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NEW RATIONAL IDENTITIES FOR ENTROPY INVOLVING ZEROS AND COEFFICIENTS OF COMPLEX POLYNOMIALS

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În această lucrare vom prezenta o metodă de obținere de identități pentru polinoame în planul complex, folosind tehnici analitice din analiza complexă. Rezultatele prezentate sunt identități rationale cu sume finite conținând zerouri şi coeficienți ai polinoamelor complexe. În ultima parte a articolului vom aplica aceste rezultate în teoria informației şi vom obtine noi identități pentru entropia Shannon.

In this paper, a method to obtain polynomial identities in the complex plane using analytic techniques from complex analysis is presented. The results given here are rational identities involving finite sums of zeros and coefficients of complex polynomials. In the last part of the paper we apply these results in information theory and we obtain new identities for Shannon Entropy.

Keywords: Rational identities; identities in the complex plane; complex polynomials; finite sums; entropy.

1. Introduction

The problem of finding relations between the zeros and coefficients of a polynomial occupies a central role in the theory of equations. The most known of such relations are Cardan-Viète's formulas.

Namely, if the complex polynomial $A(z) = \sum_{k=0}^{n} a_k z^k,\ a_n \neq 0$ has the zeros $z_1, z_2, \cdots, z_n$, then

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} z_{i_1} z_{i_2} \cdots z_{i_k} = (-1)^k \frac{a_{n-k}}{a_n}, 1 \le k \le n.$$

In this paper, taking into account the preceding relations and using complex variable techniques like the ones employed by H. W. Gould in 1956 to evaluate Stirling numbers ([1], [2]), later by G. Egorychev in 1977 [3] to obtain combinatorial identities, some rational zero and coefficient identities involving complex polynomials similar to the one given in [4] are obtained.

Finally we apply the results into information theory, where we obtain new identities for entropy. Entropy is the uncertainty of a single random variable. We

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will use the following notations: let X be a discrete random variable with the alphabet χ and the probability mass function $\mathrm{p(x) = Pr\{X = x\}, x \in \chi}$. We define the entropy *H(X)* of a random discrete variable as follows

$$H(X) = -\sum_{x\epsilon\chi} p(x) log p(x).$$

The logarithm is to the base *2*, and entropy is expressed in bits. We use the convention $0 \log 0 = 0$, easily justifiable by continuity, as $\lim_{x\to 0} x \log x = 0$.

Entropy, conditional entropy and mutual information for discrete-valued random variables are extremely useful in developing theorems in the field of information theory, as they offer information on the long term behavior of random processes ([5], [6]).

2. Rational identities for complex polynomials

Hereafter some rational identities are given. The first one, that it is an immediate consequence of the Fundamental Theorem of Algebra, is stated and proven in the following

Theorem 1. Suppose that the distinct nonzero complex numbers $z_1, z_2, \cdots, z_n$ are the zeros of the complex polynomial $A(z) = \sum_{k=0}^{n} a_k z^k, a_k \neq 0$ for $0 \leq k \leq n$. Then the following holds:

$$\frac{1}{a_0} + \sum_{k=1}^{n} \frac{1}{z_k A'(z_k)} = 0.$$

Proof. Let $B(z)$ be the complex polynomial defined by

$$B(z) = -1 + \sum_{i=1}^{n} \prod_{i\neq j} \frac{z - z_j}{z_i - z_j}.$$

Since $B(z)$ has the $n-1$ degree and $B(z_k) = 0$ for $1 \leq k \leq n$, then, according to the Fundamental Theorem of Algebra, $B(z)$ is identically zero. Therefore,

$$1 = 1 + B(0) = \sum_{i=1}^{n} \prod_{i\neq j} \frac{-z_j}{z_i - z_j} = (-1)^{n-1} z_1 z_2 \ldots z_n \sum_{i=1}^{n} \frac{1}{z_i} \prod_{j\neq i} \frac{1}{z_i - z_j}$$

or

$$\sum_{i=1}^{n} \frac{1}{z_i} \prod_{j\neq i} \frac{1}{z_i - z_j} = \frac{(-1)^{n-1}}{z_1 z_2 \ldots z_n}.$$

