U.P.B. Sci. Bull., Series A, Vol. 74, Iss. 1, 2012 ISSN 1223-7027

EXISTENCE OF POSITIVE SOLUTIONS FOR STURM-LIOUVILLE BVPS OF SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS

Yuji Liu[1], Tieshan He[2], Haiping Shi[3]

The results on the existence and uniqueness for positive solutions to Sturm-Liouville boundary value problems of singular fractional differential equations are established. The analysis relies on the Schauder's fixed point theorems. An example is given to illustrate the efficiency of the main theorems.

Keywords: Positive solution, singular fractional differential equation, Sturm-Liouville boundary value problem, fixed-point theorem

MSC2000: 92D25, 34A37, 34K15

1. Introduction

There have been many papers concerned with the existence of positive solutions of boundary value problems for fractional differential equations see [1-13].

In [11], E. R. Kaufmann and E. Mboumi studied the following boundary value problem

$$\begin{cases} D_{0^+}^{\alpha}u(t)+a(t)f(u(t))=0, \quad 0<t<1, 1<\alpha<2, \\ u(0)=0, u'(1)=0\,, \end{cases} \tag{1}$$

where $f:[0,+\infty)\to[0,\infty)$ is continuous, $a\in L^{\infty}[0,1]$, there exists a constant $m>0$ such that $a(t)\geq m$ a.e. $t\in[0,1]$. By using the Leggett-Williams fixed point theorem and the Krasnoselskii fixed point theorem, the authors in [11] proved that BVP(1) has at least one or three positive solutions under some growth conditions imposed on f.

In paper [12], the authors studied the following boundary value problem

$$\begin{cases} D_{0^+}^{\alpha}u(t)+q(t)f(t,u(t),u'(t))=0, \quad 0<t<1, \\ u(0)=0, u(1)=0\,, \end{cases} \tag{2}$$

where $f:[0,1]\times[0,+\infty)\times R\to[0,\infty)$ is continuous, q satisfies $0<\int_0^1[t(1-t)]^{\alpha-1}q(t)dt<+\infty$. By using the fixed point theorem in cones in Banach spaces, the authors in [12] proved that BVP(2) has at least three positive solutions under some assumptions.

[1] Prof., Dr., Guangdong University of Business Studies, Guangzhou, Guangdong, 510320, P.R.China, e-mail: liuyuji888@sohu.com

[2] College of Computational Science, Zhongkai University of Agriculture and Engineering, Guangzhou 510225, P.R.China

[3] Department of Mathematics, Guangdong Baiyun College, Guangzhou 510000, P.R.China.
Supported by the Natural Science Foundation of Guangdong province(No:S2011010001900) and the Guangdong Higher Education Foundation for High-level talents.

As far as we know, there has been no paper concerned with the existence and uniqueness of positive solutions of Sturm-Liouville boundary value problem for singular fractional differential equations.

To fill this gap, we discuss the Sturm-Liouville boundary value problems of the nonlinear singular fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t),D_{0+}^{\alpha-1}u(t))=0, \quad t\in(0,1), 1<\alpha<2, \\ a\lim_{t\to 0}t^{2-\alpha}u(t)-b\ \lim_{t\to 0}\ D_{0+}^{\alpha-1}u(t)=\int_0^1 g(t,u(t),D_{0+}^{\alpha-1}u(t))dt, \\ c\ D_{0+}^{\alpha-1}u(1)+du(1)=\int_0^1 h(t,u(t),D_{0+}^{\alpha-1}u(t))dt, \end{cases} \tag{3}$$

where

- D_{0+}^{α} (or $D_{0+}^{\alpha-1}$) is the Riemann-Liouville fractional derivative of order α (or $\alpha-1$),
- $a\geq 0, b\geq 0, c\geq 0, d\geq 0$ with $\delta =: ad+bd\Gamma(\alpha)+ac\Gamma(\alpha)>0$,
- f,g,h defined on $(0,1)\times[0,\infty)\times R$ are nonnegative Caratheodory functions that may be singular at $t=0$ and $t=1$, $f(t,0,0)\not\equiv 0$ on each subinterval of $[0,1]$.

We obtain the results on the existence and uniqueness of positive solutions of BVP(3). An example is given to illustrate the efficiency of the main theorems. The methods used are motivated by [13].

It is well known that boundary value problem of the following form

$$\begin{cases} u''(t)+f(t,u(t),u'(t))=0, \quad t\in(0,1), \\ au(t)-bu'(0)=0, \\ cu'(1)+du(1)=0, \end{cases}$$

is called Sturm-Liouville BVP [1,2], where $f(t,u,v)$ is continuous and nonnegative on $[0,1]\times[0,\infty)\times R$, $a,b,c,d\in R$. It comes from the situation involving nonlinear elliptic problems in annular regions. Hence BVP(3) is called Sturm-Liouville boundary value problem for singular fractional differential equation.

The remainder of this paper is as follows: in Section 2, some definitions and preliminary results are presented. In Section 3, the main theorems are given. In Section 4, an example is given to illustrate the main results.

2. **Preliminary results**

For the convenience of the readers, the necessary definitions from the fractional calculus theory are presented. These definitions and results can be found in the literatures [5,6,8,9,11]. Then the preliminary results are given.

Definition 2.1[6]. The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $g:(0,\infty)\to R$ is given by $I_{0+}^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)ds$, provided that the right-hand side exists.

Definition 2.2[6]. The Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $g:(0,\infty)\to R$ is given by

$$D_{0+}^{\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^{n+1}}{dt^{n+1}}\int_0^t\frac{g(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n-1<\alpha\leq n$, provided the right-hand side is point-wise defined on $(0,\infty)$.

Definition 2.3. $F:(0,1)\times R^2\to R$ is called a Caratheodory function if

1) $t\to F(t,t^{\alpha-2}(1-t)^{\alpha-2}x,y)$ is measurable on $(0,1)$ for each $(x,y)\in R^2$,

2) $(x,y)\to F(t,t^{\alpha-2}(1-t)^{\alpha-2}x,y)$ is continuous on R^2 for almost all $t\in(0,1)$,

3) for each $r > 0$ there exists $\phi_r \in L^1(0,1)$ such that $|x|, |y| \le r$ implies that $|F(t, t^{\alpha-2}(1-t)^{\alpha-2}x, y)| \le \phi_r(t)$ holds for almost all $t \in (0,1)$.

Lemma 2.1[6]. Let $n-1 < \alpha \le n$, $u \in C^0(0,\infty) \bigcap L^1(0,\infty)$. Then $I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$, where $C_i \in R$, $i = 1, 2, \ldots n$.

It is easy to prove for $\sigma \ge 0, \mu > -1$ that $I_{0+}^{\sigma} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\sigma+1)} t^{\mu+\sigma}$, $D_{0+}^{\sigma} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\sigma+1)} t^{\mu-\sigma}$.

For our construction, we let

$$X = \left\{ x : (0,1) \to R \begin{array}{c} x \in C(0,1), \\ D_{0+}^{\alpha-1} x \in C(0,1), \\ \text{there exist the limits} \\ \lim_{t\to 0} t^{2-\alpha}(1-t)^{2-\alpha} x(t), \\ \lim_{t\to 1} t^{2-\alpha}(1-t)^{2-\alpha} x(t), \\ \lim_{t\to 0} D_{0+}^{\alpha-1} x(t) \\ \lim_{t\to 1} D_{0+}^{\alpha-1} x(t) \end{array} \right\}.$$

For $x \in X$, $\|x\| = \max\left\{\sup_{t\in(0,1)} t^{2-\alpha}(1-t)^{2-\alpha}|u(t)|,\ \sup_{t\in(0,1)} |D_{0+}^{\alpha-1}u(t)|\right\}$. Then X is a Banach space (similar proof can be find in [16]). Choose $P = \{u \in X$, $u(t) \ge 0, 0 < t < 1\}$. Then P is a nontrivial closed cone of X. We seek solutions of BVP(3) that lie in the cone P.

