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ON SEMI-INFINITE PROGRAMMING PROBLEMS AND STRONG KKT TYPE SUFFICIENT OPTIMALITY CONDITIONS

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In present manuscript, we consider a multiobjective non-smooth semi-infinite programming problem with vanishing constraints MOSIPVC. Under generalized invexity assumptions, we establish the strong Karush-Kuhn-Tucker (KKT) type sufficient optimality condition for MOSIPVC. Example is also given in order to support the theorem.

Keywords: Generalized convexity; vanishing constraints; optimality conditions; KKT conditions

Mathematics Subject Classification: 90C33, 49J52

1. Introduction

In Multi-objective programming [2, 19, 20], a semi-infinite problem arises when multiple objective functions need to be optimized over a domain of validity (feasible region) described by an infinite number of constraints. If a MOSIP has a single objective function, it is called a semi-infinite programming problem (SIP). The problem of SIP has played a vital role in many fields of modern research, such as control of air pollution [24], robot trajectory planning [23], engineering design [21], and transportation theory [16]. For more information on SIP and its applications see books [7, 22] and for more details on MOSIP one can see the recent articles [3, 14, 15].

Achtziger and Kanzow [1] presented a mathematical program using vanishing constraints MPVC and investigated that many structural topology optimization problems can be reformulated using MPVC. Hoheisel and Kanzow [9] introduced the stationary concepts for MPVC and obtained first-order sufficient optimality conditions and second-order necessary and sufficient optimality conditions for MPVC. Hoheisel and Kanzow [10] set the optimality condition for weak constraint qualifications. Mishra *et al.* [17] invesigate various constraint qualifications and specified a KKT-type necessary optimality condition for multi-objective MPVC.

The idea of strong KKT conditions was introduced in order to avoid the case where some of the Lagrange multipliers associated with the components of multiobjective functions vanish. Golestani and Nobakhtian [8] investigated the strong KKT optimality conditions for multi-objective optimization that is not smooth. Later, Kanzi [15] presented strong KKT optimality condition for MOSIP. Pandey and Mishra [18] gave strong KKT type sufficient optimality conditions for non-smooth MOSIP with equilibrium constraints. For more information on MPVC, see [5, 11, 12].

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Motivated by the work of Guu et. al. [6], we present strong KKT optimality conditions for the MOSIP with vanishing constraints MOSIPVC that do not include any constraint qualifications. This article is structured as follows. In Section 2, we give some well-known definitions and results which will be used further. In Section 3, we give stationary points and establish strong KKT type optimality conditions for MOSIPVC. Section 4, summarizes the findings of this article.

2. Definitions and preliminaries

We consider the following MOSIPVC:

$$\begin{aligned} MOSIPVC\ \ &\min f(v) := (f_1(v), f_2(v), , , , f_m(v)) \\ \text{subject to } &A_r(v) \leq 0,\ r \in \Omega \\ &C_\delta(v) \geq 0,\ \delta \in L; \text{where } L := \{1, ..., l\}, \\ &B_\delta(v)C_\delta(v) \leq 0,\ \delta \in L; \text{where } L := \{1, ..., l\}, \end{aligned}$$

where $f_\delta : \mathbb{R}^n \to \mathbb{R}, A_r : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, B_\delta : \mathbb{R}^n \to \mathbb{R}, C_\delta : \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz functions and Ω is an index set, which is arbitrary (possibly infinite). Let $M := \{v \in \mathbb{R}^n, A_r(v) \leq 0, r \in \Omega, C_\delta(v) \geq 0, B_\delta(v)C_\delta(v) \leq 0,\ \delta = 1, ..., l\}$ indicates the feasible region of MOSIPVC. A point $\bar{v} \in M$ is known as weakly efficient solution for the MOSIPVC if there does not exists any $v \in M$ such that

$$f_\delta(v) < f_\delta(\bar{v}),\ \forall \delta \in N; \text{where } N := \{1, 2, ..., m\}.$$

