FIXED POINT RESULTS FOR NONLINEAR CONTRACTIONS WITH GENERALIZED Ω-DISTANCE MAPPINGS

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Khojasteh et.al. [F. Khojasteh, S. Shukla and S. Radenovic, A new approach to the study of fixed point theory for simulation functions, Filomat 29:6 (2015)] defined a new class of mappings namely simulation functions in which they used it to unify several fixed point results in the literature. In this paper we introduce the notion of $(\Omega,\phi,\mathscr{Z})_s$-contraction with respect to ζ through generalized Ω-distance mappings which introduced by Abodayeh et.al. [K. Abodayeh, A. Bataihah and W. Shatanawi, Generalized Ω-distance mappings and some fixed point theorems, U.P.B. Sci. Bull. Series A, Vol. 79, Iss.2, 2017] and we prove some fixed point results. Also, we give an example to support our main result.

Keywords: fixed point, simulation mappings, G-metric spaces, generalized Omega-distance

1. Introduction

The fixed point theory considered as a main tool in pure and applied mathematics since it gives a solution for the equation $f(x) = x$ for a self mapping f under some considerations. In fact the fixed point theory has been studied in various directions for instance see [12]–[34].

The concept of b-metric spaces was introduced by Bakhtin [3] which has became well known by Czerwik [4]. In 2014 Aghanjani *et.al.* [2] introduced the concept of G_b-metric spaces (or generalized b-metric spaces) using the concepts of G-metric spaces and b-metric spaces and studied some fixed point results, for more fixed point results on G_b-metric spaces we refer the reader to see [5, 6].

2. Preliminaries

The concept of G_b-metric spaces is defined as follows:

Definition 2.1. [2] *Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \to \mathbf{R}^+$ be a function satisfies:*
(G_b1) $G(x,y,z) = 0 \; if \; x = y = z$;
(G_b2) $G(x,x,y) > 0 \; for \; all \; x,y \in X, \; with \; x \neq y$;
(G_b3) $G(x,y,y) \leq G(x,y,z) \; for \, all \, x,y,z \in X, \, with \, y \neq z$;
(G_b4) $G(x,y,z) = G(p\{x,y,z\})$, *where p is a permutation of x,y,z (symmetry);*
(G_b5) $G(x,y,z) \leq s[G(x,a,a) + G(a,y,z)] \; \forall \, x,y,z,a \in X (rectangle \, inequality)$.
Then the function G is called generalized b metric and the pair (X,G) is called a generalized b metric space or G_b-metric space.

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Aghanjani *et.al.* [2] remarked that the class of G_b-metric spaces is larger than that of G-metric spaces.
The following example shows that G_b-metric on X need not be G-metric on X.

Example 2.1. [2] *Let (X,G) be a G-metric space and $p > 1$. Define $G_* : X \times X \times X \to \mathbf{R}^+$ by $G_*(x,y,z) = G(x,y,z)^p$. Then G_* is G_b-metric on X with $s = 2^{p-1}$.*

Now, we present some definitions and propositions in G_b-metric space.

Definition 2.2. [2] *Let X be a G_b-metric space. A sequence (x_n) in X is said to be*
(1) G_b-convergent to $x \in X$ if for any $\varepsilon > 0$, there exists $k \in \mathbf{N}$ such that $G(x,x_n,x_m) < \varepsilon\ \forall n,m \geq k$.
(2) G_b-Cauchy if for any $\varepsilon > 0$, there exists $k \in \mathbf{N}$ such that $G(x_n,x_m,x_l) < \varepsilon\ \forall n,m,l \geq k$.

Proposition 2.1. [2] *Let X be a G_b-metric space. Then the following are equivalent:*
(1) The sequence (x_n) is G_b-convergent to x.
(2) $G(x_n,x_n,x) \to 0$ as $n \to \infty$.
(3) $G(x_n,x,x) \to 0$ as $n \to \infty$.