From the preceding and taking into account Cardan-Viète formulas, we get

$$\sum_{i=1}^{n}\frac{1}{z_i}\prod_{j\neq i}\frac{1}{z_i-z_j}=\sum_{k=1}^{n}\frac{a_n}{z_kA'(z_k)}=\frac{(-1)^{n-1}a_n}{(-1)^na_0},$$

from which the statement immediately follows and the result is proven. □

In what follows, the results presented show how to apply complex variable techniques in order to generate new identities.

Theorem 2. Let $A(z)=\sum_{k=0}^{n}a_k\,z^k, a_k\neq 0$ for $0\leq k\leq n$ be a polynomial with complex coefficients having the simple zeros $z_1,\ z_2,\cdots,\ z_n$. Then holds

$$\sum_{k=1}^{n}\frac{z_k^{n-1}}{A'(z_k)}=\frac{1}{a_n}.$$

Proof. Consider the integral $I=\frac{1}{2\pi i}\oint_\gamma\ \frac{z^{n-1}}{A(z)}dz$, where γ is a circle centered in the origin and of radius $r>\max_{1\leq k\leq n}\{|z_k|\}$. Integrating inside the γ contour we have

$$I_1=\frac{1}{2\pi i}\oint_\gamma\ \frac{z^{n-1}}{A(z)}dz=\sum_{k=1}^{n}Res\left(\frac{z^{n-1}}{A(z)},z=z_k\right)$$
$$=\frac{1}{a_n}\sum_{k=1}^{n}z_k^{n-1}\prod_{j\neq k}\frac{1}{z_k-z_j}=\sum_{k=1}^{n}\frac{z_k^{n-1}}{A'(z_k)}.$$

On the other hand, evaluating the integral outside γ contour we have

$$I_2=\frac{1}{2\pi i}\oint_\gamma\ \frac{z^{n-1}}{A(z)}dz=Res\left(\frac{z^{n-1}}{A(z)},z=\infty\right)=Res\left(\frac{-1}{z^2}f\left(\frac{1}{z}\right),z=0\right),$$

where $f(z)=\frac{z^{n-1}}{A(z)}$. Therefore,

$$I_2=Res\left(-\frac{1}{z^2}\frac{1}{z^{n-1}A\left(\frac{1}{z}\right)},z=0\right)=-Res\left(\frac{1}{z^{n+1}A\left(\frac{1}{z}\right)},z=0\right)$$
$$=-Res\left(\frac{1}{z(a_n+a_{n-1}z+\cdots+a_0z^n},z=0\right)=-\frac{1}{a_n}.$$

Applying Cauchy's theorem on contour integrals, we have $I_1+I_2=0$ and we are done. □

Theorem 3. Suppose that the distinct nonzero complex numbers $z_1,z_2,\cdots,z_n$ are the zeros of the monic polynomial with complex coefficients $A(z)=z^n+\sum_{k=0}^{n-1}a_kz^k$. Then, for all $3\leq l\leq n+1$ holds

$$\sum_{k=1}^{n} \frac{1+z_k^{l-1}}{z_k^2 A'(z_k)} + (-1)^n \sum_{k=1}^{n} \frac{1}{z_k} \prod_{j=1}^{n} \frac{1}{z_j} = 0.$$

Proof. We will evaluate the integral $I = \oint_\gamma \frac{1+z^{l-1}}{z^2 A(z)} dz$ over the interior and exterior domains limited by γ, a circle centered at the origin with the radius $r < \min_{1 \le k \le n}\{|z_k|\}$. Integrating in the region outside of the γ contour we have

$$I_1 = \frac{1}{2\pi i} \oint_\gamma \frac{z^{l-1}+1}{z^2 A(z)} dz = \sum_{k=1}^{n} Res\left\{\frac{z^{l-1}}{z^2 A(z)}, z = z_k\right\}$$
$$= \sum_{k=1}^{n} \frac{1+z_k^{l-1}}{z_k^2 A'(z_k)}.$$