Let $x \in X$, $x(t) \ge 0, \forall t \in (0,1)$. Consider the boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, x(t), D_{0+}^{\alpha-1} x(t)) = 0, \quad t \in (0,1), 1 < \alpha < 2, \\ a \lim_{t\to 0} t^{2-\alpha} u(t) - b\ \lim_{t\to 0}\ D_{0+}^{\alpha-1} u(t) = \int_0^1 g(t, x(t), D_{0+}^{\alpha-1} x(t)) dt, \\ c\ D_{0+}^{\alpha-1} u(1) + du(1) = \int_0^1 h(t, x(t), D_{0+}^{\alpha-1} x(t)) dt, \end{cases}$$

From Lemma 2.1, there exist $c_1, c_2 \in R$ such that

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), D_{0+}^{\alpha-1} x(s)) ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2},$$

$$D_{0+}^{\alpha-1} u(t) = -\int_0^t f(s, x(s), D_{0+}^{\alpha-1} x(s)) ds + c_1 \Gamma(\alpha).$$

Then the boundary conditions imply $ac_2 - bc_1\Gamma(\alpha) = \int_0^1 g(t, x(t), D_{0+}^{\alpha-1} x(t)) dt$,

$$\begin{aligned} & c\left(-\int_0^1 f(s, x(s), D_{0+}^{\alpha-1} x(s)) ds + c_1 \Gamma(\alpha)\right) \\ & +d\left(-\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), D_{0+}^{\alpha-1} x(s)) ds + c_1 + c_2\right) \\ = & \int_0^1 h(t, x(t), D_{0+}^{\alpha-1} x(t)) dt. \end{aligned}$$

It follows that

$$\begin{aligned}
c_1 &= \frac{1}{\delta}\left(a\int_0^1 h(t,x(t),D_{0^+}^{\alpha-1}x(t))dt + ac\int_0^1 f(t,x(t),D_{0^+}^{\alpha-1}x(t))dt\right.\\
&\left.+\frac{ad}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(t,x(t),D_{0^+}^{\alpha-1}x(t))dt - d\int_0^1 g(t,x(t),D_{0^+}^{\alpha-1}x(t))dt\right),\\
c_2 &= \frac{1}{\delta}\left((c\Gamma(\alpha)+d)\int_0^1 g(t,x(t),D_{0^+}^{\alpha-1}x(t))dt + b\Gamma(\alpha)\int_0^1 h(t,x(t),D_{0^+}^{\alpha-1}x(t))dt\right.\\
&\left.+bc\Gamma(\alpha)\int_0^1 f(t,x(t),D_{0^+}^{\alpha-1}x(t))dt + bd\int_0^1(1-s)^{\alpha-1}f(t,x(t),D_{0^+}^{\alpha-1}x(t))dt\right).
\end{aligned}$$

Hence

$$\begin{gathered}
u(t) = -\tfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s),D_{0^+}^{\alpha-1}x(s))ds\\
+\tfrac{1}{\delta}\left(a\int_0^1 h(t,x(t),D_{0^+}^{\alpha-1}x(t))dt + ac\int_0^1 f(t,x(t),D_{0^+}^{\alpha-1}x(t))dt\right.\\
\left.+\tfrac{ad}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(t,x(t),D_{0^+}^{\alpha-1}x(t))dt - d\int_0^1 g(t,x(t),D_{0^+}^{\alpha-1}x(t))dt\right)t^{\alpha-1}\\
+\tfrac{1}{\delta}\left((c\Gamma(\alpha)+d)\int_0^1 g(t,x(t),D_{0^+}^{\alpha-1}x(t))dt + b\Gamma(\alpha)\int_0^1 h(t,x(t),D_{0^+}^{\alpha-1}x(t))dt\right.\\
\left.+bc\Gamma(\alpha)\int_0^1 f(t,x(t),D_{0^+}^{\alpha-1}x(t))dt + bd\int_0^1(1-s)^{\alpha-1}f(t,x(t),D_{0^+}^{\alpha-1}x(t))dt\right)t^{\alpha-2}\\
= \tfrac{c\Gamma(\alpha)t^{\alpha-2}+dt^{\alpha-2}-dt^{\alpha-1}}{\delta}\int_0^1 g(t,x(t),D_{0^+}^{\alpha-1}x(t))dt\\
+\tfrac{b\Gamma(\alpha)t^{\alpha-2}+at^{\alpha-1}}{\delta}\int_0^1 h(t,x(t),D_{0^+}^{\alpha-1}x(t))dt\\
+\int_0^t\left(\tfrac{act^{\alpha-1}+bdt^{\alpha-2}(1-s)^{\alpha-1}+bc\Gamma(\alpha)t^{\alpha-2}}{\delta}+\tfrac{adt^{\alpha-1}(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}-\tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right)\times\\
f(s,x(s),D_{0^+}^{\alpha-1}x(s))ds\\
+\int_t^1\left(\tfrac{act^{\alpha-1}+bdt^{\alpha-2}(1-s)^{\alpha-1}+bc\Gamma(\alpha)t^{\alpha-2}}{\delta}+\tfrac{adt^{\alpha-1}(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}\right)\times\\
f(s,x(s),D_{0^+}^{\alpha-1}x(s))ds\\
=\int_0^1 G(t,s)f(s,x(s),D_{0^+}^{\alpha-1}x(s))ds\\
\tfrac{c\Gamma(\alpha)t^{\alpha-2}+dt^{\alpha-2}-dt^{\alpha-1}}{\delta}\int_0^1 g(t,x(t),D_{0^+}^{\alpha-1}x(t))dt\\
+\tfrac{b\Gamma(\alpha)t^{\alpha-2}+at^{\alpha-1}}{\delta}\int_0^1 h(t,x(t),D_{0^+}^{\alpha-1}x(t))dt,
\end{gathered}$$

where

$$G(t,s)=\begin{cases}\tfrac{act^{\alpha-1}+bdt^{\alpha-2}(1-s)^{\alpha-1}+bc\Gamma(\alpha)t^{\alpha-2}}{\delta}+\tfrac{adt^{\alpha-1}(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}-\tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0<s\le t<1,\\ \tfrac{act^{\alpha-1}+bdt^{\alpha-2}(1-s)^{\alpha-1}+bc\Gamma(\alpha)t^{\alpha-2}}{\delta}+\tfrac{adt^{\alpha-1}(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}, & 0<t\le s<1.\end{cases} \tag{4}$$

Define the operator T on P, by

$$\begin{aligned}
(Tu)(t) &= \int_0^1 G(t,s)f(s,u(s),D_{0^+}^{\alpha-1}u(t))ds\\
&+\frac{c\Gamma(\alpha)t^{\alpha-2}+d(t^{\alpha-2}-t^{\alpha-1})}{\delta}\int_0^1 g(t,u(t),D_{0^+}^{\alpha-1}u(t))dt\\
&+\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{\delta}\int_0^1 h(t,u(t),D_{0^+}^{\alpha-1}u(t))dt
\end{aligned}$$

for $u\in P\subset X$.

Lemma 2.2. **(i)** G satisfies the following inequalities:

$$t^{2-\alpha}(1-t)^{2-\alpha}G(t,s) \le \frac{ad+ac\Gamma(\alpha)}{\delta\Gamma(\alpha)} + \frac{bd+bc\Gamma(\alpha)}{\delta}, \text{ for all } s,t \in (0,1),$$

and $G(t,s) \ge 0$ for all $t \in (0,1), s \in (0,1)$;

(ii) x is a positive solution of BVP(3) if and only if x is a fixed point of T in P;

(iii) $T : P \to P$ is well defined;

(iv) T is completely continuous.

Proof. (i) One sees from (4) that

$$\begin{aligned} t^{2-\alpha}(1-t)^{2-\alpha}G(t,s) &\le \frac{act+bd(1-s)^{\alpha-1}+bc\Gamma(\alpha)}{\delta} + \frac{adt(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)} \\ &\le \frac{ad+ac\Gamma(\alpha)}{\delta\Gamma(\alpha)} + \frac{bd+bc\Gamma(\alpha)}{\delta}. \end{aligned}$$

Similarly, we have from (4) that

$$\begin{aligned} G(t,s) &= \frac{bc\Gamma(\alpha)t^{\alpha-2}}{\delta} + \frac{ac\Gamma(\alpha)t^{\alpha-1}+bd\Gamma(\alpha)t^{\alpha-2}(1-s)^{\alpha-1}+adt^{\alpha-1}(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)} \\ &\quad - \frac{[ac\Gamma(\alpha)+bd\Gamma(\alpha)+ad](t-s)^{\alpha-1}}{\delta\Gamma(\alpha)} \\ &= \frac{bc\Gamma(\alpha)t^{\alpha-2}}{\delta} + \frac{ac\Gamma(\alpha)[t^{\alpha-1}-(t-s)^{\alpha-1}]+bd\Gamma(\alpha)[t^{\alpha-2}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}]}{\delta\Gamma(\alpha)} \\ &\quad + \frac{ad[t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}]}{\delta\Gamma(\alpha)} \ge 0. \end{aligned}$$

The proof of (i) is completed.

