Let $F\colon X \to W_\alpha(X)$ be a fuzzy mapping and $g\colon X \to X$ a self mapping on X, then*

- **(c):** the hybrid pair (g, F) is called *$w-$fuzzy compatible* if
$$g(F_x)_\alpha \subseteq (F_{gx})_\alpha$$
whenever $x \in C_\alpha(g, F)$.
- **(d):** mapping g is called *$F-$fuzzy weakly commuting* at some point $x \in X$ if $g^2(x) \in (F_{gx})_\alpha$.

Lemma 2.1 ([11]). *Let X be a nonempty set and $g : X \longrightarrow X$. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \longrightarrow X$ is one to one.*

Lemma 2.2 ([12]). *Let (X, d) be a metric space, $x, y \in X$ and $A, B \in W(X)$:*

(1) *if $p_\alpha(x, A) = 0$, then $x_\alpha \subset A$;*
(2) *$p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$;*
(3) *if $x_\alpha \subset A$, then $p_\alpha(x, B) \leq D_\alpha(A, B)$.*

Theorem 2.1 ([9]). *Let (X, d) be a complete metric space, F a fuzzy mapping from X to $W_\alpha(X)$, where $\alpha \in (0, 1)$. If there exists $q \in (0, 1)$ such that*

$$D_\alpha(Fx, Fy) \leq qd(x, y),$$

holds for each $x, y \in X$. Then there exists $x \in X$ such that x_α is a fixed fuzzy point

Lemma 2.3 ([17]). *Let (X, d) be a complete metric space and F a fuzzy mapping from X into $W(X)$ and $x_0 \in X$. Then there exists a $x_1 \in X$ such that $\{x_1\} \subset Fx_0$.*

Recently Khojasteh et al. [15] proved the following new type of fixed point theorem.

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ (set of closed and bounded subsets of X) be a multivalued mapping. Let T satisfy the following:*

$$H(Tx, Ty) \leq \left(\frac{d(x, Ty) + d(y, Tx)}{1 + \delta(x, Tx) + \delta(y, Ty)} \right) d(x, y)$$

for all $x, y \in X$. Then T has a fixed point.

Throughout this article we use the following notations:

$$\begin{aligned}
M_\alpha^F(x, y) &= \max\{d(x, y), p_\alpha(x, F_x), p_\alpha(y, F_y), \frac{p_\alpha(x, F_y) + p_\alpha(y, F_x)}{2}\}, \\
N_\alpha^F(x, y) &= \min\{p_\alpha(x, F_x), p_\alpha(y, F_y), p_\alpha(x, F_y), p_\alpha(y, F_x)\}, \\
\beta^F(x, y) &= \frac{p_\alpha(x, F_y) + p_\alpha(y, F_x)}{1 + p_\alpha(x, F_x) + p_\alpha(y, F_y)} \\
M_\alpha^{g,F}(x, y) &= \max\{d(gx, gy), p_\alpha(gx, F_x), p_\alpha(gy, F_y), \frac{p_\alpha(gx, F_y) + p_\alpha(gy, F_x)}{2}\}, \\
N_\alpha^{g,F}(x, y) &= \min\{p_\alpha(gx, F_x), p_\alpha(gy, F_y), p_\alpha(gx, F_y), p_\alpha(gy, F_x)\}, \\
\beta^{g,F}(x, y) &= \frac{p_\alpha(gx, F_y) + p_\alpha(gy, F_x)}{1 + p_\alpha(gx, F_x) + p_\alpha(gy, F_y)}.
\end{aligned}$$

3. **Main Results**

In this section we prove fixed fuzzy point theorem on a complete metric space.

Theorem 3.1. *Let (X, d) be a complete metric space and $F : X \to W_\alpha(X)$ a fuzzy mapping. Suppose that there exists an $L \geq 0$ such that*

$$D_\alpha(F_x, F_y) \leq \beta^F(x, y) M_\alpha^F(x, y) + L N_\alpha^F(x, y) \tag{3.1}$$

for all $x, y \in X$. Then there exists a point $x \in X$ such that $x_\alpha \subset F_x$.

