ON THE STABILITY OF A SYSTEM OF TWO DIFFERENCE EQUATIONS

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Using the centre manifold theory, we study the stability of the zero equilibrium of a system of two difference equations in the special case when one of the eigenvalues is equal to 1 *and the other eigenvalue has the absolute value less than* 1.

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1. **Introduction**

The study of the difference equations and systems of difference equations, especially those involving exponential terms, is a topic in current mathematical research (see, e.g., [1], [2], [3], [6], [8], [9], [13], [15], [16], [18], [19], [20], [22], [23], [28], [29], [30], [31], [34]). This is also due to the fact that the difference equations have many applications in biology, population dynamics, genetics, economy, physics and other applied sciences (see, e.g., [5], [7], [10], [14], [17], [26], [32], [33], [35]).

In their study of the dynamics of a perennial grass, Tilman and Wedin [33] investigated the following model

$$\begin{cases} B_{t+1} = cN\frac{e^{a-bL_t}}{1+e^{a-bL_t}}, \\ L_{t+1} = \frac{L_t^2}{L_t+d} + ckN\frac{e^{a-bL_t}}{1+e^{a-bL_t}}, \end{cases}$$

where B represents the living biomass, L the litter mass, N the total soil nitrogen, t the time (measured in years) and constants $a, b, c, d > 0$ and $0 < k < 1$.

Motivated by the second equation of the previous system, Papaschinopoulos, Schinas and Ellina [21] studied the dynamics of the solutions of the biological model described by the following difference equation

$$x_{n+1} = \frac{ax_n^2}{x_n + b} + c\frac{e^{k-dx_n}}{1 + e^{k-dx_n}},$$

where $0 < a < 1, b, c, d, k$ are positive constants and the initial condition x_0 is a positive real number. Also, Psarros, Papaschinopoulos and Schinas [25]

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investigated the stability of the zero equilibria of the following two systems of difference equations:

$$\begin{cases} x_{n+1} = a_1 \frac{y_n}{b_1+y_n} + c_1 \frac{x_n e^{k_1-d_1 x_n}}{1+e^{k_1-d_1 x_n}}, \\ y_{n+1} = a_2 \frac{x_n}{b_2+x_n} + c_2 \frac{y_n e^{k_2-d_2 y_n}}{1+e^{k_2-d_2 y_n}}, \end{cases} \quad \text{and} \quad \begin{cases} x_{n+1} = a_1 \frac{x_n}{b_1+x_n} + c_1 \frac{y_n e^{k_1-d_1 y_n}}{1+e^{k_1-d_1 y_n}}, \\ y_{n+1} = a_2 \frac{y_n}{b_2+y_n} + c_2 \frac{x_n e^{k_2-d_2 x_n}}{1+e^{k_2-d_2 x_n}}, \end{cases}$$

where a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, k_1 and k_2 are real constants and the initial values x_0 and y_0 are real numbers.

Psarros, Papaschinopoulos and Schinas [24] provided some conditions for the semistability of the zero equilibria of the following two close-to-symmetric systems of difference equations:

$$\begin{cases} x_{n+1} = a x_n + b y_n e^{-x_n}, \\ y_{n+1} = c y_n + d x_n e^{-y_n}, \end{cases} \quad \text{and} \quad \begin{cases} x_{n+1} = a y_n + b x_n e^{-y_n}, \\ y_{n+1} = c x_n + d y_n e^{-x_n}, \end{cases}$$

where a, b, c and d are positive constants and the initial conditions x_0 and y_0 are positive numbers. Mylona, Psarros, Papaschinopoulos and Schinas [15] provided some conditions for the stability of the zero equilibria of the same two systems of difference equations.

Flondor, Olteanu and Ştefan [11] proved the asymptotic stability of the unique equilibrium point of the following system of two differential equations:

$$\begin{cases} \frac{\mathrm{d}m}{\mathrm{d}t} = b - m\left(d + A\mu\right), \\ \frac{\mathrm{d}\mu}{\mathrm{d}t} = \beta - \mu\left(\delta + Bm\right). \end{cases}$$

The previous system models a certain class of (open) enzymatic reactions. Their approach uses Lyapunov theory, invariant regions and controllability.

Motivated by those presented above, in this paper we will study the stability of the zero equilibrium of the following system of difference equations:

$$\begin{cases} x_{n+1} = (1-a_1)x_n - b_1 x_n y_n + c_1 y_n e^{-d_1 x_n}, \\ y_{n+1} = (1-a_2)y_n - b_2 x_n y_n + c_2 x_n e^{-d_2 y_n}, \end{cases} \quad n = 0, 1, 2, ...,$$

where the parameters $a_1, c_1, a_2, c_2 \in (0,1)$, b_1, b_2 are negative numbers and the initial conditions x_0 and y_0 are positive numbers.

