NEW CONTRACTIONS AND SOME FIXED POINT RESULTS WITH APPLICATION BASED ON EXTENDED QUASI b-METRIC SPACES

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The conceptions of metric spaces and extended b-metric spaces play a major role in proving many theorems of existence solution of such equations as integral or differential equations. The notions of quasi metric spaces and quasi b-metric space are a modification of metric and b-metric spaces. In our article, we introduce the concepts of extended (α, ξ)-contraction and generalized extended (α, ξ)-contraction for a self-mapping l_1 in an extended quasi b-metric space. Also, we prove that every extended (α, ξ)-contraction has a unique fixed point under a set of conditions. As well as, we prove that every generalized extended (α, ξ)-contraction has a unique solution under some specified conditions. Moreover, an application and an example were added to highlight the importance of our work. Our work modify many exciting results in the literature.

Keywords: fixed point, b-metric, extended b-metric.

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1. Introduction and Preliminary

The fixed point theory plays a pivotal role to solve equations in mathematics and other sciences since there is a major similarity between solving equations and finding fixed point for functions under suitable conditions for example the solution of the equation

$$x = G(x, h(x)),$$

where $G : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, and $h : \mathbf{R} \to \mathbf{R}$ is the fixed point for the function $f : \mathbf{R} \to \mathbf{R}$ which defined by $f(x) = G(x, h(x))$, where $\mathbf{R}$ is the set of real numbers.

The concept of metric spaces was appeared at the beginning of twentieth century by Maurice Fréchet. Later on, many generalizations of metric spaces and corresponding fixed point results were proposed by many mathematicians see [1]-[23], and [28]-[41] and references therein.

In the present work, we mean by M a non-empty set, l_1 a self mapping on M, $\mathbf{R}$ the set of reals and $\mathbf{N}$ the set of naturals.

Let $\xi : M \times M \to [1, \infty)$ and $d_\xi : M \times M \to [0, \infty)$ be given functions, and consider the following axioms for all $m_1, m_2, m_3 \in M$:

(ξ_1) $d_\xi(m_1, m_2) = 0$ iff $m_1 = m_2$,

(ξ_2) $d_\xi(m_1, m_2) = d_\xi(m_2, m_1)$,

(ξ_3) $d_\xi(m_1, m_3) \leq \xi(m_1, m_3) [d_\xi(m_1, m_2) + d_\xi(m_2, m_3)]$.

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If d_ξ satisfying ξ_1, ξ_2 and ξ_3, then (d_ξ, M) is called an extended b-metric space [25], and if d_ξ satisfying just ξ_1 and ξ_3, then (d_ξ, M) is called an extended quasi b-metric space [29].

Example 1.1. *Define* $\xi : \{1,2,3\} \times \{1,2,3\} \to [1,\infty)$ *by* $\xi(m_1, m_2) = \max\{m_1, m_2\}$ *and* $q_\xi : \{1,2,3\} \times \{1,2,3\} \to [0,\infty)$ *by*
$q_\xi(1,1) = q_\xi(2,2) = q_\xi(3,3) = 0$ *and* $q_\xi(1,2) = q_\xi(1,3) = q_\xi(2,3) = 6$ *and* $q_\xi(2,1) = q_\xi(3,1) = q_\xi(3,2) = 5$. *Then* $(\{1,2,3\}, q_\xi)$ *is an extended quasi b-metric space which is not an extended b-metric space.*

Note:

(1) If $\xi(m) = s \geq 1$ in an extended b-metric space (d_ξ, M), then (d_ξ, M) is a b-metric space,

(2) If $\xi(m) = s \geq 1$ in an extended quasi b-metric space (d_ξ, M), then (d_ξ, M) is a quasi b-metric space.

From now on, (M, q_ξ) refers to an extended quasi b-metric space and $(M, \check{q}_\xi)$ refers to a quasi b-metric space.

The concept of extended b-metric spaces introduced in 2017 by Kamran et al. [25] in which they studied some fixed point results. For more results and theorems, see [35, 26, 27, 38].

Note that one can induce an extended b-metric from an extended quasi b-metric as following: Define $d_\xi : M \times M \to [0,\infty)$ by

$$d_\xi(m_1, m_2) = \max\{q_\xi(m_2, m_1), q_\xi(m_1, m_2)\},$$

then d_ξ is an extended b-metric on M.