Proposition 2.2. [2] *Let X be a G_b-metric space. The sequence (x_n) is G_b-Cauchy iff for any $\varepsilon > 0$, there exists $k \in \mathbf{N}$ such that $G(x_n,x_m,x_m) < \varepsilon\ \forall n,m \geq k$.*

Definition 2.3. [2] *A G_b-metric space X is called G_b-complete if every G_b-Cauchy sequence is G_b-convergent in X.*

Very recently, Abodayeh *et.al.* [1] defined the concept of generalized Ω-distance mappings (or Ω_b-distance) related to G_b-metric spaces and proved some fixed point theorems (see also [19]).
The notion of a generalized Ω-distance mapping is given by:

Definition 2.4. [1] *Let X be a G_b-metric space. Then a mapping $\Omega : X \times X \times X \to [0,\infty)$ is called a generalized Ω-distance mapping or an Ω_b-distance mapping on X if the following conditions are satisfied:*
(1) $\Omega(x,y,z) \leq s\,[\Omega(x,a,a) + \Omega(a,y,z)]\,\forall x,y,z,a \in X$ and $s \geq 1$,
(2) for any $x,y \in X, \Omega(x,y,.), \Omega(x,.,y) : X \to X$ are lower semi continuous,
(3) for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\Omega(x,a,a) \leq \delta$ and $\Omega(a,y,z) \leq \delta$ imply $G_b(x,y,z) \leq \varepsilon$.

Example 2.2. [1] *Let $X = \mathbf{R}$. Consider the G_b-metric $G : X \times X \times X \to [0,\infty)$ defined by $G(x,y,z) = (|x-y| + |y-z| + |x-z|)^2\ \forall\, x,y,z \in \mathbf{R}$. Define $\Omega : X \times X \times X \to [0,\infty)$ by $\Omega(x,y,z) = (|x-y| + |x-z|)^2\ \forall\, x,y,z \in \mathbf{R}$. Then Ω is a generalized Ω-distance mapping with $s = 2$.*

Definition 2.5. [1] *Let (X,G) be a G_b-metric space and Ω be an Ω_b-distance mapping on X. Then we say that X is Ω-bounded if there exists $M > 0$ such that $\Omega(x,y,z) \leq M$ for all $x,y,z \in X$.*

Lemma 2.1. [1] *Let X be a G_b-metric space and Ω_b be a generalized Ω-distance mapping on X. Let (x_n), (y_n) be sequences in X and let (α_n), (β_n) be sequences in $[0,\infty)$ converging to zero and let $x,y,z,a \in X$. Then we have the following:*
(1) If $\Omega_b(y_n,x_n,x_n) \leq \alpha_n$ and $\Omega_b(x_n,y_m,z) \leq \beta_n$ for any $m > n \in \mathbf{N}$, then $G(y_n,y_m,z) \to 0$ and hence $y_n \to z$.
(2) If $\Omega_b(y,x_n,x_n) \leq \alpha_n$ and $\Omega_b(x_n,y,z) \leq \beta_n$ for $n \in \mathbf{N}$, then $G(y,y,z) < \varepsilon$ and hence $y = z$.
(3) If $\Omega_b(x_n,x_m,x_l) \leq \alpha_n$ for any $m,n,l \in \mathbf{N}$ with $n \leq m \leq l$, then (x_n) is a G_b-Cauchy sequence.
(4) If $\Omega_b(x_n,a,a) \leq \alpha_n$ for any $n \in \mathbf{N}$, then (x_n) is a G_b-Cauchy sequence.

Khojasteh *et.al.* [8] in 2015 introduced the concept of simulation mappings in which they used it to unify several fixed point results in the literature.

Definition 2.6. [8] *Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbf{R}$ be a function. Then ζ is called a simulation function if it satisfies the following conditions:*
$(\zeta 1)$ $\zeta(0,0) = 0$.

($\zeta 2$) *$\zeta(t,s) < s-t$ for all $s,t > 0$.*
($\zeta 3$) *If (t_n and s_n) are sequences in $[0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n,s_n) < 0$.*

The set of all simulation functions are denoted by $\mathscr{Z}$
Now, we give some examples of simulation functions. In the following ζ is defined from $[0,\infty)\times[0,\infty)$ to $\mathbf{R}$.