2. **Preliminaries**

Let us consider the following two-dimensional discrete dynamical system:

$$\begin{cases} x_{n+1} = f_1\left(x_n, y_n\right), \\ y_{n+1} = f_2\left(x_n, y_n\right), \end{cases} \quad n = 0, 1, 2, ..., \tag{1}$$

where $f_1 : I \times J \to I$ and $f_2 : I \times J \to J$ are continuously differentiable functions and I, J are some intervals of real numbers. Furthermore, a solution, $\{(x_n, y_n)\}_{n\geq 0}$, of the system (1) is uniquely determined by the initial conditions $(x_0, y_0) \in I \times J$ (see, e.g., [27], Theorem 4-1, p. 167).

We first recall the following definition ([7], p. 2).

Definition 2.1. *An* equilibrium point *of the two-dimensional discrete dynamical system* (1) *is a point* $(\overline{x},\overline{y})$ *that satisfies*

$$\begin{cases}\overline{x} = f_1(\overline{x},\overline{y}),\\ \overline{y} = f_2(\overline{x},\overline{y}).\end{cases}$$

We continue with the following definition ([7], Definition 2.1, p. 2).

Definition 2.2. *Let* $(\overline{x},\overline{y})$ *be an equilibrium point of the two-dimensional discrete dynamical system* (1).

(1) *The equilibrium point* $(\overline{x},\overline{y})$ *is* stable *if, for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that, for every initial condition* (x_0, y_0)*, if* $\|(x_0,y_0)-(\overline{x},\overline{y})\| < \delta$ *then* $\|(x_n,y_n)-(\overline{x},\overline{y})\| < \varepsilon$*, for all* $n > 0$*, where* $\|\cdot\|$ *is the usual Euclidean norm in* $\mathbb{R}^2$.
(2) *The equilibrium point* $(\overline{x},\overline{y})$ *is* unstable *if it is not stable.*
(3) *The equilibrium point* $(\overline{x},\overline{y})$ *is* asymptotically stable *if there exists* $\eta > 0$ *such that if* $\|(x_0,y_0)-(\overline{x},\overline{y})\| < \eta$*, then the solution with initial data* (x_0,y_0) *verifies* $(x_n,y_n)\to(\overline{x},\overline{y})$*, as* $n\to\infty$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a map written in the following form

$$\begin{cases}x \mapsto Ax + f(x,y),\\ y \mapsto By + g(x,y),\end{cases} \tag{2}$$

where $(x,y) \in \mathbb{R}^2$, A and B are real numbers such that $|A| = 1$ and $|B| < 1$, the functions f and g are of class C^2 and the functions f, g, and their first order partial derivatives are zero at the origin.

The following theorem ([10], Theorem 5.1, p. 243) establishes the existence of a (non-unique) centre manifold (i.e., a curve $y = h(x)$) on which the dynamics of the system (2) is given by the map on the centre manifold.

Theorem 2.1. *There is a* C^r *centre manifold for system* (2) *that can be represented locally, for a sufficiently small* δ*, as*

$$M_c = \left\{(x,y) \in \mathbb{R}^2 \mid y = h(x), |x| < \delta, h(0) = 0, h'(0) = 0\right\}.$$

Furthermore, the dynamics restricted to the center manifold M_c *are given locally by the map*

$$x \mapsto Ax + f(x,h(x)), \quad x \in \mathbb{R}.$$

If the system (2) is C^k-smooth, with $k < \infty$, then the center manifold is C^k-smooth ([12], Theorem 5A.1, p. 68).

In order to determine the expression of the function $h(x)$ we have to solve the equation

$$h(Ax + f(x,h(x))) = Bh(x) + g(x,h(x)).$$

For functions $\psi : \mathbb{R} \to \mathbb{R}$, define

$$F(\psi(x)) = \psi(Ax + f(x,\psi(x))) - B\psi(x) - g(x,\psi(x)),$$

so that $F(h(x)) = 0$.

Mostly, the function $h(x)$ cannot be found explicitly, and, therefore, it is approximated using power series. The following theorem ([4], Theorem 7, p. 35) provides the theoretical justification for this approximation.

Theorem 2.2. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a C^1 map with $\psi(0) = \psi'(0) = 0$. Suppose that*

$$F(\psi(x)) = O(|x|^r),$$

as $x \to 0$, for some $r > 1$. Then

$$h(x) = \psi(x) + O(|x|^r), \quad \text{as } x \to 0.$$

We continue with the following theorem ([4], Theorem 8, p. 35), which shows that the dynamics of the centre manifold M_c completely determines the dynamics of (2).