Let $\bar{v} \in M$. Then the index sets given below will be used further

$$\begin{aligned} \Omega(\bar{v}) &:= \{r \in \Omega;\ A_r(\bar{v}) = 0\}, \\ \alpha_+(\bar{v}) &:= \{\delta \in L;\ C_\delta(\bar{v}) > 0\}, \\ \alpha_0(\bar{v}) &:= \{\delta \in L;\ C_\delta(\bar{v}) = 0\}. \end{aligned}$$

Moreover, index set $\alpha_+(\bar{v})$ can also be divided as follows:

$$\begin{aligned} \alpha_{+0}(\bar{v}) &:= \{\delta \in L;\ C_\delta(\bar{v}) > 0, B_\delta(\bar{v}) = 0\}, \\ \alpha_{+-}(\bar{v}) &:= \{\delta \in L;\ C_\delta(\bar{v}) > 0, B_\delta(\bar{v}) < 0\}. \end{aligned}$$

Similarly, $\alpha_0(\bar{v})$ can also be written as:

$$\begin{aligned} \alpha_{0+}(\bar{v}) &:= \{\delta \in L;\ C_\delta(\bar{v}) = 0, B_\delta(\bar{v}) > 0\}, \\ \alpha_{00}(\bar{v}) &:= \{\delta \in L;\ C_\delta(\bar{v}) = 0, B_\delta(\bar{v}) = 0\}, \\ \alpha_{0-}(\bar{v}) &:= \{\delta \in L;\ C_\delta(\bar{v}) = 0, B_\delta(\bar{v}) < 0\}. \end{aligned}$$

Definition 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz around $\bar{v}$ in the direction $u \in \mathbb{R}^n$ then Clarke directional derivative and Clarke subdifferentials of f at $\bar{v}$ is given by*

$$\begin{aligned} f^0(\bar{v}, u) &:= \lim_{v \to \bar{v}} \sup_{t \downarrow 0} \frac{f(v + tu) - f(v)}{t}, \\ \partial_c f(\bar{v}) &:= \{\xi \in \mathbb{R}^n : f^0(\bar{v}, u) \geq \langle \xi, u \rangle,\ \forall u \in \mathbb{R}^n\}. \end{aligned}$$

Theorem 2.1. *[4] Let Φ and Ψ be locally Lipschitz from $\mathbb{R}^n$ to $\mathbb{R}$ around $\bar{v}$. Then the following properties hold:*

(1) $\Phi^0(\bar{v}, u) = \max\{\langle \zeta, u \rangle : \zeta \in \partial_c \Phi(\bar{v}),\ \forall u \in \mathbb{R}^n\}$,

(2) $\partial_c(\gamma\Phi)(\bar{v}) = \gamma \partial_c \Phi(\bar{v}), \forall \gamma \in \mathbb{R}$,

(3) $\partial_c(\Phi + \Psi)(\bar{v}) \subseteq \partial_c \Phi(\bar{v}) + \partial_c \Psi(\bar{v})$.

Based on the definitions of p-invex function and generalized p-invex functions introduced by Joshi [13] in the framework of convexificators, we are introducing the definition of p-invex function and generalized p-invex function in terms of Clarke subdifferentials.

Definition 2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz function around $\bar{v}$. Then*
1. f is called p-invex at $\bar{v}$ if, $\forall\ v \in \mathbb{R}^n, p \in \mathbb{R}/\{0\}$ and any $\xi \in \partial_c f(\bar{v})$, there exist $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, such that

$$f(v) - f(\bar{v}) \geq \frac{1}{p}\langle \xi, e^{p\psi(v,\bar{v})} - 1\rangle.$$

2. f is called strictly p-invex at $\bar{v}$ if, $\forall\ v \in \mathbb{R}^n, p \in \mathbb{R}/\{0\}$ and any $\xi \in \partial_c f(\bar{v})$, there exist $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, such that

$$f(v) - f(\bar{v}) > \frac{1}{p}\langle \xi, e^{p\psi(v,\bar{v})} - 1\rangle.$$

3. f is called quasi-p-invex at $\bar{v}$ if, $\forall\ v \in \mathbb{R}^n, p \in \mathbb{R}/\{0\}$ and any $\xi \in \partial_c f(\bar{v})$, there exist $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, such that

$$f(v) \leq f(\bar{v}) \implies \frac{1}{p}\langle \xi, e^{p\psi(v,\bar{v})} - 1\rangle \leq 0.$$