Proof. Let u_0 be an arbitrary element of X. As $(F_{u_0})_\alpha$ is nonempty and compact so there exists $u_1 \in (F_{u_0})_\alpha$ such that $d(u_0, u_1) = p_\alpha(u_0, F_{u_0})$. If $u_0 = u_1$, then $u_0 = u_1 \in (F_{u_0})_\alpha$ and the proof is finished. Suppose that $u_0 \neq u_1$. Since $(Fu_1)_\alpha$ is nonempty and compact, there exists $u_2 \in (F_{u_1})_\alpha$ such that

$$d(u_1, u_2) = p_\alpha(u_1, F_{u_1}) \leq D_\alpha(F_{u_0}, F_{u_1}).$$

If $u_1 = u_2$, then $u_1 = u_2 \in (F_{u_1})_\alpha$ and the proof is finished. Suppose that $u_1 \neq u_2$, then by given assumption we have

$$\begin{aligned} d(u_1,u_2) &\leq D_\alpha(F_{u_0},F_{u_1}) \leq \beta^F(u_0,u_1)M_\alpha^F(u_0,u_1) + LN_\alpha^F(u_0,u_1) \\ &\leq \left(\frac{p_\alpha(u_0,F_{u_1}) + p_\alpha(u_1,F_{u_0})}{1+\delta_\alpha(u_0,F_{u_0})+\delta_\alpha(u_1,F_{u_1})}\right) M_\alpha^F(u_0,u_1) + LN_\alpha^F(u_0,u_1) \\ &\leq \left(\frac{d(u_0,u_2)+d(u_1,u_1)}{1+d(u_0,u_1)+d(u_1,u_2)}\right) M_\alpha^F(u_0,u_1) + LN_\alpha^F(u_0,u_1) \end{aligned}$$

where

$$\begin{aligned} M_\alpha^F(u_0,u_1) &= \max\{d(u_0,u_1), p_\alpha(u_0,F_{u_0}), p_\alpha(u_1,F_{u_1}), \frac{p_\alpha(u_0,F_{u_1})+p_\alpha(u_1,F_{u_0})}{2}\} \\ &= \max\{d(u_0,u_1), d(u_0,u_1), d(u_1,u_2), \frac{d(u_0,u_2)+d(u_1,u_1)}{2}\} \\ &= \max\{d(u_0,u_1), d(u_1,u_2), \frac{d(u_0,u_1)+d(u_1,u_2)}{2}\} \\ &= \max\{d(u_0,u_1), d(u_1,u_2)\} \\ N_\alpha^F(u_0,u_1) &= \min\{p_\alpha(u_0,F_{u_0}), p_\alpha(u_1,F_{u_1}), p_\alpha(u_0,F_{u_1}), p_\alpha(u_1,F_{u_0}) \\ &= \min\{p_\alpha(u_0,u_1), p_\alpha(u_1,u_2), p_\alpha(u_0,u_2), p_\alpha(u_1,u_1)\} = 0. \end{aligned}$$

Hence we obtain

$$d(u_1,u_2) \leq \left(\frac{d(u_0,u_1)+d(u_1,u_2)}{1+d(u_0,u_1)+d(u_1,u_2)}\right) \max\{d(u_0,u_1), d(u_1,u_2)\}. \tag{3.2}$$

Note that

$$d(u_1,u_2) \leq d(u_0,u_1).$$

If not, then $d(u_1,u_2) > d(u_0,u_1)$. In this case (3.2) implies

$$d(u_1,u_2) \leq \left(\frac{d(u_0,u_1)+d(u_1,u_2)}{1+d(u_0,u_1)+d(u_1,u_2)}\right) d(u_1,u_2) < d(u_1,u_2),$$

a contradiction. Hence

$$d(u_1,u_2) \leq d(u_0,u_1)$$

and consequently

$$d(u_1,u_2) \leq \left(\frac{d(u_0,u_1)+d(u_1,u_2)}{1+d(u_0,u_1)+d(u_1,u_2)}\right) d(u_0,u_1). \tag{3.3}$$