Theorem 2.3. *If the equilibrium point $(0,0)$ of the map $x \mapsto Ax + f(x, h(x))$ is stable/asymptotically stable/unstable, then the equilibrium point $(0,0)$ of the system (2) is stable/asymptotically stable/unstable.*

Finally, to determine the nature of the equilibrium point of the previous map we will use the following result ([10], Theorem 1.5, p. 28).

Theorem 2.4. *Let $\overline{x}$ be an equilibrium point of a one-dimensional map G such that $G'(\overline{x}) = 1$. If $G'(x)$, $G''(x)$ and $G'''(x)$ are continuous at $\overline{x}$, then the following statements hold true:*

(1) *if $G''(\overline{x}) \neq 0$, then $\overline{x}$ is unstable;*
(2) *if $G''(\overline{x}) = 0$ and $G'''(\overline{x}) > 0$, then $\overline{x}$ is unstable;*
(3) *if $G''(\overline{x}) = 0$ and $G'''(\overline{x}) < 0$, then $\overline{x}$ is asymptotically stable.*

3. **Main results**

As we have stated in the first section, we will study the stability of the zero equilibrium of the following system of difference equations:

$$\begin{cases} x_{n+1} = (1-a_1)x_n - b_1 x_n y_n + c_1 y_n e^{-d_1 x_n}, \\ y_{n+1} = (1-a_2)y_n - b_2 x_n y_n + c_2 x_n e^{-d_2 y_n}, \end{cases} \quad n = 0, 1, 2, ..., \tag{3}$$

where the parameters $a_1, c_1, a_2, c_2 \in (0,1)$, b_1, b_2 are negative numbers and the initial conditions x_0 and y_0 are positive numbers.

The following theorem represents the main result of the paper.

Theorem 3.1. *Let us consider:*

$$\alpha = \frac{a_1}{c_1}, \quad \beta = -\frac{a_2}{c_1}, \quad \eta = \frac{\alpha}{(1-\lambda_2)(\beta - \alpha)} \left[\alpha (b_1 + c_1 d_1) - (b_2 + c_2 d_2)\right],$$

$$\gamma = \frac{2\alpha}{\beta - \alpha} \left[-\beta (b_1 + c_1 d_1) + b_2 + c_2 d_2\right],$$

and

$$\delta = \frac{3}{\beta-\alpha}\left(-2\beta^2\eta b_1 - 2\alpha\beta\eta b_1 - 2\alpha\beta\eta c_1 d_1 + \alpha\beta c_1 - 2\beta^2\eta c_1 d_1 + \right.$$
$$\left. +2\beta\eta b_2 + 2\alpha\eta b_2 + 2\beta\eta c_2 d_2 - \alpha^2 c_2 + 2\alpha\eta c_2 d_2\right).$$

If $a_1 a_2 = c_1 c_2$, then the following statements hold true:

(1) *if $\gamma \neq 0$, then the zero equilibrium of* (3) *is unstable;*
(2) *if $\gamma = 0$ and $\delta > 0$, then the zero equilibrium of* (3) *is unstable;*
(3) *if $\gamma = 0$ and $\delta < 0$, then the zero equilibrium of* (3) *is asymptotically stable.*

Proof. We consider the following functions:

$$\begin{cases} f_1(x,y) = (1-a_1)x - b_1 xy + c_1 y e^{-d_1 x}, \\ f_2(x,y) = (1-a_2)y - b_2 xy + c_2 x e^{-d_2 y}. \end{cases}$$

The first order partial derivatives of the functions f_1 and f_2 are

$$\frac{\partial f_1}{\partial x} = 1 - a_1 - b_1 y - c_1 d_1 y e^{-d_1 x}, \quad \frac{\partial f_1}{\partial y} = -b_1 x + c_1 e^{-d_1 x},$$

$$\frac{\partial f_2}{\partial x} = -b_2 y + c_2 e^{-d_2 y} \quad \text{and} \quad \frac{\partial f_2}{\partial y} = 1 - a_2 - b_2 x - c_2 d_2 x e^{-d_2 y}.$$