(ii) For $x \in P \subset X$, we find that

$$\max\left\{\sup_{t\in(0,1)} t^{2-\alpha}(1-t)^{2-\alpha}|x(t)|, \sup_{t\in(0,1)} |D_{0^+}^{\alpha-1}x(t)|\right\} = r < +\infty.$$

Since f, g, h are Caratheodory functions, then there exists $\phi_r \in L^1(0,1)$ such that

$$\begin{aligned} |f(t,x(t),D_{0^+}^{\alpha-1}x(t))| &= |f(t,t^{\alpha-2}(1-t)^{\alpha-2}[t^{2-\alpha}(1-t)^{2-\alpha}x(t)],D_{0^+}^{\alpha-1}x(t))| \le \phi_r(t), \\ |g(t,x(t),D_{0^+}^{\alpha-1}x(t))| &= |g(t,t^{\alpha-2}(1-t)^{\alpha-2}[t^{2-\alpha}(1-t)^{2-\alpha}x(t)],D_{0^+}^{\alpha-1}x(t))| \le \phi_r(t), \\ |h(t,x(t),D_{0^+}^{\alpha-1}x(t))| &= |h(t,t^{\alpha-2}(1-t)^{\alpha-2}[t^{2-\alpha}(1-t)^{2-\alpha}x(t)],D_{0^+}^{\alpha-1}x(t))| \le \phi_r(t). \end{aligned} \tag{5}$$

It follows from (i) that G is bounded. Hence $t \to (Tx)(t)$ is well defined and

$$\begin{aligned} &t^{2-\alpha}(1-t)^{2-\alpha}(Tx)(t) \\ &= t^{2-\alpha}(1-t)^{2-\alpha}\Big[\int_0^1 G(t,s)f(s,x(s),D_{0^+}^{\alpha-1}x(s))ds \\ &\quad + \frac{c\Gamma(\alpha)t^{\alpha-2}+d(t^{\alpha-2}-t^{\alpha-1})}{\delta}\int_0^1 g(t,x(t),D_{0^+}^{\alpha-1}x(t))dt \\ &\quad + \frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{\delta}\int_0^1 h(t,x(t),D_{0^+}^{\alpha-1}x(t))dt\Big] \\ &= (1-t)^{2-\alpha}\frac{c\Gamma(\alpha)+d(1-t)}{\delta}\int_0^1 g(t,x(t),D_{0^+}^{\alpha-1}x(t))dt \\ &\quad + (1-t)^{2-\alpha}\frac{at+b\Gamma(\alpha)}{\delta}\int_0^1 h(t,x(t),D_{0^+}^{\alpha-1}x(t))dt \\ &\quad + (1-t)^{2-\alpha}\int_0^t\left[\frac{act+bd(1-s)^{\alpha-1}+bc\Gamma(\alpha)}{\delta} + \frac{adt(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)} - \frac{t^{2-\alpha}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right] \times \\ &\qquad f(s,x(s),D_{0^+}^{\alpha-1}x(s))ds \\ &\quad + (1-t)^{2-\alpha}\int_t^1\left[\frac{act+bd(1-s)^{\alpha-1}+bc\Gamma(\alpha)}{\delta} + \frac{adt(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}\right] f(s,x(s),D_{0^+}^{\alpha-1}x(s))ds. \end{aligned}$$

Furthermore, we have

$$\begin{array}{c} D_{0^+}^{\alpha-1}(Tx)(t) = -\int_0^t f(s,x(s),D_{0^+}^{\alpha-1}x(s))ds + \frac{ac\Gamma(\alpha)}{\delta}\int_0^1 f(s,x(s),D_{0^+}^{\alpha-1}x(s))ds \\ +\frac{ad}{\delta}\int_0^1(1-s)^{\alpha-1}f(s,x(s),D_{0^+}^{\alpha-1}x(s))ds - \frac{d\Gamma(\alpha)}{\delta}\int_0^1 g(t,x(t),D_{0^+}^{\alpha-1}x(t))dt \\ +\frac{a\Gamma(\alpha)}{\delta}\int_0^1 h(t,x(t),D_{0^+}^{\alpha-1}x(t))dt. \end{array}$$

Then

$$\left\{\begin{array}{l} D_{0^+}^{\alpha}(Tx)(t) = -f(t,x(t),D_{0^+}^{\alpha-1}x(t)), \\ a\lim_{t\to 0} t^{2-\alpha}(Tx)(t) - b\ \lim_{t\to 0}\ D^{\alpha-1}(Tx)(t) = \int_0^1 g(t,x(t),D_{0^+}^{\alpha-1}x(t))dt, \\ c\ D_0^{\alpha-1}(Tx)(1) + d(Tx)(1) = \int_0^1 h(t,x(t),D_{0^+}^{\alpha-1}x(t))dt. \end{array}\right. \quad (6)$$

From (6), we see that x is a positive solution of BVP(3) if and only if x is a fixed point of T in P. The proof of (ii) is complete.

(iii) For $x \in P$, from (ii), we have $Tx \in X$. From (i) and (ii), we get $(Tx)(t) \geq 0$ for all $t \in (0,1)$. Then $Tx \in P$.

(iv) We prove that T is completely continuous.

Step 1. We prove that T is continuous. This needs three steps. Let $\{y_n\}_{n=0}^{\infty}$ be a sequence such that $y_n \to y_0$ in X. Then

$$r = \sup_{n\in\mathbb{N}}||y_n|| = \sup_{n\in\mathbb{N}}\left\{\max\left\{\sup_{t\in(0,1)} t^{2-\alpha}(1-t)^{2-\alpha}|x(t)|,\ \sup_{t\in(0,1)}|D_{0^+}^{\alpha-1}x(t)|\right\}\right\} < \infty.$$

So there exists $\phi_r \in L^1(0,1)$ such that

$$\begin{array}{c} |f(t,y_n(t),D_{0^+}^{\alpha-1}y_n(t))| = |f(t,t^{\alpha-2}(1-t)^{\alpha-2}[t^{2-\alpha}(1-t)^{2-\alpha}y_n(t)],D_{0^+}^{\alpha-1}y_n(t))| \leq \phi_r(t), \\ |g(t,y_n(t),D_{0^+}^{\alpha-1}y_n(t))| = |g(t,t^{\alpha-2}(1-t)^{\alpha-2}[t^{2-\alpha}(1-t)^{2-\alpha}y_n(t)],D_{0^+}^{\alpha-1}y_n(t))| \leq \phi_r(t), \\ |h(t,y_n(t),D_{0^+}^{\alpha-1}y_n(t))| = |h(t,t^{\alpha-2}(1-t)^{\alpha-2}[t^{2-\alpha}(1-t)^{2-\alpha}y_n(t)],D_{0^+}^{\alpha-1}y_n(t))| \leq \phi_r(t) \end{array}$$

hold for all $t \in (0,1), n = 0,1,2,\cdots$. Then for $t \in (0,1)$, from (i) and (ii) we have

$$\begin{aligned} & t^{2-\alpha}(1-t)^{2-\alpha}|(Ty_n)(t) - (Ty_0)(t)| \\ \leq\ & \int_0^1 t^{2-\alpha}G(t,s)|f(s,y_n(s),D_{0^+}^{\alpha-1}y_n(s)) - f(s,y_0(s),D_{0^+}^{\alpha-1}y_0(s))|ds \\ & +\frac{c\Gamma(\alpha)+d(1-t)}{\delta}\int_0^1 |g(s,y_n(s),D_{0^+}^{\alpha-1}y_n(s)) - g(s,y_0(s),D_{0^+}^{\alpha-1}y_0(s))|ds \\ & +\frac{at+b\Gamma(\alpha)}{\delta}\int_0^1 |h(s,y_n(s),D_{0^+}^{\alpha-1}y_n(s)) - h(s,y_0(s),D_{0^+}^{\alpha-1}y_0(s))|ds \\ \leq\ & \left(\frac{ad+ac\Gamma(\alpha)}{\delta\Gamma(\alpha)} + \frac{bd+bc\Gamma(\alpha)}{\delta}\right)\int_0^1|f(s,y_n(s),D_{0^+}^{\alpha-1}y_n(s)) - f(s,y_0(s),D_{0^+}^{\alpha-1}y_0(s))|ds \\ & +\frac{c\Gamma(\alpha)+d}{\delta}\int_0^1 |g(s,y_n(s),D_{0^+}^{\alpha-1}y_n(s)) - g(s,y_0(s),D_{0^+}^{\alpha-1}y_0(s))|ds \\ & +\frac{a+b\Gamma(\alpha)}{\delta}\int_0^1 |h(s,y_n(s),D_{0^+}^{\alpha-1}y_n(s)) - h(s,y_0(s),D_{0^+}^{\alpha-1}y_0(s))|ds \end{aligned}$$

$$\leq \quad 2\left(\frac{ad+ac\Gamma(\alpha)}{\delta\Gamma(\alpha)}+\frac{bd+bc\Gamma(\alpha)}{\delta}\right)\int_0^1 \phi_r(s)ds$$
$$+2\left(\frac{c\Gamma(\alpha)+d}{\delta}+\frac{a+b\Gamma(\alpha)}{\delta}\right)\int_0^1 \phi_r(s)ds.$$