Integrating in the region inside of the γ contour and taking into account the Cardan-Viète formulas, we get

$$I_2 = \frac{1}{2\pi i} \oint_\gamma \frac{z^{l-1}+1}{z^2 A(z)} dz = Res\left\{\frac{z^{l-1}+1}{z^2 A(z)}, z = 0\right\}$$
$$= \frac{1}{1!} \lim_{z \to 0} \frac{d}{dz}\left(\frac{z^{l-1}+1}{A(z)}\right)$$
$$= -\frac{A'(0)}{A^2(0)} = -\frac{a_1}{a_0^2}$$
$$= -(-1)^{n-1} \sum_{k=1}^{n} \frac{1}{z_k} \prod_{j=1}^{n} \frac{1}{z_j}.$$

Applying again Cauchy's result on contour integrals we have $I_1 + I_2 = 0$ and the proof is complete. □

The same procedure can be applied to obtain many identities. In the following, we give several zero identities that have been obtained applying this technique. We begin with a result presented in [4].

Theorem 4. Suppose that $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ is a polynomial of degree n, $n \ge 2$. $A(z)$ has n distinct zeros $z_1, z_2, \cdots, z_n$, and ζ is a complex number such that $\min_{1 \le k \le n}\{|z_k|\} > \max_{1 \le k \le n}\{|z_k/\zeta|\}$. Then we have

$$\sum_{k=1}^{n} \frac{1}{A'(z_k)} \left(\frac{1}{A(\zeta z_k)} + \frac{1}{\zeta A\left(\frac{z_k}{\zeta}\right)}\right) = 0.$$

Proof. We will evaluate the integral $I = \frac{1}{2\pi i}\oint_\gamma \frac{1}{A(z)A(\zeta z)}dz$ over the interior and exterior domains limited by γ, where $\gamma = \{z \in \mathbb{C} |\, |z| < r\}$, with $\min_{1\le k\le n}\{|z_k|\} > r > \max_{1\le k\le n}\{|z_k/\zeta|\}$. Integrating in the region outside of the γ contour we have:

$$I_1 = \frac{1}{2\pi i}\oint_\gamma \frac{1}{A(z)A(\zeta z)}dz = \sum_{k=1}^{n} Res\left\{\frac{1}{A(z)A(\zeta z)}, z = z_k\right\}$$
$$= \sum_{k=1}^{n}\frac{1}{A(\zeta z_k)}\prod_{j\ne k}^{n}\frac{1}{z_k - z_j} = \sum_{k=1}^{n}\frac{1}{A(\zeta z_k)A'(z_k)}.$$

Integrating in the region inside of the γ contour we get

$$I_2 = \frac{1}{2\pi i}\oint_\gamma \frac{1}{A(z)A(\zeta z)}dz = \sum_{k=1}^{n} Res\left\{\frac{1}{A(z)A(\zeta z)}, z = z_k/\zeta\right\}$$
$$= \sum_{k=1}^{n}\frac{1}{\zeta A(z_k/\zeta)}\prod_{j\ne k}^{n}\frac{1}{z_k - z_j} = \sum_{k=1}^{n}\frac{1}{\zeta A(z_k/\zeta)A'(z_k)}.$$

Taking into account Cauchy's result on contour integrals, we have $I_1 + I_2 = 0$ and this completes the proof. □

We continue generalizing the previous result as follows:

Theorem 5. Assume that the zeros $z_1, z_2, \cdots, z_n$ of the polynomial $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ are distinct nonzero complex numbers. Let ζ_1 and ζ_2 be two complex numbers such that $\max_{1\le k\le n}\left\{\left|\frac{z_k}{\zeta_2}\right|\right\} < \min_{1\le k\le n}\{|z_k/\zeta_1|\}$. Then the following hold:

$$\sum_{k=1}^{n}\frac{1}{A'(z_k)}\left(\frac{1}{\zeta_1 A\left(\frac{z_k\zeta_2}{\zeta_1}\right)} + \frac{1}{\zeta_2 A\left(\frac{z_k\zeta_1}{\zeta_2}\right)}\right) = 0,$$

$$\sum_{k=1}^{n}\frac{1}{A'(z_k)}\left(\frac{A\left(\frac{z_k}{\zeta_1}\right)}{\zeta_1 A\left(\frac{z_k\zeta_2}{\zeta_1}\right)} + \frac{A\left(\frac{z_k}{\zeta_2}\right)}{\zeta_2 A\left(\frac{z_k\zeta_1}{\zeta_2}\right)}\right) = 0,$$

$$\sum_{k=1}^{n}\frac{1}{A'(z_k)}\left(\frac{A(\zeta_1 z_k)}{\zeta_2 A\left(\frac{z_k\zeta_1}{\zeta_2}\right)} + \frac{A(\zeta_2 z_k)}{\zeta_1 A\left(\frac{z_k\zeta_2}{\zeta_1}\right)}\right) = 0.$$

Proof. We will begin by proving the last identity as follows, evaluating the integral $I = \frac{1}{2\pi i}\oint_\gamma \frac{A(\zeta_1\zeta_2 z)}{A(\zeta_1 z)A(\zeta_2 z)}dz$ over the interior and exterior domains limited by γ, a circle centered in the origin with the radius r, such that $\max_{1\le k\le n}\left\{\left|\frac{z_k}{\zeta_2}\right|\right\} < r < \min_{1\le k\le n}\{|z_k/\zeta_1|\}$. Integrating in the region inside of the γ contour we have

$$I_1 = \frac{1}{2\pi i}\oint_\gamma \frac{A(\zeta_1\zeta_2 z)}{A(\zeta_1 z)A(\zeta_2 z)}dz = \sum_{k=1}^{n} Res\left\{\frac{A(\zeta_1\zeta_2 z)}{A(\zeta_1 z)A(\zeta_2 z)}, z = \frac{z_k}{\zeta_2}\right\}$$

$$= \sum_{k=1}^{n} \frac{A(\zeta_1 z_k)}{\zeta_2 A\left(\frac{z_k\zeta_1}{\zeta_2}\right)}\prod_{j\neq k}\frac{1}{z_k - z_j} = \sum_{k=1}^{n}\frac{1}{A'(z_k)}\left(\frac{A(\zeta_1 z_k)}{\zeta_2 A\left(\frac{z_k\zeta_1}{\zeta_2}\right)}\right).$$

Integrating in the region outside of the γ contour we get

$$I_2 = \frac{1}{2\pi i}\oint_\gamma \frac{A(\zeta_1\zeta_2 z)}{A(\zeta_1 z)A(\zeta_2 z)}dz = \sum_{k=1}^{n} Res\left\{\frac{A(\zeta_1\zeta_2 z)}{A(\zeta_1 z)A(\zeta_2 z)}, z = \frac{z_k}{\zeta_1}\right\}$$

$$= \sum_{k=1}^{n} \frac{A(\zeta_2 z_k)}{\zeta_1 A\left(\frac{z_k\zeta_2}{\zeta_1}\right)}\prod_{j\neq k}\frac{1}{z_k - z_j} = \sum_{k=1}^{n}\frac{1}{A'(z_k)}\left(\frac{A(\zeta_2 z_k)}{\zeta_1 A\left(\frac{z_k\zeta_2}{\zeta_1}\right)}\right).$$

According to Cauchy's result on contour integrals we have $I_1 + I_2 = 0$, and the identity is proven.