We adopt [11] and [24] to generate the following definitions:

Definition 1.1. *Let* (m_t) *be a sequence in* (M, q_ξ). *Then, the sequence* m_t *converges to* $m \in M$ *if* $\lim_{t\to\infty} q_\xi(m_t, m) = \lim_{t\to\infty} q_\xi(m, m_t) = 0$.

Definition 1.2. *Let* (m_t) *be a sequence in* (M, q_ξ). *Then, we say that*

(1) (m_t) *is left-Cauchy in* (M, q_ξ) *iff for all* $\epsilon > 0$, $\exists\ N_\epsilon \in \mathbf{N}$ *such that* $q_\xi(m_t, m_s) \leq \epsilon$, *for all* $t \geq s > N_\epsilon$,

(2) (m_t) *is right-Cauchy in* (M, q_ξ) *iff for all* $\epsilon > 0$, $\exists\ N_\epsilon \in \mathbf{N}$ *such that* $q_\xi(m_t, m_s) \leq \epsilon$, *for all* $s \geq t > N_\epsilon$,

(3) (m_t) *is a Cauchy sequence in* (M, q_ξ) *iff* (m_t) *is right and left Cauchy.*

Definition 1.3. *We say that:*

(i) (M, q_ξ) *is left-complete iff Every left-Cauchy sequence in* M *is convergent,*

(ii) (M, q_ξ) *is right-complete iff Every right-Cauchy sequence in* M *is convergent,*

(iii) (M, q_ξ) *is complete iff Every Cauchy sequence in* M *is convergent.*

2. Main Results

On (M, q_ξ), we introduce the notions of an extended (α, ξ)-contraction and a generalized extended (α, ξ)-contraction.

Definition 2.1. *On* (M, q_ξ), *we say that* l_1 *is an extended* (α, ξ)*-contraction if there is* $\alpha \in (0,1)$ *such that* $\forall\ m_1, m_2 \in M$ *and* $j \in \mathbf{N}$, *we have*

$$(1-\alpha) q_\xi(m_1, l_1^j m_1) \leq q_\xi(m_1, m_2) \text{ implies } q_\xi(l_1 m_1, l_1 m_2) \leq \alpha^2 \xi(m_1, m_2) q_\xi(m_1, m_2),$$

and

$$(1-\alpha) q_\xi(l_1^j m_1, m_1) \leq q_\xi(m_2, m_1) \text{ implies } q_\xi(l_1 m_2, l_1 m_1) \leq \alpha^2 \xi(m_2, m_1) q_\xi(m_2, m_1).$$

Definition 2.2. *On (M, q_ξ), we say that l_1 is a generalized extended (α, ξ)-contraction if $\exists\ \alpha \in (0,1)$ such that $\forall\ m_1, m_2 \in M$ and $j \in \mathbf{N}$, we have*

$$(1-\alpha)q_\xi(m_1, l_1^j m_1) \leq q_\xi(m_1, m_2) \text{ implies } q_\xi(l_1 m_1, l_1 m_2) \leq \alpha^2 \xi(m_1, m_2) A(m_1, m_2),$$

and

$$(1-\alpha)q_\xi(l_1^j m_1, m_1) \leq q_\xi(m_2, m_1) \text{ implies } q_\xi(l_1 m_2, l_1 m_1) \leq \alpha^2 \xi(m_2, m_1) A(m_1, m_2),$$

where, $A(m_1, m_2) = \max\{q_\xi(m_1, l_1 m_1), q_\xi(m_2, l_1 m_2)\}$.

Definition 2.3. *On (M, q_ξ) we say that ξ is bounded by $c \in [1, \infty)$ if $\xi(m_1, m_2) \leq c$ for all $m_1, m_2 \in M$.*

Now, we introduce our first main result.