By the dominated convergence theorem, since f, g, h are Caratheodory functions, then we get

$$\lim_{n\to+\infty}\sup_{t\in(0,1)} t^{2-\alpha}(1-t)^{2-\alpha}|(Ty_n)(t)-(Ty_0)(t)|=0.$$

Similarly, we have

$$\begin{array}{c}|D_{0^+}^{\alpha-1}(Ty_n)(t)-D_{0^+}^{\alpha-1}(Ty_0)(t)| \\ \leq \int_0^t |f(s,y_n(s),D_{0^+}^{\alpha-1}y_n(s))-f(s,y_0(s),D_{0^+}^{\alpha-1}y_0(s))|ds \\ +\frac{ac\Gamma(\alpha)}{\delta}\int_0^1 |f(s,y_n(s),D_{0^+}^{\alpha-1}y_n(s))-f(s,y_0(s),D_{0^+}^{\alpha-1}y_0(s))|ds \\ +\frac{ad}{\delta}\int_0^1(1-s)^{\alpha-1}|f(s,y_n(s),D_{0^+}^{\alpha-1}y_n(s))-f(s,y_0(s),D_{0^+}^{\alpha-1}y_0(s))|ds \\ +\frac{d\Gamma(\alpha)}{\delta}\int_0^1 |g(s,y_n(s),D_{0^+}^{\alpha-1}y_n(s))-g(s,y_0(s),D_{0^+}^{\alpha-1}y_0(s))|ds \\ +\frac{a\Gamma(\alpha)}{\delta}\int_0^1 |h(s,y_n(s),D_{0^+}^{\alpha-1}y_n(s))-h(s,y_0(s),D_{0^+}^{\alpha-1}y_0(s))|ds \\ \leq 2\left(1+\frac{ac\Gamma(\alpha)}{\delta}+\frac{ad}{\delta}+\frac{d\Gamma(\alpha)}{\delta}+\frac{a\Gamma(\alpha)}{\delta}\right)\int_0^1 \phi_r(s)ds\end{array}$$

By the dominated convergence theorem, since f, g, h are Caratheodory functions, then we get $\lim_{n\to+\infty}\sup_{t\in(0,1)}|D_{0^+}^{\alpha-1}(Ty_n)(t)-D_{0^+}^{\alpha-1}(Ty_0)(t)|=0$. Hence we have $||Ty_n-Ty_0||\to 0$ as $n\to\infty$. Then T is continuous.

Step 2. T maps bounded sets into bounded sets in X.

Let $M\subset X$ be a bounded set. Then there exists $r>0$ such that $||x||\leq r$ for all $x\in M$. Hence there exists $\phi_r\in L^1(0,1)$ such that (5) holds. Then Lemma 2.2(i) implies that

$$\begin{aligned}
&t^{2-\alpha}(1-t)^{2-\alpha}|(Tx)(t)| \leq \int_0^1 t^{2-\alpha}G(t,s)|f(s,x(s),D_{0^+}^{\alpha-1}x(s))|ds \\
&+\frac{c\Gamma(\alpha)+d(1-t)}{\delta}\int_0^1 |g(s,x(s),D_{0^+}^{\alpha-1}x(s))|ds \\
+\quad &\frac{at+b\Gamma(\alpha)}{\delta}\int_0^1 |h(s,x(s),D_{0^+}^{\alpha-1}x(s))|ds \\
\leq\quad &\left(\frac{ad+ac\Gamma(\alpha)}{\delta\Gamma(\alpha)}+\frac{bd+bc\Gamma(\alpha)}{\delta}\right)\int_0^1 |f(s,x(s),D_{0^+}^{\alpha-1}x(s))|ds \\
&+\frac{c\Gamma(\alpha)+d}{\delta}\int_0^1 |g(s,x(s),D_{0^+}^{\alpha-1}x(t))|ds+\frac{a+b\Gamma(\alpha)}{\delta}\int_0^1 |h(s,x(s),D_{0^+}^{\alpha-1}x(s))|ds \\
\leq\quad &\left(\frac{ad+ac\Gamma(\alpha)}{\delta\Gamma(\alpha)}+\frac{bd+bc\Gamma(\alpha)}{\delta}\right)\int_0^1 \phi_r(s)ds \\
+\quad &\left(\frac{c\Gamma(\alpha)+d}{\delta}+\frac{a+b\Gamma(\alpha)}{\delta}\right)\int_0^1 \phi_r(s)ds
\end{aligned}$$

and

$$\begin{aligned}
|D_{0^+}^{\alpha-1}(Tx)(t)| &\leq \int_0^t |f(s,x(s),D_{0^+}^{\alpha-1}x(s))|ds+\frac{ac\Gamma(\alpha)}{\delta}\int_0^1 |f(s,x(s),D_{0^+}^{\alpha-1}x(s))|ds\\
&\quad+\frac{ad}{\delta}\int_0^1(1-s)^{\alpha-1}|f(s,x(s),D_{0^+}^{\alpha-1}x(s))|ds\\
&\quad+\frac{d\Gamma(\alpha)}{\delta}\int_0^1 |g(s,x(s),D_{0^+}^{\alpha-1}x(s))|ds\\
&\quad+\frac{a\Gamma(\alpha)}{\delta}\int_0^1 |h(s,x(s),D_{0^+}^{\alpha-1}x(s))|ds\\
&\leq \left(1+\frac{ac\Gamma(\alpha)}{\delta}+\frac{ad}{\delta}+\frac{d\Gamma(\alpha)}{\delta}+\frac{a\Gamma(\alpha)}{\delta}\right)\int_0^1\phi_r(s)ds
\end{aligned}$$

So

$$\begin{aligned}
||Tx|| &\leq \max\left\{\frac{ad+ac\Gamma(\alpha)}{\delta\Gamma(\alpha)}+\frac{bd+bc\Gamma(\alpha)}{\delta}+\frac{c\Gamma(\alpha)+d}{\delta}+\frac{a+b\Gamma(\alpha)}{\delta},\right.\\
&\quad\left.1+\frac{ac\Gamma(\alpha)}{\delta}+\frac{ad}{\delta}+\frac{d\Gamma(\alpha)}{\delta}+\frac{a\Gamma(\alpha)}{\delta}\right\}\int_0^1\phi_r(s)ds.
\end{aligned}$$

It follows that T maps bounded sets into bounded sets.

Step 3. Let $M\subset\{y\in X:||y||\leq r\}$ be a bounded set on X. We prove that both $\{t^{2-\alpha}(1-t)^{2-\alpha}Tx:x\in M\}$ and $\{D_{0^+}^{\alpha-1}Tx:x\in M\}$ are equi-continuous on $(0,1)$.

Similarly to Step 2, we can get $\phi_r\in L^1(0,1)$ such that (5) holds. Let $t_1,t_2\in(0,1)$ with $t_1<t_2$ and $y\in M$.

One can see from (4) that, for $s\in[0,t_1]$

$$\begin{aligned}
&|t_1^{2-\alpha}(1-t_1)^{2-\alpha}G(t_1,s)-t_2^{2-\alpha}(1-t_2)^{2-\alpha}G(t_2,s)|\\
=&\left|\frac{t_2^{2-\alpha}(1-t_2)^{2-\alpha}(t_2-s)^{\alpha-1}-t_1^{2-\alpha}(1-t_1)^{2-\alpha}(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\right.\\
&+\frac{ac[t_1(1-t_1)^{2-\alpha}-t_2(1-t_2)^{2-\alpha}]}{\delta}+\frac{ad[t_1(1-t_1)^{2-\alpha}-t_2(1-t_2)^{2-\alpha}](1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}\\
&\left.+\frac{bc\Gamma(\alpha)[(1-t_1)^{2-\alpha}-(1-t_2)^{2-\alpha}]+bd[(1-t_1)^{2-\alpha}-(1-t_2)^{2-\alpha}](1-s)^{\alpha-1}}{\delta}\right|\\
\leq&\frac{|t_2^{2-\alpha}(1-t_2)^{2-\alpha}-t_1^{2-\alpha}(1-t_1)^{2-\alpha}|+t_1^{2-\alpha}(1-t_1)^{2-\alpha}|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}|}{\Gamma(\alpha)}\\
&+\frac{ac|t_1(1-t_1)^{2-\alpha}-t_2(1-t_2)^{2-\alpha}|}{\delta}+\frac{ad|t_1(1-t_1)^{2-\alpha}-t_2(1-t_2)^{2-\alpha}|(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}\\
&+\frac{bc\Gamma(\alpha)|(1-t_1)^{2-\alpha}-(1-t_2)^{2-\alpha}|+bd|(1-t_1)^{2-\alpha}-(1-t_2)^{2-\alpha}|(1-s)^{\alpha-1}}{\delta}
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{|t_2^{2-\alpha}(1-t_2)^{2-\alpha}-t_1^{2-\alpha}(1-t_1)^{2-\alpha}|+|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}|}{\Gamma(\alpha)}\\
& +\frac{ac|t_1(1-t_1)^{2-\alpha}-t_2(1-t_2)^{2-\alpha}|}{\delta}+\frac{ad|t_1(1-t_1)^{2-\alpha}-t_2(1-t_2)^{2-\alpha}|}{\delta\Gamma(\alpha)}\\
& +\frac{bc\Gamma(\alpha)|(1-t_1)^{2-\alpha}-(1-t_2)^{2-\alpha}|+bd|(1-t_1)^{2-\alpha}-(1-t_2)^{2-\alpha}|}{\delta}.
\end{aligned}$$