The other two identities can be proven using the same method, this time evaluating the integrals $J = \frac{1}{2\pi i}\oint_\gamma \frac{1}{A(\zeta_1 z)A(\zeta_2 z)}dz$ and $K = \frac{1}{2\pi i}\oint_\gamma \frac{A(z)}{A(\zeta_1 z)A(\zeta_2 z)}dz$ over the interior and exterior domains limited by γ, where $\gamma = \{z\epsilon\mathbb{C}\,|\,|z| < r\}$, with $\min_{1\le k\le n}\{|z_k|\} > r > \max_{1\le k\le n}\{|z_k/\zeta|\}$ and we are done. □

We continue with another general result as follows:

Theorem 6. Assume that the zeros $z_1, z_2, \cdots, z_n$ of the polynomial $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ are distinct nonzero complex numbers. Let ζ_1 and ζ_2 be two complex numbers such that $\max_{1\le k\le n}\left\{\left|\frac{z_k}{\zeta_2}\right|\right\} < \min_{1\le k\le n}\left\{\left|\frac{z_k}{\zeta_1}\right|\right\}$ and $\max_{1\le k\le n}\left\{\left|\frac{z_k}{1}\right|\right\} < \min_{1\le k\le n}\{|z_k|\}$. Then the following identities hold:

$$\sum_{k=1}^{n}\frac{1}{A'(z_k)}\left(\frac{1}{A(z_k\zeta_2)A(z_k\zeta_1)} + \frac{1}{\zeta_1 A\left(\frac{z_k}{\zeta_1}\right)A\left(\frac{z_k\zeta_2}{\zeta_1}\right)} + \frac{1}{\zeta_2 A\left(\frac{z_k}{\zeta_2}\right)A\left(\frac{z_k\zeta_1}{\zeta_2}\right)}\right) = 0,$$

$$\sum_{k=1}^{n}\frac{1}{A'(z_k)}\left(\frac{A(\zeta_1\zeta_2 z_k)}{A(z_k\zeta_2)A(z_k\zeta_1)}+\frac{A(\zeta_1 z_k)}{\zeta_2 A\left(\frac{z_k}{\zeta_2}\right)A\left(\frac{z_k\zeta_1}{\zeta_2}\right)}+\frac{A(\zeta_2 z_k)}{\zeta_1 A\left(\frac{z_k}{\zeta_1}\right)A\left(\frac{z_k\zeta_2}{\zeta_1}\right)}\right)=0.$$

Proof. We will start to prove the first identity as follows, evaluating the integral $I=\frac{1}{2\pi i}\oint_\gamma \frac{1}{A(z)A(\zeta_1 z)A(\zeta_2 z)}dz$ over the interior and exterior domains limited by γ, a circle with the radius r centered in the origin, such that $\max_{1\le k\le n}\left\{\left|\frac{z_k}{\zeta_2}\right|\right\}<$ $r<\min_{1\le k\le n}\left\{\left|\frac{z_k}{\zeta_1}\right|\right\}$. Integrating in the region inside of the γ contour we have

$$\begin{aligned}I_1&=\frac{1}{2\pi i}\oint_\gamma \frac{1}{A(z)A(\zeta_1 z)A(\zeta_2 z)}dz\\&=\sum_{k=1}^{n}Res\left\{\frac{1}{A(z)A(\zeta_1 z)A(\zeta_2 z)},z=\frac{z_k}{\zeta_2}\right\}\\&=\sum_{k=1}^{n}\frac{1}{\zeta_2 A\left(\frac{z_k}{\zeta_2}\right)A\left(\frac{z_k\zeta_1}{\zeta_2}\right)}\prod_{j\neq k}\frac{1}{z_k-z_j}\\&=\sum_{k=1}^{n}\frac{1}{A'(z_k)}\left(\frac{1}{\zeta_2 A\left(\frac{z_k}{\zeta_2}\right)A\left(\frac{z_k\zeta_1}{\zeta_2}\right)}\right).\end{aligned}$$