Continuing this process, we construct a sequence $\{u_n\}$ in X such that $u_n \in (F_{u_{n-1}})_\alpha$, and $u_{n+1} \in (F_{u_n})_\alpha$ with

$$d(u_n,u_{n+1}) = p_\alpha(u_n,F_{u_n}) \leq D_\alpha(F_{u_{n-1}},F_{u_n}).$$

If $u_n = u_{n+1}$ for some $n \in \mathbb{N}$, then $u_n = u_{n+1} \in (F_{u_n})_\alpha$ and the proof is finished. Suppose $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$, then by the given assumption we have

$$\begin{aligned} d(u_n,u_{n+1}) &\leq D_\alpha(F_{u_{n-1}},F_{u_n}) \\ &\leq \left(\frac{p_\alpha(u_{n-1},F_{u_n}) + p_\alpha(u_n,F_{u_{n-1}})}{1+\delta_\alpha(u_{n-1},F_{u_{n-1}})+\delta_\alpha(u_n,F_{u_n})}\right) M_\alpha^F(u_{n-1},u_n) + LN_\alpha^F(u_{n-1},u_n) \\ &\leq \left(\frac{d(u_{n-1},u_{n+1})+d(u_n,u_n)}{1+d(u_{n-1},u_n)+d(u_n,u_{n+1})}\right) M_\alpha^F(u_{n-1},u_n) + LN_\alpha^F(u_{n-1},u_n), \end{aligned}$$

where

$$\begin{aligned} M^F_\alpha(u_{n-1},u_n) &= \max\{d(u_{n-1},u_n), p_\alpha(u_{n-1},F_{u_{n-1}}), p_\alpha(u_n,F_{u_n}), \frac{p_\alpha(u_{n-1},F_{u_n})+p_\alpha(u_n,F_{u_{n-1}})}{2}\} \\ &= \max\{d(u_{n-1},u_n), d(u_{n-1},u_n), d(u_n,u_{n+1}), \frac{d(u_{n-1},u_{n+1})+d(u_n,u_n)}{2}\} \\ &= \max\{d(u_{n-1},u_n), d(u_1,u_{n+1}), \frac{d(u_{n-1},u_n)+d(u_n,u_{n+1})}{2}\} = \max\{d(u_{n-1},u_n), d(u_n,u_{n+1})\} \\ N^F_\alpha(u_{n-1},u_n) &= \min\{p_\alpha(u_{n-1},F_{u_{n-1}}), p_\alpha(u_n,F_{u_n}), p_\alpha(u_{n-1},F_{u_n}), p_\alpha(u_n,F_{u_{n-1}}) \\ &= \min\{p_\alpha(u_{n-1},u_n), p_\alpha(u_n,u_{n+1}), p_\alpha(u_{n-1},u_{n+1}), p_\alpha(u_n,u_n)\} = 0. \end{aligned}$$

Hence we obtain

$$d(u_n,u_{n+1}) \le \left(\frac{d(u_{n-1},u_n)+d(u_n,u_{n+1})}{1+d(u_{n-1},u_n)+d(u_n,u_{n+1})}\right)\max\{d(u_{n-1},u_n), d(u_n,u_{n+1})\}. \quad (3.4)$$

Note that

$$d(u_n,u_{n+1}) \le d(u_{n-1},u_n).$$

If not, then $d(u_n,u_{n+1}) > d(u_{n-1},u_n)$ In this case (3.4) implies

$$d(u_n,u_{n+1}) \le \left(\frac{d(u_{n-1},u_n)+d(u_n,u_{n+1})}{1+d(u_{n-1},u_n)+d(u_n,u_{n+1})}\right) d(u_n,u_{n+1}) < d(u_n,u_{n+1}),$$

a contradiction. Hence

$$d(u_n,u_{n+1}) \le d(u_{n-1},u_n)$$

and consequently

$$d(u_n,u_{n+1}) \le \left(\frac{d(u_{n-1},u_n)+d(u_n,u_{n+1})}{1+d(u_{n-1},u_n)+d(u_n,u_{n+1})}\right) d(u_{n-1},u_n). \quad (3.5)$$

Denote

$$\beta_n = \beta^F(u_{n-1},u_n) = \frac{d(u_{n-1},u_n)+d(u_n,u_{n+1})}{1+d(u_{n-1},u_n)+d(u_n,u_{n+1})}.$$

Then from (3.5) we have

$$d(u_n,u_{n+1}) \le \beta_n d(u_{n-1},u_n) \le \beta_n\beta_{n-1} d(u_{n-2},u_{n-1}) \le ... \le (\beta_n\beta_{n-1}...\beta_1) d(u_0,u_1). \quad (3.6)$$