Since $(1-t)^{2-\alpha}$, $t(1-t)^{2-\alpha}$, t^{α} and $t^{2-\alpha}(1-t)^{2-\alpha}$ are uniformly continuous functions on $[0,1]$, then for $\epsilon>0$ there exists $\delta_1>0$ such that $|u_1-u_2|<\delta_1, u_1,u_2\in[0,1]$ imply that

$$|t_1^{2-\alpha}(1-t_1)^{2-\alpha}G(t_1,s)-t_2^{2-\alpha}(1-t_2)^{2-\alpha}G(t_2,s)|<\epsilon. \tag{7}$$

For $s\in[t_1,t_2]$, we have

$$\begin{aligned}
& |t_1^{2-\alpha}(1-t_1)^{2-\alpha}G(t_1,s)-t_2^{2-\alpha}(1-t_2)^{2-\alpha}G(t_2,s)|\\
=\ & \Bigg|\frac{ac[t_1(1-t_1)^{2-\alpha}-t_2(1-t_2)^{2-\alpha}]}{\delta}+\frac{ad[t_1(1-t_1)^{2-\alpha}-t_2(1-t_2)^{2-\alpha}](1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}\\
& +\frac{bc\Gamma(\alpha)[(1-t_1)^{2-\alpha}-(1-t_2)^{2-\alpha}]+bd[(1-t_1)^{2-\alpha}-(1-t_2)^{2-\alpha}](1-s)^{\alpha-1}}{\delta}\\
& +\frac{t_2^{2-\alpha}(1-t_2)^{2-\alpha}(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\Bigg|\\
\leq\ & \frac{2ac}{\delta}+\frac{2ad}{\delta\Gamma(\alpha)}+\frac{2bc\Gamma(\alpha)+2bd}{\delta}+\frac{2}{\Gamma(\alpha)}.
\end{aligned}$$

For $s\in[t_2,1]$, we have

$$\begin{aligned}
& |t_1^{2-\alpha}(1-t_1)^{2-\alpha}G(t_1,s)-t_2^{2-\alpha}(1-t_2)^{2-\alpha}G(t_2,s)|\\
=\ & \Bigg|\frac{ac[t_1(1-t_1)^{2-\alpha}-t_2(1-t_2)^{2-\alpha}]}{\delta}+\frac{ad[t_1(1-t_1)^{2-\alpha}-t_2(1-t_2)^{2-\alpha}](1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}\\
& +\frac{bc\Gamma(\alpha)[(1-t_1)^{2-\alpha}-(1-t_2)^{2-\alpha}]+bd[(1-t_1)^{2-\alpha}-(1-t_2)^{2-\alpha}](1-s)^{\alpha-1}}{\delta}\Bigg|\\
\leq\ & \frac{ac|t_1(1-t_1)^{2-\alpha}-t_2(1-t_2)^{2-\alpha}|}{\delta}+\frac{ad|t_1(1-t_1)^{2-\alpha}-t_2(1-t_2)^{2-\alpha}|}{\delta\Gamma(\alpha)}\\
& +\frac{bc\Gamma(\alpha)|(1-t_1)^{2-\alpha}-(1-t_2)^{2-\alpha}|+bd|(1-t_1)^{2-\alpha}-(1-t_2)^{2-\alpha}|}{\delta}.
\end{aligned}$$

Since both $(1-t)^{2-\alpha}$ and $t(1-t)^{2-\alpha}$ are uniformly continuous functions on $[0,1]$, then for $\epsilon>0$ there exists $\delta_2>0$ such that $|u_1-u_2|<\delta_2, u_1,u_2\in[0,1]$ imply that (9) holds.

For $\epsilon > 0$ it is easy to see similarly that there exists $\delta_3 > 0$ such that $|u_1 - u_2| < \delta_3, u_1, u_2 \in [0,1]$ imply that

$$\frac{c\Gamma(\alpha)|(1-t_1)^{2-\alpha} - (1-t_2)^{2-\alpha}| + d|(1-t_1)^{3-\alpha} - (1-t_2)^{3-\alpha}|}{\delta} < \epsilon,$$
$$\frac{a|t_1(1-t_1)^{2-\alpha} - t_2(1-t_2)^{2-\alpha}| + b\Gamma(\alpha)|(1-t_1)^{2-\alpha} - (1-t_2)^{2-\alpha}|}{\delta} < \epsilon,$$
$$\int_{t_1}^{t_2} \phi_r(s)ds < \epsilon.$$

For $\epsilon > 0$, for $y \in M$, $|u_1 - u_2| < \min\{\delta_1, \delta_2, \delta_3\}, u_1, u_2 \in [0,1]$, it follows that

$$\begin{aligned}
&|t_1^{2-\alpha}(1-t_2)^{2-\alpha}(Ty)(t_1) - t_1^{2-\alpha}(1-t_1)^{2-\alpha}(Ty)(t_2)| \\
= &\left|\int_0^1 t_1^{2-\alpha}(1-t_1)^{2-\alpha}G(t_1,s)f(s,y(s),D_{0^+}^{\alpha-1}y(s))ds\right. \\
&-\int_0^1 t_2^{2-\alpha}(1-t_2)^{2-\alpha}G(t_2,s)f(s,y(s),D_{0^+}^{\alpha-1}y(s))ds \\
&+\frac{c\Gamma(\alpha)[(1-t_1)^{2-\alpha} - (1-t_2)^{2-\alpha}] + d[(1-t_1)^{\alpha} - (1-t_2)^{\alpha}]}{\delta} \\
\times &\int_0^1 g(s,y(s),D_{0^+}^{\alpha-1}y(s))ds \\
&+\frac{a[t_1(1-t_1)^{2-\alpha} - t_2(1-t_2)^{2-\alpha}] + b\Gamma(\alpha)[(1-t_1)^{2-\alpha} - (1-t_2)^{2-\alpha}]}{\delta} \times \\
&\left.\int_0^1 h(s,y(s),D_{0^+}^{\alpha-1}y(s))ds\right| \\
\leq &\int_0^{t_1} \left|t_1^{2-\alpha}(1-t_1)^{2-\alpha}G(t_1,s) - t_2^{2-\alpha}(1-t_2)^{2-\alpha}G(t_2,s)\right| f(s,y(s),D_{0^+}^{\alpha-1}y(s))ds \\
&+\int_{t_1}^{t_2} \left|t_1^{2-\alpha}(1-t_1)^{2-\alpha}G(t_1,s) - t_2^{2-\alpha}(1-t_2)^{2-\alpha}G(t_2,s)\right| f(s,y(s),D_{0^+}^{\alpha-1}y(s))ds \\
&+\int_{t_2}^{1} \left|t_1^{2-\alpha}(1-t_1)^{2-\alpha}G(t_1,s) - t_2^{2-\alpha}(1-t_2)^{2-\alpha}G(t_2,s)\right| f(s,y(s),D_{0^+}^{\alpha-1}y(s))ds \\
&+\frac{c\Gamma(\alpha)|(1-t_1)^{2-\alpha} - (1-t_2)^{2-\alpha}| + d|(1-t_1)^{3-\alpha} - (1-t_2)^{3-\alpha}|}{\delta} \times \\
&\int_0^1 g(s,y(s),D_{0^+}^{\alpha-1}y(s))ds \\
&+\frac{a|t_1(1-t_1)^{2-\alpha} - t_2(1-t_2)^{2-\alpha}| + b\Gamma(\alpha)|(1-t_1)^{2-\alpha} - (1-t_2)^{2-\alpha}|}{\delta} \times \\
&\int_0^1 h(s,y(s),D_{0^+}^{\alpha-1}y(s))ds \\
\leq &\int_0^{t_1} \left|t_1^{2-\alpha}(1-t_1)^{2-\alpha}G(t_1,s) - t_2^{2-\alpha}(1-t_2)^{2-\alpha}G(t_2,s)\right| \phi_r(s)ds \\
&+\int_{t_1}^{t_2} \left|t_1^{2-\alpha}(1-t_1)^{2-\alpha}G(t_1,s) - t_2^{2-\alpha}(1-t_2)^{2-\alpha}G(t_2,s)\right| \phi_r(s)ds
\end{aligned}$$