Example 2.3. [8] *Let $h_1,h_2 : [0,\infty) \to [0,\infty)$ be two continuous functions such that $h_1(t) = h_2(t) = 0$ if and only if $t = 0$ and $h_2(t) < t \le h_1(t)$ for all $t \in [0,\infty)$ and define $\zeta : [0,\infty)\times[0,\infty) \to \mathbf{R}$ by $\zeta(t,s) = h_2(s) - h_1(t)$ for all $t,s \in [0,\infty)$. Then ζ is a simulation function.*

Example 2.4. [8] *Let $g : [0,\infty) \to [0,\infty)$ be a continuous function such that $g(t) = 0$ if and only if t=0 and define $\zeta : [0,\infty)\times[0,\infty) \to \mathbf{R}$ by $\zeta(t,s) = s - g(s) - t$ for all $t,s \in [0,\infty)$. Then ζ is a simulation function.*

Example 2.5. [11] *Let $\eta : [0,\infty) \to [0,\infty)$ be an upper semi continuous function such that $\eta(t) < t\ \forall t > 0$ and $\eta(0) = 0$ and define $\zeta : [0,\infty)\times[0,\infty) \to \mathbf{R}$ by $\zeta(t,s) = \eta(s) - t$ for all $t,s \in [0,\infty)$. Then ζ is a simulation function.*

Example 2.6. [11] *Let $\gamma : [0,\infty) \to [0,\infty)$ be a function such that $\int_0^{\varepsilon} \gamma(u)du$ exists $\forall \varepsilon > 0$ and define $\zeta : [0,\infty)\times[0,\infty) \to \mathbf{R}$ by $\zeta(t,s) = s - \int_0^t \gamma(u)du$ for all $t,s \in [0,\infty)$. Then ζ is a simulation function.*

For more work on simulation functions in fixed point theory, we refer the reader to [9]-[11] and references therein.

3. Main Result

In our main result, we use a contraction condition equipped with c-comparison functions with base s which introduced by Shatanawi [7].

Definition 3.1. [7] *Let s be a constant with $s \ge 1$. A function $\phi : [0,+\infty) \to [0,+\infty)$ is called a c-comparison function with base s if ϕ satisfies the following:*
(i) *ϕ is monotone nondecreasing.*
(ii) *$\sum_{n=0}^{\infty} s^n \phi^n(st)$ converges for all $t \ge 0$.*

Remark 3.1. [7] *If ϕ is a c-comparison function with base s, then $\phi(t) < t$ for all $t > 0$.*

The following example inspired from [7].

Example 3.1. *Let $s \ge 1$. Define $\phi_1,\phi_2 : [0,\infty) \to [0,\infty)$ by $\phi_1(t) = kt$ where $0 \le k < \frac{1}{s}$ and $\phi_2(t) = \dfrac{1}{a+s}$ where $a > 0$. Then ϕ_1 and ϕ_2 are c-comparison functions with base s.*

Now, we introduce the following definition

Definition 3.2. *Let (X,G) be a G_b-metric space equipped with a generalized Ω-distance mapping Ω with base $s \ge 1$ and $\zeta \in \mathscr{Z}$. A self mapping $T : X \to X$ is said to be $(\Omega,\phi,\mathscr{Z})_s$-contraction with respect to ζ if there is a c-comparison function ϕ with base s such that T satisfies the following condition:*

$$\zeta(s\Omega(Tx,T^2x,Ty),\phi s\Omega(x,Tx,y)) \ge 0\ \forall x,y \in X. \tag{1}$$

Lemma 3.1. *Let (X,G) be a G_b-metric space equipped with a generalized Ω-distance mapping Ω with base $s \ge 1$. Let $\zeta \in \mathscr{Z}$ and ϕ be a c-comparison function with base s. Suppose that $T : X \to X$ is $(\Omega,\phi,\mathscr{Z})_s$-contraction with respect to ζ. If T has a fixed point (say) $u \in X$, then it is unique.*