Hence

$$\frac{\partial f_1}{\partial x}(0,0) = 1 - a_1, \quad \frac{\partial f_1}{\partial y}(0,0) = c_1, \quad \frac{\partial f_2}{\partial x}(0,0) = c_2, \quad \frac{\partial f_2}{\partial y}(0,0) = 1 - a_2,$$

and the Jacobian matrix at the zero equilibrium is

$$J_0 = \begin{pmatrix} 1-a_1 & c_1 \\ c_2 & 1-a_2 \end{pmatrix}.$$

To determine the eigenvalues of the matrix A, we calculate the following determinant

$$\begin{aligned} \det(J_0 - \lambda I_2) &= (1-a_1-\lambda)(1-a_2-\lambda) - c_1 c_2 \\ &= \lambda^2 + (a_1 + a_2 - 2)\lambda + (1 - a_1 - a_2 + a_1 a_2 - c_1 c_2). \end{aligned}$$

Taking into account, from the hypothesis, that $a_1 a_2 = c_1 c_2$, we obtain that the eigenvalues of the matrix J_0 are $\lambda_1 = 1$ and $\lambda_2 = 1 - a_1 - a_2$. Again, from the hypothesis we have that $|\lambda_2| < 1$.

We rewrite the initial system as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = J_0 \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{pmatrix}, \tag{4}$$

where

$$f(x,y) = -b_1 xy + c_1 y e^{-d_1 x} - c_1 y$$

and

$$g(x,y) = -b_2 xy + c_2 x e^{-d_2 y} - c_2 x.$$

It can be easily demonstrated that the vector $v_1 = \begin{pmatrix} 1 \\ \frac{a_1}{c_1} \end{pmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_1 = 1$ and the vector $v_2 = \begin{pmatrix} 1 \\ -\frac{a_2}{c_1} \end{pmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_2 = 1 - a_1 - a_2$. Then a matrix which diagonalizes matrix J_0 is

$$T = \begin{pmatrix} 1 & 1 \\ \alpha & \beta \end{pmatrix},$$

where $\alpha = \frac{a_1}{c_1}$ and $\beta = -\frac{a_2}{c_1}$.

Using the previous matrix T, we write

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = T \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \tag{5}$$

equivalent to

$$\begin{cases} x_n = u_n + v_n, \\ y_n = \alpha u_n + \beta v_n. \end{cases} \tag{6}$$

With this, the relationship (4) becomes

$$T \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = J_0 T \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} f(u_n + v_n, \alpha u_n + \beta v_n) \\ g(u_n + v_n, \alpha u_n + \beta v_n) \end{pmatrix}.$$

We multiply our last relationship with the inverse of the matrix T and obtain

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = D \begin{pmatrix} u_n \\ v_n \end{pmatrix} + T^{-1} \begin{pmatrix} f(u_n + v_n, \alpha u_n + \beta v_n) \\ g(u_n + v_n, \alpha u_n + \beta v_n) \end{pmatrix},$$

where $D = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $T^{-1} = \frac{1}{\beta - \alpha} \begin{pmatrix} \beta & -1 \\ -\alpha & 1 \end{pmatrix}$.

Hence

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} \widehat{f}(u_n, v_n) \\ \widehat{g}(u_n, v_n) \end{pmatrix},$$

where

$$\begin{aligned} \widehat{f}(u_n, v_n) &= \frac{1}{\beta - \alpha} \left[\beta f(u_n + v_n, \alpha u_n + \beta v_n) - g(u_n + v_n, \alpha u_n + \beta v_n) \right] \\ &= \frac{1}{\beta - \alpha} \left[\beta \left(-b_1 (u_n + v_n)(\alpha u_n + \beta v_n) + \right. \right. \\ &\quad \left. + c_1 (\alpha u_n + \beta v_n) e^{-d_1 (u_n + v_n)} - c_1 (\alpha u_n + \beta v_n) \right) - \\ &\quad - \left(-b_2 (u_n + v_n)(\alpha u_n + \beta v_n) + c_2 (u_n + v_n) e^{-d_2 (\alpha u_n + \beta v_n)} - \right. \\ &\quad \left. \left. - c_2 (u_n + v_n) \right) \right] \end{aligned}$$

and

$$\begin{aligned}
\widehat{g}(u_n, v_n) &= \frac{1}{\beta-\alpha}\left[-\alpha f(u_n+v_n, \alpha u_n+\beta v_n) + g(u_n+v_n, \alpha u_n+\beta v_n)\right] \\
&= \frac{1}{\beta-\alpha}\left[-\alpha\left(-b_1(u_n+v_n)(\alpha u_n+\beta v_n) + \right.\right. \\
&\quad + c_1(\alpha u_n+\beta v_n)e^{-d_1(u_n+v_n)} - c_1(\alpha u_n+\beta v_n)\big) + \\
&\quad + \big(-b_2(u_n+v_n)(\alpha u_n+\beta v_n) + c_2(u_n+v_n)e^{-d_2(\alpha u_n+\beta v_n)} - \\
&\quad - c_2(u_n+v_n))]
\end{aligned}$$

Based on the Theorem 2.2, we consider

$$v = h(u) = \phi(u) + O\left(u^4\right),$$

with $\phi(u) = \eta u^2 + \theta u^3$, where η and θ are real numbers.