$$
\begin{aligned}
& +\int_{t_2}^{1}\left|t_1^{2-\alpha}(1-t_1)^{2-\alpha}G(t_1,s)-t_2^{2-\alpha}(1-t_2)^{2-\alpha}G(t_2,s)\right|\phi_r(s)ds \\
& +\frac{c\Gamma(\alpha)|(1-t_1)^{2-\alpha}-(1-t_2)^{2-\alpha}|+d|(1-t_1)^{3-\alpha}-(1-t_2)^{3-\alpha}|}{\delta}\int_0^1\phi_r(s)ds \\
& +\frac{a|t_1(1-t_1)^{2-\alpha}-t_2(1-t_2)^{2-\alpha}|+b\Gamma(\alpha)|(1-t_1)^{2-\alpha}-(1-t_2)^{2-\alpha}|}{\delta}\int_0^1\phi_r(s)ds \\
< \; & \epsilon\int_0^1\phi_r(s)ds+\epsilon\left(\frac{2ac}{\delta}+\frac{2ad}{\delta\Gamma(\alpha)}+\frac{2bc\Gamma(\alpha)+2bd}{\delta}+\frac{2}{\Gamma(\alpha)}\right) \\
& +\epsilon\int_0^1\phi_r(s)ds+\epsilon\int_0^1\phi_r(s)ds+\epsilon\int_0^1\phi_r(s)ds.
\end{aligned}
$$

Then

$$|t_1^{2-\alpha}(1-t_2)^{2-\alpha}(Ty)(t_1)-t_1^{2-\alpha}(1-t_1)^{2-\alpha}(Ty)(t_2)|\to 0$$

uniformly as $t_1\to t_2$. Therefore, $\{t^{2-\alpha}(1-t)^{2-\alpha}Tx : x\in M\}$ is equicontinuous on $(0,1)$.

On the other hand, we have

$$
\begin{aligned}
|D_{0^+}^{\alpha-1}(Ty)(t_1)-D_{0^+}^{\alpha-1}(Ty)(t_1)| \;&\leq\; \int_{t_1}^{t_2} f(s,y(s),D_{0^+}^{\alpha-1}y(s))ds \\
&\leq\; \int_{t_1}^{t_2}\phi_r(s)ds.
\end{aligned}
$$

As $t_1\to t_2$, the right-hand side of the above inequality tends to zero uniformly. Therefore, $\{D_{0^+}^{\alpha-1}Tx : x\in M\}$ is equicontinuous on $(0,1)$.

The Arzela-Ascoli theorem implies that $\overline{T(M)}$ is relatively compact. Thus, the operator $T : P\to P$ is completely continuous.

3. **Main theorems**

Now, we prove the main results. Let

$$
\begin{aligned}
\Delta_1 &= \frac{ad+ac\Gamma(\alpha)}{\delta\Gamma(\alpha)}+\frac{bd+bc\Gamma(\alpha)}{\delta}+\frac{a+b\Gamma(\alpha)}{\delta}+\frac{a+b\Gamma(\alpha)}{\delta}, \\
\Delta_2 &= 1+\frac{ac\Gamma(\alpha)}{\delta}+\frac{ad}{\delta}+\frac{d\Gamma(\alpha)}{\delta}+\frac{a\Gamma(\alpha)}{\delta}.
\end{aligned}
$$

Theorem 3.1. Suppose that there exist $\phi,\psi\in L^1(0,1)$ such that

$$
\begin{aligned}
|f(t,t^{\alpha-2}(1-t)^{\alpha-2}u,v)-f(t,t^{\alpha-2}(1-t)^{\alpha-2}u_1,v_1)| &\leq \phi(t)|u-u_1|+\psi(t)|v-v_1|, \\
|g(t,t^{\alpha-2}(1-t)^{\alpha-2}u,v)-f(t,t^{\alpha-2}(1-t)^{\alpha-2}u_1,v_1)| &\leq \phi(t)|u-u_1|+\psi(t)|v-v_1|, \\
|h(t,t^{\alpha-2}(1-t)^{\alpha-2}u,v)-f(t,t^{\alpha-2}(1-t)^{\alpha-2}u_1,v_1)| &\leq \phi(t)|u-u_1|+\psi(t)|v-v_1|
\end{aligned}
\tag{8}
$$

hold for all $t\in(0,1), u,u_1\in[0,\infty), v,v_1\in R$. Then BVP(3) has a unique positive solution if

$$\max\{\Delta_1,\ \Delta_2\}\int_0^1[\phi(s)+\psi(s)]ds<1. \tag{9}$$

Proof. We shall prove that under the assumptions (8) and (9), T is a contraction operator. Indeed, by the definition of $G(t,s)$ for $x,y\in P$, from Lemma 2.2

and (8), we have the estimate

$$\begin{aligned}
& t^{2-\alpha}(1-t)^{2-\alpha}|(Tx)(t)-(Ty)(t)| \\
\leq & \int_0^1 t^{2-\alpha}(1-t)^{2-\alpha}G(t,s)|f(s,x(s),D_{0^+}^{\alpha-1}x(s))-f(s,y(s),D_{0^+}^{\alpha-1}y(s))|ds \\
& +\frac{c\Gamma(\alpha)(1-t)^{2-\alpha}+d[(1-t)^{2-\alpha}-t(1-t)^{2-\alpha}]}{\delta} \\
\times & \int_0^1 |g(t,x(t),D_{0^+}^{\alpha-1}x(t))-g(t,y(t),D_{0^+}^{\alpha-1}x(t))|dt \\
& +\frac{at(1-t)^{2-\alpha}+b\Gamma(\alpha)(1-t)^{2-\alpha}}{\delta}\int_0^1 |h(t,x(t),D_{0^+}^{\alpha-1}x(t))-h(t,y(t),D_{0^+}^{\alpha-1}y(t))|dt \\
\leq & \left(\frac{ad+ac\Gamma(\alpha)}{\delta\Gamma(\alpha)}+\frac{bd+bc\Gamma(\alpha)}{\delta}\right)\times \\
& \int_0^1[\phi(s)s^{2-\alpha}(1-s)^{2-\alpha}|x(s)-y(s)|+\psi(s)|D_{0^+}^{\alpha-1}x(s)-D_{0^+}^{\alpha-1}y(s)|ds \\
& +\frac{c\Gamma(\alpha)+d}{\delta}\int_0^1[\phi(s)s^{2-\alpha}(1-s)^{2-\alpha}|x(s)-y(s)|+\psi(s)|D_{0^+}^{\alpha-1}u(s)-D_{0^+}^{\alpha-1}v(s)|ds \\
& +\frac{a+b\Gamma(\alpha)}{\delta}\int_0^1[\phi(s)s^{2-\alpha}(1-s)^{2-\alpha}|x(s)-y(s)|+\psi(s)|D_{0^+}^{\alpha-1}x(s)-D_{0^+}^{\alpha-1}y(s)|ds \\
\leq & \left(\frac{ad+ac\Gamma(\alpha)}{\delta\Gamma(\alpha)}+\frac{bd+bc\Gamma(\alpha)}{\delta}+\frac{a+b\Gamma(\alpha)}{\delta}+\frac{a+b\Gamma(\alpha)}{\delta}\right)\int_0^1[\phi(s)+\psi(s)]ds||x-y||,
\end{aligned}$$

and

$$\begin{aligned}
& |D_{0^+}^{\alpha-1}Tx(t)-D_{0^+}^{\alpha-1}Ty(t)| \\
\leq & \left(1+\frac{ac\Gamma(\alpha)}{\delta}+\frac{ad}{\delta}\right)\int_0^1|f(s,x(s),D_{0^+}^{\alpha-1}x(s))-f(s,y(s),D_{0^+}^{\alpha-1}y(s))|ds \\
& +\frac{d\Gamma(\alpha)}{\delta}\int_0^1|g(t,x(t),D_{0^+}^{\alpha-1}x(t))-g(s,y(s),D_{0^+}^{\alpha-1}y(s))|dt \\
& +\frac{a\Gamma(\alpha)}{\delta}\int_0^1|h(t,x(t),D_{0^+}^{\alpha-1}x(t))-h(s,y(s),D_{0^+}^{\alpha-1}y(s))|dt \\
\leq & \left(1+\frac{ac\Gamma(\alpha)}{\delta}+\frac{ad}{\delta}+\frac{d\Gamma(\alpha)}{\delta}+\frac{a\Gamma(\alpha)}{\delta}\right)\times \\
& \int_0^1[\phi(s)s^{2-\alpha}|x(s)-y(s)|+\psi(s)|D_{0^+}^{\alpha-1}x(s)-D_{0^+}^{\alpha-1}y(s)|ds \\
\leq & \left(1+\frac{ac\Gamma(\alpha)}{\delta}+\frac{ad}{\delta}+\frac{d\Gamma(\alpha)}{\delta}+\frac{a\Gamma(\alpha)}{\delta}\right)\int_0^1[\phi(s)+\psi(s)]ds||x-y||.
\end{aligned}$$