4. f is called pseudo-p-invex at $\bar{v}$ if, $\forall\ v \in \mathbb{R}^n, p \in \mathbb{R}/\{0\}$ and any $\xi \in \partial_c f(\bar{v})$, there exist $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, such that

$$f(v) \leq f(\bar{v}) \implies \frac{1}{p}\langle \xi, e^{p\psi(v,\bar{v})} - 1\rangle < 0.$$

3. Sufficient optimality conditions (Strong KKT type)

Definition 3.1. *[6] A feasible point $\bar{v}$ of the MOSIPVC is known as a MOSIPVC strong (S-)stationary point if $\exists$ Lagrange multiplier $\lambda_\delta > 0, \delta \in N$ and $\mu_r \geq 0, r \in \Omega(\bar{v})$ with $\mu_r \neq 0$ for at most finitely many indices and $\eta_\delta^C, \eta_\delta^C \in \mathbb{R}, \delta \in L$ such that following holds:*

$$0 \in \sum_{\delta=1}^{m} \lambda_\delta \partial_c f_\delta(\bar{v}) + \sum_{r\in\Omega(\bar{v})} \mu_r \partial_c A_r(\bar{v}) - \sum_{\delta=1}^{l} \eta_\delta^C \partial_c C_\delta(\bar{v}) + \sum_{\delta=1}^{l} \eta_\delta^B \partial_c B_\delta(\bar{v}),$$
$$\eta_\delta^C = 0,\ \delta \in \alpha_+(\bar{v}),\ \eta_\delta^C \geq 0,\ \delta \in \alpha_{0-}(\bar{v}) \cup \alpha_{00}(\bar{v}), \eta_\delta^C \in \mathbb{R}\ \delta \in \alpha_{0+}(\bar{v}),$$
$$\eta_\delta^B = 0,\ \delta \in \alpha_{+-}(\bar{v}) \cup \alpha_0(\bar{v}) \cup \alpha_{0+}(\bar{v}),\ \eta_\delta^B \geq 0,\ \delta \in \alpha_{+0}(\bar{v}).$$

Definition 3.2. *[6] A feasible point $\bar{v}$ of the MOSIPVC is known as a MOSIPVC Mordukhovich (M-) point if $\exists$ Lagrange multiplier $\lambda_\delta > 0, \delta \in N$, and $\mu_r \geq 0, r \in \Omega(\bar{v})$ with $\mu_r \neq 0$ for at most finitely many indices and $\eta_\delta^C, \eta_\delta^C \in \mathbb{R}, \delta \in L$ such that following holds:*

$$0 \in \sum_{\delta=1}^{m} \lambda_\delta \partial_c f_\delta(\bar{v}) + \sum_{r\in\Omega(\bar{v})} \mu_r \partial_c A_r(\bar{v}) - \sum_{\delta=1}^{l} \eta_\delta^C \partial_c C_\delta(\bar{v}) + \sum_{\delta=1}^{l} \eta_\delta^B \partial_c B_\delta(\bar{v}),$$
$$\eta_\delta^C = 0,\ \delta \in \alpha_+(\bar{v}),\ \eta_\delta^C \geq 0,\ \delta \in \alpha_{0-}(\bar{v}), \eta_\delta^C \in \mathbb{R}\ \delta \in \alpha_{0+}(\bar{v}),$$
$$\eta_\delta^B = 0,\ \delta \in \alpha_{+-}(\bar{v}) \cup \alpha_{0-}(\bar{v}) \cup \alpha_{0+}(\bar{v}),\ \eta_\delta^B \geq 0,\ \delta \in \alpha_{+0}(\bar{v}) \cup \alpha_{00}(\bar{v}),$$
$$\eta_\delta^C, \eta_\delta^B = 0, \delta \in \alpha_{00}(\bar{v}).$$

Following theorem gives strong KKT type sufficient optimality condition for the MOSIPVC using generalized invexity assumptions.