Integrating in the region outside of the γ contour, we get

$$\begin{aligned}I_2&=\frac{1}{2\pi i}\oint_\gamma \frac{1}{A(z)A(\zeta_1 z)A(\zeta_2 z)}dz\\&=\sum_{k=1}^{n}Res\left\{\frac{1}{A(z)A(\zeta_1 z)A(\zeta_2 z)},z=\frac{z_k}{\zeta_1}\right\}\\&+\sum_{k=1}^{n}Res\left\{\frac{1}{A(z)A(\zeta_1 z)A(\zeta_2 z)},z=z_k\right\}\\&=\sum_{k=1}^{n}\frac{1}{A'(z_k)}\left(\frac{1}{A(z_k\zeta_2)A(z_k\zeta_1)}+\frac{1}{\zeta_1 A\left(\frac{z_k}{\zeta_1}\right)A\left(\frac{z_k\zeta_2}{\zeta_1}\right)}\right).\end{aligned}$$

According to Cauchy's result on contour integrals we have $I_1+I_2=0$ and the identity is proven.

The other identity can be proved by using the same method, this time evaluating the integral $J = \frac{1}{2\pi i}\oint_\gamma \frac{A(\zeta_1\zeta_2 z)}{A(z)A(\zeta_1 z)A(\zeta_2 z)}dz$ over the interior and exterior domains limited by γ, where $\gamma = \{z\epsilon\mathbb{C} \mid |z| < r\}$ and $\max_{1\le k\le n}\left\{\left|\frac{z_k}{\zeta_2}\right|\right\} < r < \min_{1\le k\le n}\left\{\left|\frac{z_k}{\zeta_1}\right|\right\}$ and we are done. □

3. Entropy Identities

In this section we will present new identities in information theory, obtained by using some polynomials and their zeros. We will begin by particularizing the general polynomial considered in Theorems 1 and 2, and we will obtain the following:

Theorem 7. Let the distinct non-zero complex numbers $z_1, z_2, \cdots, z_n$ be the zeros of the complex polynomial $A(z) = -\frac{1}{H(X)} + \sum_{k=1}^{n-1} a_k z^k + \frac{1}{H(X)} z^n, a_k \neq 0$ for $1 \le k \le n$. Then we will have

$$H(X) = \sum_{k=1}^{n} \frac{1}{z_k A'(z_k)} = \sum_{k=1}^{n} \frac{z_k^{n-1}}{A'(z_k)}.$$

Proof. It follows from Theorems 1 and 2, by particularizing the general polynomial $A(z)$ with $a_0 = -\frac{1}{H(X)}$ and $a_n = \frac{1}{H(X)}$. □

We will continue by presenting a consequence of Theorem 4, for which we will consider a random complex number.

Corollary 8. Let $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a polynomial of degree n with $n \ge 2$. $A(z)$ has n distinct zeros, $z_1, z_2, \cdots, z_n$, with $z_{n+1} = a \neq z_k, 1 \le k \le n$ and ζ a complex number for which $\min_{1\le k\le n+1}\{|z_k|\} > \max_{1\le k\le n+1}\left\{\left|\frac{z_k}{\zeta}\right|\right\}$. Then

$$\frac{1}{a} = \frac{(1-\zeta)A(a)A(\zeta a)A\left(\frac{a}{\zeta}\right)}{A\left(\frac{a}{\zeta}\right) - A(\zeta a)}$$

$$\sum_{k=1}^{n} \frac{1}{(z_k - a)A'(z_k)}\left[\frac{1}{(\zeta z_k - a)A(\zeta z_k)} + \frac{1}{(z_k - \zeta a)A\left(\frac{z_k}{\zeta}\right)}\right].$$