Note that $\{\beta_n\}$ is nonincreasing sequence with $\beta_n > 0$ for all $n \in \mathbb{N}$. So $\beta_1...\beta_n \le (\beta_1)^n$ and $\lim_{n\to\infty}(\beta_1)^n = 0$. It follows that

$$\lim_{n\to\infty}(\beta_1\beta_2...\beta_n) = 0.$$

Hence from (3.6) we obtain

$$\lim_{n\to\infty} d(u_n,u_{n+1}) = 0. \quad (3.7)$$

Now for all $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d(u_n,u_m) &\le d(u_n,u_{n+1}) + d(u_{n+1},u_{n+2}) + ... + d(u_{m-1},u_m) \\ &\le [(\beta_n\beta_{n-1}...\beta_1) + (\beta_n\beta_{n-1}...\beta_1) + ... + (\beta_n\beta_{n-1}...\beta_1)] d(u_0,u_1) \\ &= \sum_{k=n}^{m-1}(\beta_k\beta_{k-1}...\beta_1) d(u_0,u_1) = \sum_{k=n}^{m-1}\upsilon_k d(u_0,u_1) < \sum_{k=n}^{\infty}\upsilon_k d(u_0,u_1), \end{aligned}$$

where $\upsilon_k = (\beta_k\beta_{k-1}...\beta_1)$. Now using (3.7) we get

$$\lim_{k\to\infty}\frac{\upsilon_{k+1}}{\upsilon_k} = \lim_{k\to\infty}\frac{\beta_{k+1}\beta_k\beta_{k-1}...\beta_1}{\beta_k\beta_{k-1}...\beta_1} = \lim_{k\to\infty}\beta_{k+1} = \lim_{k\to\infty}\frac{d(u_k,u_{k+1})+d(u_{k+1},u_{k+2})}{1+d(u_k,u_{k+1})+d(u_{k+1},u_{k+2})} = 0.$$

This implies that $\sum_{k=n}^{\infty} \upsilon_k < \infty$. Hence $\{u_n\}_n$ is a Cauchy sequence and so is convergent in (X, d). That is there exists an element z such that $\lim_{n\to\infty} u_n = z$. We claim that $z \in (F_z)_\alpha$. On contrary suppose that $z \notin (F_z)_\alpha$. Then $p_\alpha(z, F_z) > 0$ and consequently

$$\begin{aligned}
p_\alpha(z, F_z) &= \lim_{n\to\infty} p_\alpha(u_{n+1}, F_z) \leq \lim_{n\to\infty} D_\alpha(F_{u_n}, F_z) \\
&\leq \lim_{n\to\infty} \left(\frac{p_\alpha(u_n, F_z) + p_\alpha(z, F_{u_n})}{1 + \delta_\alpha(u_n, F_{u_n}) + \delta_\alpha(z, F_z)} \right) M_\alpha^F(u_n, z) + LN_\alpha^F(u_n, z) \\
&\leq \lim_{n\to\infty} \left(\frac{p_\alpha(u_n, F_z) + p_\alpha(z, u_{n+1})}{1 + d(u_n, u_{n+1}) + p_\alpha(z, F_z)} \right) \lim_{n\to\infty} \max\{d(u_n, z), p_\alpha(u_n, F_{u_n}), p_\alpha(z, F_z), \\
&\quad \frac{p_\alpha(z, F_z) + p_\alpha(z, F_{u_n})}{2}\} \\
&\quad +L \lim_{n\to\infty} \max\{d(u_n, u_{n+1}), p_\alpha(z, F_z), p_\alpha(z, F_z), d(z, u_{n+1})\} \\
&\leq \lim_{n\to\infty} \left(\frac{p_\alpha(u_n, F_z) + p_\alpha(z, u_{n+1})}{1 + d(u_n, u_{n+1}) + p_\alpha(z, F_z)} \right) \lim_{n\to\infty} \max\{d(u_n, z), d(u_n, u_{n+1}), p_\alpha(z, F_z), \\
&\quad \frac{p_\alpha(z, F_z) + d(z, u_{n+1})}{2}\} \\
&\leq \left(\frac{p_\alpha(z, F_z)}{1 + p_\alpha(z, F_z)} \right) p_\alpha(z, F_z) < p_\alpha(z, F_z).
\end{aligned}$$