Theorem 3.1. *Let $\bar{v}$ be a MOSIPVC M-stationary point. Assume that $f_\delta, \delta \in N, A_r, r \in \Omega(\bar{v}), -C_\delta, B_\delta, \delta \in L$ are p-invex at $\bar{v}$ on M and at least one of them is strictly p-invex at $\bar{v}$ on M. Then $\bar{v}$ is a weakly efficient solution of the MOSIPVC.*

Proof. Given that, $\bar{v}$ is a MOSIPVC M-stationary point, i.e., $\exists\ \bar{\xi}^f_\delta \in \partial_c f_\delta(\bar{v}), \delta \in N, \bar{\xi}^A_\delta \in \partial_c A_r(\bar{v}), r \in \Omega(\bar{v})$ and $\bar{\xi}^C_\delta \in \partial_c C_\delta(\bar{v}), \bar{\xi}^B_\delta \in \partial_c B_\delta(\bar{v}), \delta \in L$ such that

$$\sum_{\delta=1}^{m} \lambda_\delta \bar{\xi}^f_\delta + \sum_{r\in\Omega(\bar{v})} \mu_r \bar{\xi}^A_\delta - \sum_{\delta=1}^{l} \eta^C_\delta \bar{\xi}^C_\delta + \sum_{\delta=1}^{l} \eta^B_\delta \bar{\xi}^B_\delta = 0. \tag{1}$$

Suppose $\bar{v}$ is not a weakly efficient solution for the MOSIPVC, i.e., $\exists\ \tilde{v} \in M$, such that

$$f_\delta(\tilde{v}) < f_\delta(\bar{v}) \forall\ \delta \in N.$$

Using MOSIPVC M-stationary point condition, one has $\lambda_\delta > 0$ for $\delta \in N$. Thus, we obtain

$$\sum_{\delta=1}^{m} \lambda_\delta f_\delta(\tilde{v}) < \sum_{\delta=1}^{m} \lambda_\delta f_\delta(\bar{v}). \tag{2}$$

Since $\bar{v}$ is a MOSIPVC M-stationary point and $\tilde{v}$ is a feasible point of the MOSIPVC, we obtain

$$\begin{aligned}
&A_r(\tilde{v}) < 0,\ \mu_r \geq 0,\ r \in \Omega(\bar{v}),\\
&-C_\delta(\tilde{v}) < 0,\ \eta^C_\delta \geq 0,\ \delta \in \alpha_{0-}(\bar{v}) \cup \alpha_+(\bar{v}),\\
&C_\delta(\tilde{v}) = 0,\ \eta^C \in \mathbb{R},\ \delta \in \alpha_{0+}(\bar{v}),\\
&B_\delta(\tilde{v}) > 0,\ \eta^B = 0,\ \delta \in \alpha_{+-}(\bar{v}) \cup \alpha_{0-}(\bar{v}) \cup \alpha_{0+}(\bar{v}),\\
&B_\delta(\tilde{v}) \leq 0,\ \eta^B > 0,\ \delta \in \alpha_{00}(\bar{v}) \cup \alpha_{+0}(\bar{v}).
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sum_{r\in\Omega(\bar{v})} \mu_r A_r(\tilde{v}) - \sum_{\delta=1}^{l} \eta^C_\delta C_\delta(\tilde{v}) + \sum_{\delta=1}^{l} \eta^B_\delta B_\delta(\tilde{v})\\
&\qquad \leq \sum_{r\in\Omega(\bar{v})} \mu_r A_r(\bar{v}) - \sum_{\delta=1}^{l} \eta^C_\delta C_\delta(\bar{v}) + \sum_{\delta=1}^{l} \eta^B_\delta B_\delta(\bar{v}).
\end{aligned} \tag{3}$$