Proof. First we show that for all $w \in X$ if $fw = w$, then $\Omega(w,w,w) = 0$. Assume that $\Omega(w,w,w) > 0$. From (1) and ($\zeta 2$), we have

$$\begin{aligned}
0 &\leq \zeta(s\Omega(Tw,T^2w,Tw),\phi s\Omega(w,Tw,w))\\
&= \zeta(s\Omega(w,w,w),\phi s\Omega(w,w,w))\\
&< \phi s\Omega(w,w,w) - s\Omega(w,w,w),\\
&< s\Omega(w,w,w) - s\Omega(w,w,w),\\
&= 0,
\end{aligned}$$

a contradiction. Hence $\Omega(w,w,w) = 0$.
Now, assume that there is $v \in X$ such that $Tv = v$ and $\Omega(u,v,v) > 0$. Since T is $(\Omega,\phi,\mathscr{Z})$-contraction with respect to ζ, then by substituting $x = u$ and $y = v$ in (1) and taking into account ($\zeta 2$), we have

$$\begin{aligned}
0 &\leq \zeta(s\Omega(Tu,T^2u,Tv),\phi s\Omega(u,Tu,v))\\
&= \zeta(s\Omega(u,u,v),\phi s\Omega(u,u,v))\\
&< \phi s\Omega(u,u,v) - s\Omega(u,u,v)\\
&< s\Omega(u,u,v) - s\Omega(u,u,v) = 0,
\end{aligned}$$

a contradiction. Hence $\Omega(u,v,v) = 0$. Thus by the definition of Ω we have $G(u,v,v) = 0$ and so $u = v$. □

Theorem 3.1. *(X,G) be a G_b-metric space equipped with a generalized Ω-distance mapping Ω with base $s \geq 1$ such that X is Ω-bounded and $\zeta \in \mathscr{Z}$. Suppose that there is a c-comparison function ϕ with base s such that the mapping $T : X \to X$ is $(\Omega,\phi,\mathscr{Z})_s$-contraction with respect to ζ that satisfies the following condition*

$$\forall\, u \in X \text{ if } Tu \neq u, \text{ then } \inf\{\Omega(x,Tx,u) : x \in X\} > 0. \tag{2}$$

Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary and define the sequence (x_n) in X inductively by $x_n = Tx_{n-1}$ $n \in \mathbf{N}$. Let $p \geq 0$ be a nonnegative integer. Then by (1), we have for all $n \in \mathbf{N}$

$$\begin{aligned}
0 &\leq \zeta(s\Omega(Tx_{n-1},T^2x_{n-1},Tx_{n+p-1}),\phi s\Omega(x_{n-1},Tx_{n-1},x_{n+p-1}))\\
&= \zeta(s\Omega(x_n,x_{n+1},x_{n+p}),\phi s\Omega(x_{n-1},x_n,x_{n+p-1}))\\
&< \phi s\Omega(x_{n-1},x_n,x_{n+p-1}) - s\Omega(x_n,x_{n+1},x_{n+p}).
\end{aligned}$$

Thus,

$$s\Omega(x_n,x_{n+1},x_{n+p}) < \phi s\Omega(x_{n-1},x_n,x_{n+p-1}). \tag{3}$$

Also, by (1) we have

$$\begin{aligned}
0 &\leq \zeta(s\Omega(Tx_{n-2},T^2x_{n-2},Tx_{n+p-2}),\phi s\Omega(x_{n-2},Tx_{n-2},x_{n+p-2}))\\
&= \zeta(s\Omega(x_{n-1},x_n,x_{n+p-1}),\phi s\Omega(x_{n-2},x_{n-1},x_{n+p-2})).\\
&< \phi s\Omega(x_{n-2},x_{n-1},x_{n+p-2}) - s\Omega(x_{n-1},x_n,x_{n+p-1}).
\end{aligned}$$

Therefore,

$$s\Omega(x_{n-1},x_n,x_{n+p-1}) < \phi s\Omega(x_{n-2},x_{n-1},x_{n+p-2}). \tag{4}$$