Since $p_\alpha(z, F_z) > 0$ therefore

$$\frac{p_\alpha(z, F_z)}{1 + p_\alpha(z, F_z)} = 1,$$

a contradiction. Hence $z \in (F_z)_\alpha$. □

Corollary 3.1. *Let (X, d) be a complete metric space and $F : X \to W_\alpha(X)$ a fuzzy mapping such that there exists an $L \geq 0$ such that*

$$D_\alpha(F_x, F_y) \leq \beta^F(x, y) d(x, y) + L d(x, y) \tag{3.8}$$

for all $x, y \in X$. Then there exists a point $x \in X$ such that $x_\alpha \subset F_x$.

Corollary 3.2. *Let (X, d) be a complete metric space and $F : X \to W_\alpha(X)$ a fuzzy mapping such that there exists an $L \geq 0$ such that*

$$D_\alpha(F_x, F_y) \leq \beta^F(x, y) \max\{d(x, y), p_\alpha(x, F_x), p_\alpha(y, F_y)\} + LN_\alpha^F(x, y) \tag{3.9}$$

for all $x, y \in X$. Then there exists a point $x \in X$ such that $x_\alpha \subset F_x$.

Corollary 3.3. *Let (X, d) be a complete metric space and $F : X \to W_\alpha(X)$ a fuzzy mapping such that*

$$D_\alpha(F_x, F_y) \leq \beta^F(x, y) \max\{d(x, y), p_\alpha(x, F_x), p_\alpha(y, F_y)\} \tag{3.10}$$

for all $x, y \in X$. Then there exists a point $x \in X$ such that $x_\alpha \subset F_x$.

Corollary 3.4. *Let (X, d) be a complete metric space and $F : X \to W_\alpha(X)$ a fuzzy mapping such that*

$$D_\alpha(F_x, F_y) \leq \beta^F(x, y)d(x, y)$$

for all $x, y \in X$. Then there exists a point $x \in X$ such that $x_\alpha \subset F_x$.

Now we present an example to explain the Theorem 3.1 as a generalization of some comparable results in the literature.

Example 3.1. *Let $X = \{0, 1, 2\}$ be endowed with metric d defined as:*

$$\begin{aligned} d(0,2) &= 15, d(0,1) = 10, d(1,2) = 5, \\ d(x,x) &= 0, d(x,y) = d(y,x) \text{ for all } x, y \in X. \end{aligned}$$

Let $\alpha \in (0, \frac{1}{3})$ and define a fuzzy mapping F from X into $W_\alpha(X)$ as:

$$F_0(x) = \begin{cases} 2\alpha \text{ if } x = 0, \\ \frac{\alpha}{3} \text{ if } x = 1, \\ 0 \text{ if } x = 2, \end{cases}, F_1(x) = \begin{cases} \frac{\alpha}{2} \text{ if } x = 0 \\ 3\alpha \text{ if } x = 1, \\ 0 \text{ if } x = 2, \end{cases}, F_2(x) = \begin{cases} \alpha \text{ if } x = 0, \\ \frac{\alpha}{5} \text{ if } x = 1, \\ \frac{\alpha}{3} \text{ if } x = 2, \end{cases}.$$

Then $(F_0)_\alpha = \{0\}, (F_1)_\alpha = \{1\}, (F_2)_\alpha = \{0\}$. Note that for all $x, y \in \{0, 2\}$, we have $D_\alpha(F_x, F_y) = H((F_x)_\alpha, (F_y)_\alpha) = 0$. For $x = 1$ and $y \in \{0, 2\}$, we obtain

$$\begin{aligned} D_\alpha(F_0, F_1) &= H((F_0)_\alpha, (F_1)_\alpha) = d(0,1) = 10, \\ D_\alpha(F_1, F_2) &= H((F_1)_\alpha, (F_2)_\alpha) = d(1,0) = 10, \end{aligned}$$