From (2) and (3), we have

$$\begin{aligned}
&\sum_{\delta=1}^{m} \lambda_\delta f_\delta(\tilde{v}) + \sum_{r\in\Omega(\bar{v})} \mu_r A_r(\tilde{v}) - \sum_{\delta=1}^{l} \eta^C_\delta C_\delta(\tilde{v}) + \sum_{\delta=1}^{l} \eta^B_\delta B_\delta(\tilde{v})\\
&\qquad < \sum_{\delta=1}^{m} \lambda_\delta f_\delta(\bar{v}) + \sum_{r\in\Omega(\bar{v})} \mu_r A_r(\bar{v}) - \sum_{\delta=1}^{l} \eta^C_\delta C_\delta(\bar{v}) + \sum_{\delta=1}^{l} \eta^B_\delta B_\delta(\bar{v}).
\end{aligned} \tag{4}$$

Using equation 3.1, we obtain

$$\begin{aligned}
0 &= \sum_{\delta=1}^{m} \lambda_\delta \bar{\xi}^f_\delta + \sum_{r\in\Omega(\bar{v})} \mu_r \bar{\xi}^A_\delta - \sum_{\delta=1}^{l} \eta^C_\delta \bar{\xi}^C_\delta + \sum_{\delta=1}^{l} \eta^B_\delta \bar{\xi}^B_\delta\\
&\in \partial_c \left(\sum_{\delta=1}^{m} \lambda_\delta f_\delta(\bar{v}) + \sum_{r\in\Omega(\bar{v})} \mu_r A_r(\bar{v}) - \sum_{\delta=1}^{l} \eta^C_\delta C_\delta(\bar{v}) + \sum_{\delta=1}^{l} \eta^B_\delta B_\delta(\bar{v}) \right).
\end{aligned} \tag{5}$$

From (1), (4) and (5), we get

$$0 > \left\langle \sum_{\delta=1}^{m} \lambda_\delta \bar{\xi}_\delta^f + \sum_{r\in\Omega(\bar{v})} \mu_r \bar{\xi}_\delta^A - \sum_{\delta=1}^{l} \eta_\delta^C \bar{\xi}_\delta^C + \sum_{\delta=1}^{l} \eta_\delta^B \bar{\xi}_\delta^B, e^{p\psi(v,\bar{v})} - 1 \right\rangle$$
$$= \frac{1}{p}\langle 0, e^{p\psi(v,\bar{v})} - 1\rangle.$$

This is a contradiction, hence proved. □

One can easily find, following corollary directly using Theorem 3.1.

Corollary 3.1. *Assume that $\bar{v}$ is a MOSIPVC S-stationary point and let $f_\delta, \delta \in N, A_r, r \in \Omega(\bar{v}), -C_\delta, B_\delta, \delta \in L$ are quasi p-invex at $\bar{v}$ on M and at least one of them is strictly quasi p-invex at $\bar{v}$ on M. Then $\bar{v}$ is a weakly efficient solution for the MOSIPVC.*

The example given below satisfies assumptions of the Theorem 3.1.

Example 3.1. *Assuming $p = 1$ and $\psi(v, \bar{v}) = 0$, consider MOSIPVC problem in $\mathbb{R}^2$as* follows;

$$\begin{aligned} \min\ & f(v) = (v_1^2, |v_1| + v_2) \\ s.t.\ & A_r(v) = -rv_1 \le 0, \\ & r \in \mathbb{N}, \text{Here } \mathbb{N} \text{ denotes the set of natural numbers} \\ & C(v) = v_1 \ge 0, \\ & C(v)B(v) = v_1(|v_1| + |v_2|) \le 0. \end{aligned}$$

Note that we have, $f_1(v) = v_1^2, f_2(v) = |v_1| + v_2$ and the feasible set of the MOSIPVC is as follows

$$M = \{(v_1, v_2) \in R^2 : -rv_1 \le 0, r \in \mathbb{N}, v_1 \ge 0, v_1(|v_1| + |v_2|) \le 0\},$$