Since ϕ in nondecreasing, then $\phi s\Omega(x_{n-1},x_n,x_{n+p-1}) < \phi^2 s\Omega(x_{n-2},x_{n-1},x_{n+p-2})$. Hence, (3) becomes

$$s\Omega(x_n,x_{n+1},x_{n+p}) < \phi^2 s\Omega(x_{n-2},x_{n-1},x_{n+p-2}). \tag{5}$$

If we apply the previous steps repeatedly, we get $s\Omega(x_n,x_{n+1},x_{n+p}) \leq \phi^n s\Omega(x_0,x_1,x_p)$. Since X is Ω-bounded, there is $M \geq 0$, such that $\Omega(x,y,z) \leq M$, $\forall x,y,z,\in X$. Thus

$$s\Omega(x_n,x_{n+1},x_{n+p}) \leq \phi^n(sM). \tag{6}$$

Now, by using the definition of Ω and (6), we have for all $l \geq m \geq n$

$$\begin{aligned}\Omega(x_n,x_m,x_l) &\leq s\Omega(x_n,x_{n+1},x_{n+1})+s^2\Omega(x_{n+1},x_{n+2},x_{n+2})+\cdots\\ &\quad +s^{m-n-1}\Omega(x_{m-2},x_{m-1},x_{m-1})+s^{m-n-1}\Omega(x_{m-1},x_m,x_l)\\ &\leq \phi^n(sM)+s\phi^{n+1}(sM)+\cdots+s^{m-n-2}\phi^{m-1}(sM)+s^{m-n-2}\phi^{m-1}(sM)\\ &\leq \phi^n(sM)+s\phi^{n+1}(sM)+\cdots\\ &= s^{-n}[s^n\phi^n(sM)+s^{n+1}\phi^{n+1}(sM)+\cdots]\\ &= s^{-n}\sum_{k=n}^{\infty}s^k\phi^k(sM).\end{aligned}$$

Since ϕ is a c-comparison function with base s, then $\left(\sum_{k=n}^{\infty}s^k\phi^k(sM) : n\in\mathbf{N}\right)$ converges to 0. Thus for any $\varepsilon>0$, there is $N\in\mathbf{N}$ such that $\sum_{k=n}^{\infty}s^k\phi^k(M)\leq s^n\varepsilon\ \forall\, n\geq N$.
Hence for $l\geq m\geq n\geq N$, we have

$$\Omega(x_n,x_m,x_l)\leq s^{-n}\sum_{k=n}^{\infty}s^k\phi^k(M)\leq\varepsilon\ \ \forall n\geq N.$$

By Lemma 2.1, (x_n) is a G_b-Cauchy sequence. Therefore there is $u\in X$ such that $\lim_{n\to\infty}x_n=u$.
Consider $\delta>0$. Then there exists $r_0\in\mathbf{N}$ such that $\Omega(x_n,x_m,x_l)\leq\delta\ \forall n,m,l\geq r_0$.
Therefore, $\lim_{l\to\infty}\Omega(x_n,x_m,x_l)\leq\lim_{l\to\infty}\delta=\delta$. $\forall n,m\geq r_0$.
By the lower semi continuity of Ω, we have $\Omega(x_n,x_m,u)\leq\liminf_{p\to\infty}\Omega(x_n,x_m,x_p)\leq\delta\ \forall m,n\geq r_0$.
Consider $m=n+1$. Then $\Omega(x_n,x_{n+1},u)\leq\liminf_{p\to\infty}\Omega(x_n,x_{n+1},x_p)\leq\delta\ \forall n\geq r_0$.
If $Tu\neq u$, then (2) implies that

$$\begin{aligned}0 &< \inf\{\Omega(x,Tx,u):x\in X\}\\ &\leq \inf\{\Omega(x_n,x_{n+1},u):n\geq r_0\}\\ &\leq \delta,\end{aligned}$$

for each $\delta>0$ which is a contradiction. Therefore $Tu=u$. The uniqueness follows from Lemma 3.1. □