and for $x = 1, y = 2$ we have

$$\begin{aligned} M_\alpha^F(1,2) &= \max\{d(1,2), p_\alpha(1,F_1), p_\alpha(2,F_2), \frac{p_\alpha(1,F_2) + p_\alpha(2,F_1)}{2}\} \\ &= \max\{d(1,2), d(1,1), d(2,0), \frac{d(1,0) + d(2,1)}{2}\} \\ &= \max\{5, 0, 15, \frac{15}{2}\} = 15, \\ N_\alpha^F(1,2) &= \min\{p_\alpha(1,F_1), p_\alpha(2,F_2), p_\alpha(1,F_2), p_\alpha(2,F_1)\} \\ &= \min\{d(1,1), d(2,0), d(1,0), d(2,1)\} = \min\{0, 15, 10, 5\} = 0 \\ \beta^F(1,2) &= \frac{p_\alpha(1,F_2) + p_\alpha(2,F_1)}{1 + \delta_\alpha(1,F_1) + \delta_\alpha(2,F_2)} = \frac{d(1,0) + d(2,1)}{1 + d(1,1) + d(2,0)} = \frac{15}{16}. \end{aligned}$$

So

$$D_\alpha(F_1, F_2) = 10 \leq \frac{225}{16} = \beta^F(1,2)M_\alpha(1,2) + LN_\alpha^F(1,2).$$

Now for $x = 0$ and $y = 1$, we obtain

$$\begin{aligned}
M_\alpha^F(0,1) &= \max\{d(0,1), p_\alpha(1,F_1), p_\alpha(0,F_0), \frac{p_\alpha(1,F_0)+p_\alpha(0,F_1)}{2}\} \\
&= \max\{d(0,1), d(1,1), d(0,0), \frac{d(1,0)+d(0,1)}{2}\} \\
&= \max\{10, 0, 0, \frac{20}{2}\} = 10. \\
N_\alpha^F(0,1) &= \min\{p_\alpha(1,F_1), p_\alpha(0,F_0), p_\alpha(1,F_0), p_\alpha(0,F_1)\} \\
&= \min\{d(1,1), d(0,0), d(1,0), d(0,1)\} \\
&= = \min\{0, 0, 10, 10\} = 0, \\
\beta^F(0,1) &= \frac{p_\alpha(1,F_0)+p_\alpha(0,F_1)}{1+\delta_\alpha(1,F_1)+\delta_\alpha(0,F_0)} = \frac{d(1,0)+d(0,1)}{1+d(1,1)+d(0,0)} = 20.
\end{aligned}$$

Hence

$$D_\alpha(F_0, F_1) = 10 \le 200 = \beta^F(0,1)M_\alpha(0,1) + LN_\alpha^F(0,1).$$

Consequently

$$D_\alpha(F_x, F_y) \le \beta^F(x,y)M_\alpha(x,y) + LN_\alpha^F(x,y)$$

is satisfied for all $x, y \in X$. Hence all the conditions of Theorem 3.1 are satisfied. Moreover for $x = 0$, we have $x_\alpha \subset F(x)$ as $(F0)0 \ge \alpha$. Hence $\{0\} \subset (F0)_\alpha$.This implies that $x = 0$ is the fixed fuzzy point of fuzzy mapping F.

Remark 3.1. *Let fuzzy mapping F from X into $W_\alpha(X)$ be defined as in above example. Since*

$$\begin{aligned}
D_\alpha(F_1, F_2) &= d(1,0) = 10, d(1,2) = 5 \\
\beta^F(1,2) &= \frac{p_\alpha(1,F_2)+p_\alpha(2,F_1)}{1+\delta_\alpha(1,F_1)+\delta_\alpha(2,F_2)} = \frac{15}{16},
\end{aligned}$$

therefore

$$D_\alpha(F_1, F_2) = 10 \not< \frac{75}{16} = \beta^F(1,2)d(1,2).$$

Hence Theorem 2.2 doesn't hold true in this example that shows Theorem 3.1 is a proper generalization of Theorem 2.2.

Remark 3.2. *Let fuzzy mapping F from X into $W_\alpha(X)$ be defined above. Since*

$$D_\alpha(F_1, F_2) = d(1,0) = 10, d(1,2) = 5$$

and for any choice of $q \in \,]0,1[$

$$D_\alpha(F_1, F_2) \not\le qd(1,2).$$

Hence Theorem 2.1 does not holds true in this case. Hence Theorem 3.3 is a proper generalization of results given in [9, 12, 15, 20].