According to the Theorem 2.3, the study of the stability of the zero equilibrium of the system (3) is reduced to the study of the stability of the zero equilibrium of the equation

$$u_{n+1} = u_n + \widehat{f}(u_n, \phi(u_n)). \tag{7}$$

Now, consider the map

$$G(u) = u + \widehat{f}(u, \phi(u)),$$

which leads to

$$\begin{aligned}
G(u) &= u + \widehat{f}\left(u, \eta u^2 + \theta u^3\right) \\
&= u + \frac{1}{\beta-\alpha}\left[\beta\left(-b_1\left(u+\eta u^2+\theta u^3\right)\left(\alpha u+\beta\eta u^2+\beta\theta u^3\right) + \right.\right. \\
&\quad + c_1\left(\alpha u+\beta\eta u^2+\beta\theta u^3\right)e^{-d_1\left(u+\eta u^2+\theta u^3\right)} - \\
&\quad - c_1\left(\alpha u+\beta\eta u^2+\beta\theta u^3\right)\big) - \left(-b_2\left(u+\eta u^2+\theta u^3\right)\left(\alpha u+\beta\eta u^2+\beta\theta u^3\right)\right. \\
&\quad \left.\left. + c_2\left(u+\eta u^2+\theta u^3\right)e^{-d_2\left(\alpha u+\beta\eta u^2+\beta\theta u^3\right)} - c_2\left(u+\eta u^2+\theta u^3\right)\right)\right]
\end{aligned}$$

Using the power series expansion, we find that

$$\begin{aligned}
G(u) &= u + \frac{\alpha}{\beta-\alpha}\left[-\beta(b_1+c_1d_1) + b_2 + c_2d_2\right]u^2 + \\
&\quad + \frac{1}{\beta-\alpha}\left(-\beta^2\eta b_1 - \alpha\beta\eta b_1 - \alpha\beta\eta c_1d_1 + \frac{1}{2}\alpha\beta c_1 - \beta^2\eta c_1d_1 + \right. \\
&\quad \left. + \beta\eta b_2 + \alpha\eta b_2 + \beta\eta c_2d_2 - \frac{1}{2}\alpha^2 c_2 + \alpha\eta c_2d_2\right)u^3 + O(u^4).
\end{aligned}$$

Then

$$G'(0) = 1, \quad G''(0) = \frac{2\alpha}{\beta-\alpha}\left[-\beta(b_1+c_1d_1) + b_2 + c_2d_2\right]$$

and

$$\begin{aligned} G'''(0) &= \frac{3}{\beta-\alpha}\left(-2\beta^2\eta b_1 - 2\alpha\beta\eta b_1 - 2\alpha\beta\eta c_1 d_1 + \alpha\beta c_1 - 2\beta^2\eta c_1 d_1 + \right.\\ &\left. +2\beta\eta b_2 + 2\alpha\eta b_2 + 2\beta\eta c_2 d_2 - \alpha^2 c_2 + 2\alpha\eta c_2 d_2\right). \end{aligned}$$

Now we will determine the expression of η. According to the comments following the Theorem 6 from [4], the map h satisfies the centre manifold equation

$$h\left(u + \widehat{f}\,(u, h(u))\right) = \lambda_2 h(u) + \widehat{g}\,(u, h(u)).$$

As before, using the power series expansion, we find that

$$\eta = \frac{\alpha}{(1-\lambda_2)(\beta-\alpha)}\left[\alpha\left(b_1 + c_1 d_1\right) - (b_2 + c_2 d_2)\right].$$

Finally, according to the Theorem 2.4, we have:

- if $G''(0) \neq 0$, then the zero equilibrium of (7) is unstable, therefore, due to the Theorem 2.3, the zero equilibrium of the system (3) is unstable;
- if $G''(0) = 0$ and $G'''(0) > 0$, then the zero equilibrium of (7) is unstable, therefore, due to the Theorem 2.3, the zero equilibrium of the system (3) is unstable;
- if $G''(0) = 0$ and $G'''(0) < 0$, then the zero equilibrium of (7) is asymptotically stable, therefore, due to the Theorem 2.3, the zero equilibrium of the system (3) is asymptotically stable.

The conclusions of the Theorem 3.1 follow immediately. □

4. **Conclusions**

Using centre manifold theory, we provided some conditions for the stability of the zero equilibrium of the system (3).

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