Example 3.2. *Let $X=[0,1]$ and let $G:X\times X\times X\to[0,\infty)$, $\Omega:X\times X\times X\to[0,\infty)$, $T:X\to X$ and $\zeta:[0,\infty)\times[0,\infty)\to\mathbf{R}$ be defined as follow:*
$G(x,y,z)=(|x-y|+|y-z|+|x-z|)^2$, $\Omega(x,y,z)=(|x-y|+|x-z|)^2$, $Tx=ax$, $\zeta(u,v)=bv-u$ and $\phi(t)=ct$ where $a,b\in[0,1)$, $c\in[0,\frac{1}{2})$ and $a^2\leq bc$. Then
(1) (X,G) is a complete G_b-metric space and Ω is a generalized Ω-distance on X with base $s=2$,
(2) $\zeta\in\mathscr{Z}$, ϕ is a c-comparison function with base $s=2$,
(3) T is $(\Omega,\phi,\mathscr{Z})_s$-contraction with respect to ζ,
(4) for every $u\in X$ if $Tu\neq u$, then $\inf\{\Omega(x,Tx,u):x\in X\}>0$.

Proof. We shall prove (3) and (4).
To prove that T is $(\Omega,\phi,\mathscr{Z})_s$-contraction with respect to ζ, let $x,y\in X$. Then

$$\begin{aligned}&\zeta(s\Omega(Tx,T^2x,Ty),\phi s\Omega(x,Tx,y))\\ &=\zeta(2\Omega(Tx,T^2x,Ty),2c\Omega(x,Tx,y))\\ &=2bc(|x-ax|+|x-y|)^2-2(|ax-a^2x|+|ax-ay|)^2\\ &=2bc((1-a)|x|+|x-y|)^2-2a^2((1-a)|x|+|x-y|)^2\\ &=2(bc-a^2)(|x|+|x-y|)\\ &\geq 0.\end{aligned}$$

To prove (4), given $u\in X$ such that $Tu\neq u$. Then $u\neq 0$. Therefore

$$\begin{aligned}\inf\{\Omega(x,Tx,u):x\in X\} &= \inf\{\Omega(x,ax,u):x\in X\}\\ &= \inf\{|x-ax|+|x-u|:x\in X\}\\ &= \inf\{(1-a)|x|+|x-u|:x\in X\}\\ &= (1-a)u>0.\end{aligned}$$

Thus all hypotheses of Theorem 3.1 hold true. Hence T has a unique fixed point in X. Here the unique fixed point of T is 0. □

Now, we utilized our main result to derive the following results. To facilitate our work, we let $\mathscr{H}=\{h:[0,\infty)\to[0,\infty): h\ is\ a\ continuous\ function$ with $h^{-1}(\{0\})=\{0\}\}$.

Corollary 3.1. *Let (X,G) be a complete G_b-metric space and Ω be a generalized Ω-distance mapping on X with base $s\geq 1$. Let $T:X\to X$ be a self mapping and ϕ be a c-comparison function with base s. Assume that there are $h_1,h_2\in\mathscr{H}$ where $h_2(t)<t\leq h_1(t)$ $\forall t>0$ such that T satisfies the following condition:*

$$h_1 s\Omega(Tx,T^2x,Ty)\leq h_2\phi s\Omega(x,Tx,y)\ \forall x,y,z\in X. \tag{7}$$

Also, suppose that for all $u\in X$ if $Tu\neq u$, then $\inf\{\Omega(x,Tx,u):x\in X\}>0$.

Then T has a unique fixed point in X.

Proof. Define $\zeta:[0,\infty)\times[0,\infty)\to\mathbf{R}$ by $\zeta(u,v)=h_2(v)-h_1(u)$. Clearly $\zeta\in\mathscr{Z}$ and T is $(\Omega,\phi,\mathscr{Z})_s$-contraction with respect to ζ. Hence the result follows from Theorem 3.1. □

By choosing $h_1(t)=t$ and $h_2(t)=\lambda t$ where $0\leq\lambda<1$ in Corollary 3.1 we have the following:

Corollary 3.2. *Let (X,G) be a complete G_b-metric space and Ω be a generalized Ω-distance mapping on X with base $s\geq 1$. Let $T:X\to X$ be a self mapping and ϕ be a c-comparison function with base s. Assume that there is $\lambda\in[0,1)$ such that T satisfies the following condition:*

$$\Omega(Tx,T^2x,Ty)\leq\frac{\lambda}{s}\phi s\Omega(x,Tx,y)\ \forall x,y\in X. \tag{8}$$

Also, suppose that for all $u\in X$ if $Tu\neq u$, then $\inf\{\Omega(x,Tx,u):x\in X\}>0$.