one can see that $\bar{v} = (0, 0)$ is a feasible point of the MOSIPVC, $\Omega(\bar{v}) = \mathbb{N}$ and $I_{00}(\bar{v}) = \{1\}$. The feasible point $\bar{v}$ is a MOSIPVC M-stationary point with $\lambda_1 > 0, \lambda_2 = 1, \mu_1 = 1, \mu_2 = \frac{1}{2}, \mu_3 = \mu_4 =, ... = 0, \eta^C = -1, \eta^B = 0, \xi^{f_1} = (0, 0) \in \partial_c f_1(\bar{v}) = \{(0, 0)\}, \xi^{f_2} = (0, 1) \in \partial_c f_2(\bar{v}) = [-1, 1] \times \{1\}, \xi_1^{A_r} = (-r, 0) \in \partial_c A_r(\bar{v}) = \{(-r, 0)\}, \xi^C = (1, 0) \in \partial_c C(\bar{v}) = \{(1, 0)\}$ and $\xi^B = (0, 1) \in \partial_c B(\bar{v}) = [-1, 1] \times [-1, 1]$.

By relaxing the generalized p-invexity requirement in Theorem 3.1, a strong KKT-type sufficient optimality condition can also be obtained for MOSIPVC.

Theorem 3.2. *Let $\bar{v}$ be a MOSIPVC M-stationary point. Suppose that $f_\delta, \delta \in N, A_r, r \in \Omega(\bar{v}), -C_\delta, B_\delta, \delta \in L$ are quasi-p-invex at $\bar{v}$ on M and at least one of them is strictly quasi-p-invex at $\bar{v}$ on M. Then MOSIPVC attains $\bar{v}$ as a weakly efficient solution.*

Theorem 3.3. *Let $\bar{v}$ be a MOSIPVC M-stationary point. Assume that each $f_\delta, \delta \in N$, is p-invex at $\bar{v}$ on M and $\sum_{r\in\Omega(\bar{v})} \mu_r A_r(v) - \sum_{\delta=1}^{l} \eta_\delta^C C_\delta(v) + \sum_{\delta=1}^{l} \eta_\delta^B B_\delta(v)$ is p-invex at $\bar{v}$ on M. Then $\bar{v}$ is a weakly efficient solution for the MOSIPVC.*

Proof. Assuming that MOSIPVC does not have $\bar{v}$ as a weakly efficient solution, that is, there is a feasible point $\tilde{v}$ such that

$$f_\delta(\tilde{v}) < f_\delta(\bar{v}),\ \forall \delta \in N.$$

By strict p-invexity of f_δ, we have

$$\frac{1}{p}\langle \xi_\delta^f, e^{p\psi(\tilde{v},\bar{v})} - 1\rangle < 0,\ \xi_\delta^f \in \partial_c f_\delta(\bar{v}), \delta \in N. \tag{6}$$

Using M-stationary condition, we obtain $\lambda_\delta > 0,\ \delta \in N$. Thus, we have

$$\frac{1}{p}\left\langle \sum_{\delta=1}^{m} \lambda_\delta \xi_\delta^f, e^{p\psi(\tilde{v},\bar{v})} - 1 \right\rangle < 0. \tag{7}$$

Since, MOSIPVC has $\bar{v}$ as a M-stationary point, from (1) and (7), we obtain

$$\frac{1}{p}\left\langle \sum_{r\in\Omega(\bar{v})} \mu_r \bar{\xi}_r^A - \sum_{\delta=1}^{l} \eta_\delta^C \bar{\xi}_\delta^C + \sum_{\delta=1}^{l} \eta_\delta^B \bar{\xi}_\delta^B, e^{p\psi(\tilde{v},\bar{v})} - 1 \right\rangle > 0. \tag{8}$$

From (3), we have

$$\begin{aligned} &\sum_{r\in\Omega(\bar{v})} \mu_r A_r(\tilde{v}) - \sum_{\delta=1}^{l} \eta_\delta^C C_\delta(\tilde{v}) + \sum_{\delta=1}^{l} \eta_\delta^B B_\delta(\tilde{v}) \\ &\qquad \leq \sum_{r\in\Omega(\bar{v})} \mu_r A_r(\bar{v}) - \sum_{\delta=1}^{l} \eta_\delta^C C_\delta(\bar{v}) + \sum_{\delta=1}^{l} \eta_\delta^B B_\delta(\bar{v}). \end{aligned} \tag{9}$$