Remark 3.3. *Let F be a fuzzy mapping from X into $W_\alpha(X)$ and $T : X \to CB(X)$ (set of all compact subsets of X). Define*

$$(F_x)(z) = \begin{cases} \alpha, & \text{if } z \in Tx \\ 0, & \text{otherwise} \end{cases} \tag{3.11}$$

for each $x \in X$. Note that

$$(F_x)_\alpha = \{z : F(_x)(z) \geq \alpha\} = Tx. \tag{3.12}$$

Now we present multivalued version of Theorem 3.1 which itself is a new result in complete metric spaces and is a generalization of resuts given in [15].

Theorem 3.2. *Let (X, d) be a complete metric space and $T : X \to K(X)$ (set of all compact subsets of X) a multivalued mapping. Then T has a fixed point provided that T satisfy the following:*

$$H(Tx, Ty) \leq \left(\frac{d(x, Ty) + d(y, Tx)}{1 + \delta(x, Tx) + \delta(y, Ty)} \right) M^T(x, y) + LN^T(x, y) \tag{3.13}$$

for all $x, y \in X$ where

$$\begin{aligned} M^T(x, y) &= \max\{d(x, y), d(x, Fx), d(y, Fy), \frac{d(x, Fy) + d(y, Fx)}{2}\}, \\ N^T(x, y) &= \min\{d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx)\}. \end{aligned}$$

Proof. It follows from Remark 3.3. □

Theorem 3.3. *Let (X, d) be a complete metric space and $g : X \to X$ a self map on X, $F : X \to W_\alpha(X)$ a fuzzy mapping. Suppose that there exists an $L \geq 0$ such that*

$$D_\alpha(F_x, F_y) \leq \beta^{g,F}(x, y)M_\alpha^{g,F}(x, y) + LN_\alpha^{g,F}(x, y) \tag{3.14}$$

Then $C_\alpha(g, F) \neq \phi$ provided that $(F_{(X)})_\alpha \subseteq g(X)$ for each α. Moreover F and g have common fixed fuzzy point if any of the following conditions holds:

- **(f):** F and g are *$w -$ fuzzy compatible*, $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $x \in C_\alpha(F, g)$, $u \in X$ and g is continuous at u.
- **(g):** g is *$F -$ fuzzy weakly commuting* for some $x \in C_\alpha(g, F)$, and $g^2x = gx$.
- **(h):** g is continuous at x for some $x \in C_\alpha(g, F)$ and for some $u \in X$, such that $\lim_{n\to\infty} g^n u = x$.

Proof. By Lemma 2.1, there exists $E \subseteq X$ such that $g : E \longrightarrow X$ is one to one and $g(E) = g(X)$. Define a mapping $\mathcal{A} : g(E) \longrightarrow W_\alpha(X)$ by

$$\mathcal{A}_{gx} = F_x \text{ for all } gx \in g(E). \tag{3.15}$$

As g is one to one on E, so $\mathcal{A}$ is well defined. Therefore (3.14) becomes

$$\begin{aligned} D_\alpha(\mathcal{A}_{gx}, \mathcal{A}_{gy}) &= D_\alpha(F_x, F_y) \leq \beta^{g,F}(x, y)M_\alpha^{g,F}(x, y) + LN_\alpha^{g,F}(x, y) \\ &= \beta^{\mathcal{A}}(gx, gy)M_\alpha^{\mathcal{A}}(gx, gy) + LN_\alpha^{\mathcal{A}}(gx, gy) \end{aligned}$$

for all $gx, gy \in g(E)$. Hence $\mathcal{A}$ satisfies (3.1) and all the conditions of Theorem 3.1. Using Theorem 3.1 with mapping $\mathcal{A}$, it follows that $\mathcal{A}$ has fixed fuzzy point $u \in g(E)$. Now it is left to prove that F and g have coincidence fuzzy point. Since $\mathcal{A}$ has fixed fuzzy point $u_\alpha \subset \mathcal{A}_u$, therefore $u \in (\mathcal{A}_u)_\alpha$. As $(F_{(X)})_\alpha \subseteq g(X)$, there exists $u_1 \in X$ such that $gu_1 = u$, thus it follows that