It follows that

$$\begin{aligned}||Tx-Ty|| &\leq \max\left\{\frac{ad+ac\Gamma(\alpha)}{\delta\Gamma(\alpha)}+\frac{bd+bc\Gamma(\alpha)}{\delta}+\frac{a+b\Gamma(\alpha)}{\delta}+\frac{a+b\Gamma(\alpha)}{\delta},\right.\\ &\left. 1+\frac{ac\Gamma(\alpha)}{\delta}+\frac{ad}{\delta}+\frac{d\Gamma(\alpha)}{\delta}+\frac{a\Gamma(\alpha)}{\delta}\right\}\int_0^1[\phi(s)+\psi(s)]ds||x-y||\\ &= \max\{\Delta_1,\ \Delta_2\}\int_0^1[\phi(s)+\psi(s)]ds||x-y||.\end{aligned}$$

Hence the contraction mapping principle implies that BVP(3) has a unique positive solution x_0. Since $f(t,0,0)\not\equiv 0$ on each subinterval of $(0,1)$, then x_0 is positive on $(0,1)$. So x_0 is a positive solution of BVP(3). The proof is completed.

Theorem 3.2. Suppose that there exists $\phi\in L^1(0,1)$ such that

$$\begin{array}{l}\overline{\lim}_{x\to\infty}\sup_{t\in(0,1)}\frac{f(t,t^{\alpha-2}(1-t)^{\alpha-2}x,y)}{\phi(t)(|x|+|y|)}<\frac{1}{2\max\{\Delta_1,\ \Delta_2\}\int_0^1\phi(s)ds}\\ \overline{\lim}_{x\to\infty}\sup_{t\in(0,1)}\frac{g(t,t^{\alpha-2}(1-t)^{\alpha-2}x,y)}{\phi(t)(|x|+|y|)}<\frac{1}{2\max\{\Delta_1,\ \Delta_2\}\int_0^1\phi(s)ds},\\ \overline{\lim}_{x\to\infty}\sup_{t\in(0,1)}\frac{h(t,t^{\alpha-2}(1-t)^{\alpha-2}x,y)}{\phi(t)(|x|+|y|)}<\frac{1}{2\max\{\Delta_1,\ \Delta_2\}\int_0^1\phi(s)ds}.\end{array}\tag{10}$$

Then, BVP(3) has at least one positive solution.

Proof. It follows from (10) that there exist M and H

$$0<M<\frac{1}{2\max\{\Delta_1,\ \Delta_2\}\int_0^1\phi(s)ds},\quad H>0$$

such that

$$\begin{aligned}&0\leq\frac{f(t,t^{\alpha-2}(1-t)^{\alpha-2}x,y)}{\phi(t)(|x|+|y|)}\leq M<\frac{1}{2\max\{\Delta_1,\ \Delta_2\}},t\in(0,1),|x|+|y|>H,\\ &0\leq\frac{g(t,t^{\alpha-2}(1-t)^{\alpha-2}x,y)}{\phi(t)(|x|+|y|)}\leq M<\frac{1}{2\max\{\Delta_1,\ \Delta_2\}},t\in(0,1),|x|+|y|>H,\\ &0\leq\frac{h(t,t^{\alpha-2}(1-t)^{\alpha-2}x,y)}{\phi(t)(|x|+|y|)}\leq M<\frac{1}{2\max\{\Delta_1,\ \Delta_2\}},t\in(0,1),|x|+|y|>H.\end{aligned}$$

On the other hand, there exists $\phi_M\in L^1(0,1)$ such that

$$\begin{aligned}&0\leq f(t,t^{\alpha-2}(1-t)^{\alpha-2}x,y)\leq\phi_M(t),t\in(0,1),|x|+|y|\leq H,\\ &0\leq g(t,t^{\alpha-2}(1-t)^{\alpha-2}x,y)\leq\phi_M(t),t\in(0,1),|x|+|y|\leq H,\\ &0\leq h(t,t^{\alpha-2}(1-t)^{\alpha-2}x,y)\leq\phi_M(t),t\in(0,1),|x|+|y|\leq H.\end{aligned}$$

It follows that

$$\begin{aligned}&0\leq f(t,t^{\alpha-2}x,y)\leq M(|x|+|y|)\phi(t)+\phi_M(t),t\in(0,1),x\in[0,\infty),y\in R,\\ &0\leq g(t,t^{\alpha-2}x,y)\leq M(|x|+|y|)\phi(t)+\phi_M(t),t\in(0,1),x\in[0,\infty),y\in R,\\ &0\leq h(t,t^{\alpha-2}x,y)\leq M(|x|+|y|)\phi(t)+\phi_M(t),t\in(0,1),x\in[0,\infty),y\in R.\end{aligned}$$

From $\phi_M \in L^1(0,1)$ and denote

$$\begin{aligned}\phi_0(t) &= \int_0^1 G(t,s)\phi_M(s)ds + \frac{c\Gamma(\alpha)t^{\alpha-2} + d(t^{\alpha-2} - t^{\alpha-1})}{\delta}\int_0^1 \phi_M(t)dt \\ &\quad + \frac{at^{\alpha-1} + b\Gamma(\alpha)t^{\alpha-2}}{\delta}\int_0^1 \phi_M(t)dt,\end{aligned}$$

we can see $\phi_0 \in X$. Choose $R > 0$ sufficiently large such that

$$R \geq \frac{2\max\{\Delta_1,\ \Delta_2\}\int_0^1 \phi_M(s)ds + 2\max\{\Delta_1,\ \Delta_2\}^2 M\int_0^1 \phi(s)ds\int_0^1 \phi_M(s)ds}{1 - 2\max\{\Delta_1,\ \Delta_2\}M\int_0^1 \phi(s)ds}. \quad (11)$$

Let

$$B_R = \{x \in P : ||x - \phi_0|| \leq R\}.$$

It is easy to see that B_R is a convex, bounded, and closed subset of the Banach space X. For $x \in B_R$, by the methods used in the proof of Lemma 2.2(iv)(Step 2), we have

$$\begin{aligned}||x|| &\leq ||x - \phi_0|| + ||\phi_0|| \\ &\leq R + \max\{\Delta_1,\ \Delta_2\}\int_0^1 \phi_M(s)ds.\end{aligned}$$

When the methods in the proof of Lemma 2.2(iv)(Step 2) are used, we get

$$\begin{aligned}&||Tx - \phi_0|| \leq ||Tx|| + ||\phi_0||| \\ \leq\ & \max\{\Delta_1,\ \Delta_2\}\int_0^1 \left[M\phi(t)\left(t^{2-\alpha}(1-t)^{2-\alpha}|x(t) + |D_{0^+}^{\alpha-1}x(t)|\right) + \phi_M(t)\right]dt \\ & + \max\{\Delta_1,\ \Delta_2\}\int_0^1 \phi_M(s)ds \\ \leq\ & 2\max\{\Delta_1, ;\Delta_2\}M\int_0^1 \phi(t)dt||x|| + 2\max\{\Delta_1,\ \Delta_2\}\int_0^1 \phi_M(t)dt \\ \leq\ & 2\max\{\Delta_1, ;\Delta_2\}M\int_0^1 \phi(t)dt\left(R + \max\{\Delta_1,\ \Delta_2\}\int_0^1 \phi_M(t)dt\right) \\ & +2\max\{\Delta_1,\ \Delta_2\}\int_0^1 \phi_M(t)dt \leq R.\end{aligned}$$

So, we have $TB_R \subset B_R$. Since T is completely continuous, the Schauder fixed point theorem essures that operator T has at least one fixed point in B_R and then BVP(3) has at least one positive solution. The proof is complete.