Then T has a unique fixed point in X.

Corollary 3.3. *Let (X,G) be a complete G_b-metric space and Ω be a generalized Ω-distance mapping on X with base $s\geq 1$. Let $T:X\to X$ be a self mapping and ϕ be a c-comparison function with base s. Assume that there is $g\in\mathscr{H}$ such that T satisfies the following condition:*

$$s\Omega(Tx,T^2x,Ty)\leq\phi s\Omega(x,Tx,y)-g\phi s\Omega(x,Tx,y)\ \forall x,y\in X. \tag{9}$$

Also, suppose that for all $u\in X$ if $Tu\neq u$, then $\inf\{\Omega(x,Tx,u):x\in X\}>0$.

Then T has a unique fixed point in X.

Proof. Define $\zeta:[0,\infty)\times[0,\infty)\to\mathbf{R}$ by $\zeta(u,v)=v-g(v)-u$. Clearly $\zeta\in\mathscr{Z}$ and T is $(\Omega,\phi,\mathscr{Z})_s$-contraction with respect to ζ. Hence the result follows from Theorem 3.1. □

Corollary 3.4. *Let (X,G) be a complete G_b-metric space and Ω be a generalized Ω-distance mapping on X with base $s\geq 1$. Let $T:X\to X$ be a self mapping and ϕ be a c-comparison function with base s. Assume that there is an upper semi continuous function η such that T satisfies the following condition:*

$$s\Omega(Tx,T^2x,Ty)\leq\eta\phi s\Omega(x,Tx,y)\ \forall x,y\in X. \tag{10}$$

Also, suppose that for all $u\in X$ if $Tu\neq u$, then $\inf\{\Omega(x,Tx,u):x\in X\}>0$.

Then T has a unique fixed point in X.

Proof. Define $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\zeta(u,v) = \eta(v) - u$. Clearly $\zeta \in \mathscr{Z}$ and T is $(\Omega,\phi,\mathscr{Z})_s$-contraction with respect to ζ. Hence the result follows from Theorem 3.1. □

Corollary 3.5. *Let (X,G) be a complete G_b-metric space and Ω be a generalized Ω-distance mapping on X with base $s \geq 1$. Let $T : X \to X$ be a self mapping and ϕ be a c-comparison function with base s. Assume that there is a function $\gamma : [0,\infty) \to [0,\infty)$ where $\int_0^\varepsilon \gamma(t)dt$ exists and $\int_0^\varepsilon \gamma(t)dt > \varepsilon$ $\forall \varepsilon > 0$ such that T satisfies the following condition:*

$$\int_0^{s\Omega(Tx,T^2x,Ty)} \gamma(u)du \leq \phi s\Omega(x,Tx,y)\ \forall x,y \in X. \tag{11}$$

Also, suppose that for all $u \in X$ if $Tu \neq u$, then $\inf\{\Omega(x,Tx,u) : x \in X\} > 0$.
Then T has a unique fixed point in X.

Proof. Define $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\zeta(u,v) = v - \int_0^u \gamma(t)dt$. Clearly $\zeta \in \mathscr{Z}$ (see Example 2.6) and T is $(\Omega,\phi,\mathscr{Z})_s$-contraction with respect to ζ. Hence the result follows from Theorem 3.1. □

4. **Conclusion**

In this paper, we introduced and studied some fixed point theorems in the setting of generalized Ω-distance mappings [1] using contraction conditions depend on simulation functions [8] in which our work gives a more general cases in the study of fixed point theory. Also, an example is introduced to support our main result.

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