$$gu_1 \in (\mathcal{A}_{gu_1})_\alpha = (F_{u_1})_\alpha. \tag{3.16}$$

This implies that $u_1 \in X$ is coincidence fuzzy point of F and g. Hence $C_\alpha(g,F) \neq \phi$. Suppose now that (a) holds. Then for some $x_\alpha \in C_\alpha(g,F)$, we have $\lim_{n\to\infty} g^n x = u$, where $u \in X$. Since g is continuous at u, so we have that u is a fixed points of g. As F and g are *$w-$fuzzy compatible* and $(g^n x)_\alpha \in C_\alpha(g,F)$ for all $n \geq 1$. That is $g^n x \in (F_{g^{n-1}x})_\alpha$ for all $n \geq 1$. Now we show that $gu \in (F_u)_\alpha$. Assume on contrary that $gu \notin (F_u)_\alpha$, then by Lemma 2.2 $p_\alpha(gu,F_u) > 0$

$$\begin{aligned}
p_\alpha(gu,F_u) &\leq p_\alpha(gu,g^n x) + p_\alpha(g^n x,F_u) \leq p_\alpha(gu,g^n x) + D_\alpha(F_{g^{n-1}x},F_u)\\
&\leq p_\alpha(gu,g^n x) + \beta^{g,F}(x,y)M_\alpha^{g,F}(x,y) + LN_\alpha^{g,F}(x,y)\\
&\leq p_\alpha(gu,g^n x) + \frac{p_\alpha(gu,F_{g^{n-1}x}) + p_\alpha(g^n x,F_u)}{1+p_\alpha(gg^{n-1}x,F_{g^{n-1}x}) + p_\alpha(gu,F_u)}\\
&\quad \max\{d(gg^{n-1}x,gu), p_\alpha(g^n x,F_{g^{n-1}x}), p_\alpha(gu,F_u), \frac{p_\alpha(gu,F_{g^{n-1}x}) + p_\alpha(g^n x,F_u)}{2}\}\\
&\quad + L\min\{p_\alpha(g^n x,F_{g^{n-1}x}), p_\alpha(gu,F_u), p_\alpha(gu,F_{g^{n-1}x}), p_\alpha(g^n x,F_u)\}\\
&\leq p_\alpha(gu,g^n x) + \frac{p_\alpha(gu,g^n x) + p_\alpha(g^n x,F_u)}{1+p_\alpha(g^n x,g^n x) + p_\alpha(gu,F_u)}\\
&\quad \max\{d(g^n x,gu), p_\alpha(g^n x,g^n x), p_\alpha(gu,F_u), \frac{p_\alpha(gu,g^n x) + p_\alpha(g^n x,F_u)}{2}\}\\
&\quad + L\min\{p_\alpha(g^n x,g^n x), p_\alpha(gu,F_u), p_\alpha(gu,g^n x), p_\alpha(g^n x,F_u)\}.
\end{aligned}$$

On taking limit as $n \to \infty$, we get

$$p_\alpha(gu,F_u) \leq \frac{p_\alpha(gu,F_u)}{1+p_\alpha(gu,F_u)} p_\alpha(gu,F_u) < p_\alpha(gu,F_u) \tag{3.17}$$

a contradiction. Hence $u = gu \in (F_u)_\alpha$. That is, u_α is common fixed fuzzy point of F and g. Suppose now that (b) holds. If for some $x_\alpha \in C_\alpha(F,g)$, g is *$F-$fuzzy weakly commuting* and $g^2 x = gx$ then $gx = g^2 x \in (F_{gx})_\alpha$. Hence $(gx)_\alpha$ is a common fixed fuzzy point of F and g. Suppose now that (c) holds and assume that for some $x_\alpha \in C_\alpha(F,g)$ and for some $u \in X$, $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$. By continuity of g at x and y, we get $x = gx \in (F_x)_\alpha$. The result follows. □

4. **Conclusion**

In attempt to model the real world problems, we have to deal with uncertainties and vagueness of the data, tools or conditions in the form of constraints . Fuzzy set theory has provided many important tools in mathematics and related disciplines to resolve the issues of uncertainty and ambiguity. Fuzzy sets and mappings play important roles in the process of systems and fuzzy optimization. Fixed point theorems for fuzzy mappings obtained in this article can further be used in solving the real world problems involving fuzzy situations.

We presented a new fixed point theorem in the context of fuzzy mappings which generalize the comparable results [9, 12, 15, 20] in the existing literature. An example is given to prove that the generalization is proper and important one. These results obtained here can be applied in functional equations involving fuzzy situations.

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