Proof. We will use the polynomial $\hat{A}(z) = (z - a)A(z)$ with its distinct zeros $z_1, z_2, \cdots, z_{n+1}$, where $z_{n+1} = a$. We apply the result of Theorem 4 for the polynomial $\hat{A}$, taking into account that $\hat{A}'(z) = A(z) + (z - a)A'(z)$ and we obtain

$$\frac{1}{A(a)}\left(\frac{1}{a(\zeta-1)A(\zeta a)}+\frac{1}{a\zeta\left(\frac{1}{\zeta}-1\right)A\left(\frac{a}{\zeta}\right)}\right)$$
$$+\sum_{k=1}^{n}\frac{1}{(z_k-a)A'(z_k)}\left(\frac{1}{(\zeta z_k-a)A(\zeta z_k)}+\frac{1}{\zeta\left(\frac{z_k}{\zeta}-a\right)A\left(\frac{z_k}{\zeta}\right)}\right)=0,$$

which is equivalent to

$$\frac{1}{(1-\zeta)aA(a)}\left(\frac{1}{A(\zeta a)}-\frac{1}{A\left(\frac{a}{\zeta}\right)}\right)$$
$$=\sum_{k=1}^{n}\frac{1}{(z_k-a)A'(z_k)}\left(\frac{1}{(\zeta z_k-a)A(\zeta z_k)}+\frac{1}{(z_k-\zeta a)A\left(\frac{z_k}{\zeta}\right)}\right),$$

and the conclusion follows. □

We continue by presenting a result that binds the zeros of a polynomial and the entropy of a random variable.

Theorem 9. Let $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a polynomial of degree n, with $n \geq 2$. $A(z)$ has n distinct zeros $z_1, z_2, \cdots, z_n$, such that $z_{n+1} = H(X) \neq z_k$, $1 \leq k \leq n$ and ζ a complex number with $\min_{1\leq k\leq n+1}\{|z_k|\} > \max_{1\leq k\leq n+1}\left\{\left|\frac{z_k}{\zeta}\right|\right\}$. Then

$$\frac{1}{H(X)}=\frac{(1-\zeta)A\big(H(X)\big)A\big(\zeta H(X)\big)A\left(\frac{H(X)}{\zeta}\right)}{A\left(\frac{H(X)}{\zeta}-A(\zeta H(X))\right)}$$
$$\sum_{k=1}^{n}\frac{1}{(z_k-H(X))A'(z_k)}\left[\frac{1}{(\zeta z_k-H(X))A(\zeta z_k)}+\frac{1}{\big(z_k-\zeta H(X)\big)A\left(\frac{z_k}{\zeta}\right)}\right].$$

Proof. It follows directly by replacing a with $H(X)$ in the last corollary. □

We will try to present a more detailed result, in order to provide a new identity related to entropy.

Theorem 10. Let $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a polynomial of degree n with $n \geq 2$. $A(z)$ has n distinct zeros $z_1, z_2, \cdots, z_n$ with $z_{n+1} = H_x = p(x) log \frac{1}{p(x)} \neq z_k, 1 \leq k \leq n$. Let ζ be a complex number such that $\min_{1 \leq k \leq n+1}\{|z_k|\} > \max_{1 \leq k \leq n+1}\left\{\left|\frac{z_k}{\zeta}\right|\right\}$. Then we will have

$$H(X) = \sum_{x \in X} \left\{ \frac{A\left(\frac{H_x}{\zeta}\right) - A(\zeta H_x)}{(1-\zeta) A(H_x) A(\zeta H_x) A\left(\frac{H_x}{\zeta}\right)} \left[\sum_{k=1}^{n} \frac{1}{(z_k - H_x) A'(z_k)} \left(\frac{1}{(\zeta z_k - H_x) A(\zeta z_k)} + \frac{1}{(z_k - \zeta H_x) A\left(\frac{z_k}{\zeta}\right)} \right) \right] \right\}^{-1},$$

where $p(x) > 0$ is a mass probability function.

Proof. It follows by summating, over $x \in X$, the result obtained applying Corollary 8 for $a = H_x = p(x) log \frac{1}{p(x)}$. □

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