Theorem 3.3. Suppose that

$$\begin{array}{l}|f(t, t^{\alpha-2}(1-t)^{\alpha-2}x, y)| \leq \phi(t)w(|x| + |y|), t \in (0,1], x, y \in R, \\ |g(t, t^{\alpha-2}(1-t)^{\alpha-2}x, y)| \leq \phi(t)w(|x| + |y|), t \in (0,1], x, y \in R, \\ |h(t, t^{\alpha-2}(1-t)^{\alpha-2}x, y)| \leq \phi(t)w(|x| + |y|), t \in (0,1], x, y \in R\end{array} \quad (12)$$

with $\phi \in L^1(0,1)$ and $w \in C(R, [0,\infty))$ nondecreasing. If there exists a constant $\mu > 0$ such that

$$\frac{\mu}{\max\{\Delta_1,\ \Delta_2\}w(2\mu)\int_0^1 \phi(s)ds} \geq 1, \quad (13)$$

then, BVP(3) has at least one positive solution.

Proof. We consider the BVP of the form

$$\begin{cases} D_{0^+}^{\alpha}u(t)+\lambda f(t,u(t),D_{0^+}^{\alpha-1}u(t))=0, \quad t\in(0,1),\\ a\lim_{t\to 0}t^{2-\alpha}u(t)-b\ \lim_{t\to 0}\ D^{\alpha-1}u(t)=\lambda\int_0^1 g(t,u(t),D_{0^+}^{\alpha-1}u(t))dt,\\ c\ D_0^{\alpha-1}u(1)+du(1)=\lambda\int_0^1 h(t,u(t),D_{0^+}^{\alpha-1}u(t))dt,\end{cases} \tag{14}$$

for $0<\lambda<1$. Solving BVP(14) is equivalent to solving the fixed point problem $x=\lambda Tx$.

Let

$$U=\{x\in X:||x||\le\mu\}.$$

We claim that $x\neq\lambda Tx$ for all $x\in\partial U$ and $\lambda\in(0,1)$. In fact, if $x=\lambda Tx$ for some $x\in\partial U$ and $\lambda\in(0,1)$, when the methods in the proof of Lemma 2.2(iv)(Step 2) are used, we have

$$\begin{aligned} ||x|| &= \max\left\{\sup_{t\in(0,1)}\lambda t^{2-\alpha}(1-t)^{2-\alpha}(Tx)(t),\ \sup_{t\in(0,1)}|D_{0^+}^{\alpha-1}\lambda(Tx)(t)|\right\}\\ &< \max\{\Delta_1,\ \Delta_2\}\int_0^1\phi(s)w\left(s^{2-s}(1-s)^{2-\alpha}|x(s)|+|D_{0^+}^{\alpha-1}x(s)|\right)ds\\ &\le \max\{\Delta_1,\ \Delta_2\}\,w\,(2\mu)\int_0^1\phi(s)ds\\ &= \max\{\Delta_1,\ \Delta_2\}\,w\,(2\mu)\int_0^1\phi(s)ds. \end{aligned}$$

So

$$\mu<\max\{\Delta_1,\ \Delta_2\}\,w\,(2\mu)\int_0^1\phi(s)ds.$$

It follows that

$$\frac{\mu}{\max\{\Delta_1,\ \Delta_2\}\,w\,(2\mu)\int_0^1\phi(s)ds}<1,$$

which contradicts with (13). Since T is completely continuous, by Schauder's fixed point theorem [5], we see that BVP(3) has at least one positive solution x. The proof is complete.

4. **An example**

In this section, we give an example to illustrate the main theorem.

Example 4.1. Let$\lambda>0$. Consider the following BVP

$$\begin{cases} D_{0^+}^{\frac{3}{2}}u(t)+2t^{-\frac{1}{2}}+2(1-t)^{-2}+\lambda\left(t-\frac{1}{2}\right)^4t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}u(t)+\mu D_{0^+}^{\frac{1}{2}}u(t)=0, t\in(0,1],\\ \lim_{t\to 0}t^{\frac{1}{2}}u(t)-\lim_{t\to 0}\ D_{0^+}^{\frac{1}{2}}u(t)=0,\\ D_{0^+}^{\frac{1}{2}}u(1)+u(1)=0,\end{cases} \tag{15}$$

where $\lambda,\mu>0$.

Corresponding to BVP(2), we find that $\alpha=\frac{3}{2}$, $a=b=c=d=1$ and

$$f(t,x,y)=2t^{-\frac{1}{2}}+\lambda\left(t-\frac{1}{2}\right)^4t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}x+\mu y.$$

Choose $\phi(t) = \lambda\left(t - \frac{1}{2}\right)^4$ and $\psi(t) = \mu$. One sees that $\delta = 2\Gamma(3/2) + 1 > 0$, $f(t,0,0) \not\equiv 0$ on each subinterval of $(0,1)$. On can see that f satisfies

$$|f(t, t^{\frac{3}{2}-2}(1-t)^{\frac{3}{2}-2}u, v) - f(t, t^{\frac{3}{2}-2}(1-t)^{\frac{3}{2}-2}u_1, v_1)| \le \phi(t)|u - u_1| + \psi(t)|v - v_1|$$

for all $t \in (0,1), u, u_1 \in [0,\infty), v, v_1 \in R$. Hence Theorem 3.1 implies that BVP(15) has a unique positive solution if

$$\max\left\{(1+\Gamma(3/2))\left(\frac{1}{\delta\Gamma(3/2)} + \frac{3}{\delta}\right),\ 1 + \frac{1}{\delta} + \frac{3\Gamma(3/2)}{\delta}\right\}\int_0^1 \left[\lambda\left(s - \frac{1}{2}\right)^4 + \mu\right] ds < 1.$$

Acknowledgement. The authors of this paper would like to thank the anonymous referees and the editors for their helpful remarks and suggestions in improving the presentation and quality of the paper.

REFERENCES

[1] *R.P. Agarwal, M. Benchohra, S. Hamani*, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta Appl. Math., **109**(2010)973-1033.

[2] *A. Arara, M. Benchohra, N. Hamidi, and J.J. Nieto*, Fractional order differential equations on an unbounded domain, Nonlinear Anal., **72**(2010)580-586,

[3] *D. Guo, and J. Sun*, Nonlinear Integral Equations, Shandong Science and Technology Press, Jinan, 1987 (in Chinese).

[4] *A. A. Kilbas, O.I. Marichev, and S. G. Samko*, Fractional Integral and Derivatives (Theory and Applications), Gordon and Breach, Switzerland, 1993.

[5] *A. A. Kilbas, and J.J. Trujillo*, Differential equations of fractional order: methods, results and problems-I, Applicable Anal., **78**(2001)153–192.

[6] *A. Arara, M. Benchohra, N. Hamidi, and J. Nieto*, Fractional order differential equations on an unbounded domain, Nonlinear Anal., **72**(2010)580-586.

[7] *K. S. Miller, B. Ross*, An Introduction to the Fractional Calculus and Fractional Differential Equation, Wiley, New York, 1993.

[8] *S. Z. Rida, H.M. El-Sherbiny, and A. Arafa*, On the solution of the fractional nonlinear Schrodinger equation, Physics Lett. A, **372**(2008)553-558.

[9] *S. Samko, A. Kilbas, and O. Marichev*, Fractional Integral and Derivative, Theory and Applications, Gordon and Breach, 1993.

[10] *S. Zhang*, The existence of a positive solution for a nonlinear fractional differential equation, J. Math. Anal. Appl. **252**(2000)804–812.

[11] *E. Kaufmann, E. Mboumi*, Positive solutions of a boundary value problem for a nonlinear fractional differential equation, Electronic J. Qualitative Theory Differ. Equations, **3**(2008)1-11.

[12] *R. Dehghant and K. Ghanbari*, Triple positive solutions for boundary value problem of a nonlinear fractional differential equation, Bull. Iranian Math. Soc., **33**(2007)1-14.

[13] *Y. Liu*, Positive solutions for singular FDES, U.P.B. Sci. Series A, **73**(2011)89-100.

[14] *L. Erbe and M. Tang*, Existence and multiplicity of positive solutions to nonlinear boundary value problems, Differ. Eqs. Dynam. Systems, **4**(1996)313-320.

[15] *L. Erbe and M. Tang*, On the existence of positive solutions of ordinary differential equations, Proc. Amer. Math. Soc. **120**(1994)743-748.

[16] *D. Guo*, Second order integral-differential equations of Volterra type on unbounded domain in Banach spaces, Nonlinear Anal., **41**(2000)465-476.

[17] *C. Constantin*, Integral Equations and Stability of Feedback Systems, Academic Press, New York, 1973.