Considering the p-invexity of

$$\sum_{r\in\Omega(\bar{v})} \mu_r A_r(v) - \sum_{\delta=1}^{l} \eta_\delta^C C_\delta(v) + \sum_{\delta=1}^{l} \eta_\delta^B B_\delta(v),$$

at $\bar{v}$ on M, we obtain

$$\frac{1}{p}\left\langle \sum_{r\in\Omega(\bar{v})} \mu_r \bar{\xi}_r^A - \sum_{\delta=1}^{l} \eta_\delta^C \bar{\xi}_\delta^C + \sum_{\delta=1}^{l} \eta_\delta^B \bar{\xi}_\delta^B, e^{p\psi(\tilde{v},\bar{v})} - 1 \right\rangle \leq 0. \tag{10}$$

which is a contradiction to (8). Hence, MOSIPVC attains $\bar{v}$ as a weakly efficient solution. Hence the result is proved □

Corollary 3.2. *Let $\bar{v}$ be a MOSIPVC S-stationary point. Assume that each $f_\delta, \delta \in N$, is p-invex at $\bar{v}$ on M and $\sum_{r\in\Omega(\bar{v})} \mu_r A_r(v) - \sum_{\delta=1}^{l} \eta_\delta^C C_\delta(v) + \sum_{\delta=1}^{l} \eta_\delta^B B_\delta(v)$ is p-invex at $\bar{v}$ on M. Then $\bar{v}$ is a weakly efficient solution for the MOSIPVC.*

Now, we provide an example to satisfy Theorem 3.3.

Example 3.2. *Assuming $p = 1$ and $\psi(v, \bar{v}) = 0$. considering that MOSIPVC problem is in $\mathbb{R}^2$ as* follows;

$$\begin{aligned} &\min\ f(v) = (|v_1|, |v_2|) \\ &s.t.\ \ A_r(v) = -rv_1^2 \leq 0, \\ &\qquad r \in \mathbb{N}, \text{Here } \mathbb{N} \text{ denotes the set of natural numbers} \\ &\qquad C(v) = v_1^2 + v_2 \geq 0, \\ &\qquad C(v)B(v) = |v_1|(v_1^2 + v_2) \leq 0. \end{aligned}$$

Note that we have, $f_1(v) = |v_1|, f_2(v) = |v_2|$ and the feasible set of the MOSIPVC is as follows

$$M = \{(v_1, v_2) \in R^2 : -rv_1^2 \leq 0, r \in \mathbb{N}, v_1^2 + v_2 \geq 0, |v_1|(v_1^2 + v_2) \leq 0\},$$

one can see that $\bar{v} = (0,0)$ *is a feasible point of the MOSIPVC,* $\Omega(\bar{v}) = \mathbb{N}$ *and* $I_{00}(\bar{v}) = \{1\}$. *The feasible point* $\bar{v}$ *is a MOSIPVC M-stationary point with* $\lambda_1 > 0, \lambda_2 = 1, \mu_1 = 1, \mu_2 = \mu_3 = \mu_4 =, ... = 0, \eta_1^C = -1, \eta_1^B = 0, \xi^{f_1} = (0,0) \in \partial_c f_1(\bar{v}) = [-1,1] \times \{0\}, \xi^{f_2} = (0,-1) \in \partial_c f_2(\bar{v}) = \{0\} \times [-1,1], \xi_1^{A_r} = (0,0) \in \partial_c A_r(\bar{v}) = \{(0,0)\}, \xi^C = (0,1) \in \partial_c C(\bar{v}) = \{(0,1)\}$ *and* $\xi^B = (1,0) \in \partial_c B(\bar{v}) = [-1,1] \times [-1,1]$. *Now, one can easily see that Theorem 3.3 is verified.*

4. Results and discussion

This article considers MOSIPVC. Under the assumption of generalized convexity, we introduce a stationary condition for MOSIPVC and establish a sufficient optimum for the strong KKT type of MOSIPVC. We extend KKT's notion of strong optimality to constraint-free MOSIPVC. Moreover, the findings of this article can be extended to strong KKT type necessary optimality conditions for the MOSIPVC in which constraint qualification will be involved.

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