Theorem 2.1. *Suppose (M, q_ξ) is complete and l_1 is continuous on M. Assume that there is $\alpha \in (0,1)$ such that l_1 is an extended (α, ξ)-contraction, where ξ is bounded by $\frac{1}{\alpha}$. Then l_1 has a unique fixed point in M.*

Proof. Begin with $m_0 \in M$ to construct the sequence (m_t) in M inductively by putting $m_{t+1} = l_1 m_t$ for $t \in \mathbf{N} \cup \{0\}$. To show that, (m_t) is a right Cauchy sequence, given $t, s \in \mathbf{N}$ with $s > t$. Let $s = t + j$ for some $j \in \mathbf{N}$. Then, we have

$$\begin{aligned}(1-\alpha)q_\xi(m_{t-1}, m_{s-1}) &= (1-\alpha)q_\xi(m_{t-1}, l_1^j m_{t-1}) \\ &\leq q_\xi(m_{t-1}, m_{s-1}).\end{aligned}$$

So,

$$\begin{aligned} q_\xi(m_t, m_s) &= q_\xi(l_1 m_{t-1}, l_1 m_{s-1}) \\ &\leq \alpha^2 \xi(m_{t-1}, m_{s-1}) q_\xi(m_{t-1}, m_{s-1}) \\ &\leq \alpha^4 [\xi(m_{t-1}, m_{s-1}) \xi(m_{t-2}, m_{s-2})] q_\xi(m_{t-2}, m_{s-2}) \\ &\vdots \\ &\leq \alpha^{2t} \left(\prod_{r=1}^{t} \xi(m_{r-1}, m_r) \right) q_\xi(m_0, m_j) \\ &< \alpha^t q_\xi(m_0, m_j). \end{aligned} \tag{2.1}$$

Thus, $\lim\limits_{t,s\to\infty} q_\xi(m_t, m_s) = 0$, and so, (m_t) is a right Cauchy sequence. To show that, (m_t) is a left Cauchy sequence, given $t, s \in \mathbf{N}$ with $s < t$. Let $t = s + i$ for some $i \in \mathbf{N}$. Then we have

$$\begin{aligned}(1-\alpha)q_\xi(m_{t-1}, m_{s-1}) &= (1-\alpha)q_\xi(l_1^i m_{s-1}, m_{s-1}) \\ &\leq q_\xi(m_{t-1}, m_{s-1}).\end{aligned}$$

So,

$$\begin{aligned} q_\xi(m_t, m_s) &= q_\xi(l_1 m_{t-1}, l_1 m_{s-1}) \\ &\leq \alpha^2 \xi(m_{t-1}, m_{s-1}) q_\xi(m_{t-1}, l_1 m_{t-1}) \\ &\vdots \\ &\leq \alpha^{2s} \left(\prod_{j=1}^{s} \xi(m_{j-1}, m_j) \right) q_\xi(m_i, m_0) \\ &< \alpha^s q_\xi(m_i, m_0). \end{aligned}$$

Thus, we get that $\lim\limits_{s,t\to\infty} q_\xi(m_t, m_s) = 0$, and so, (m_t) is a right Cauchy sequence. Consequently, (m_t) is a Cauchy sequence. The completeness of q_ξ informs us that $\exists\ \beta^* \in M$ such that $m_t \to \beta^*$.

Since l_1 is a continuous function, then $m_{t+1} = l_1 m_t \to l_1 \beta^*$. The uniqueness of the limit

ensures that $l_1\beta^* = \beta^*$.
To prove the uniqueness of β^*, assume $\exists\ m_* \in M$ with $l_1 m_* = m_*$. Then we have

$$\begin{aligned}(1-\alpha)q_\xi(\beta^*,\beta^*) &= (1-\alpha)q_\xi(\beta^*, l_1^i\beta^*)\\ &\leq q_\xi(\beta^*, m_*).\end{aligned}$$

Then,

$$q_\xi(\beta^*, m_*) = q_\xi(l_1\beta^*, l_1 m_*) \leq \alpha^2\xi(\beta^*, m_*)q_\xi(\beta^*, m_*) \leq \alpha q_\xi(\beta^*, m_*). \tag{2.2}$$

Thus, $q_\xi(\beta^*, m_*) = 0$, and so, $\beta^* = m_*$.

□

Theorem 2.2. *Suppose (M, q_ξ) is complete. Assume that there exists $\alpha \in (0,1)$ such that ξ is bounded by $\frac{1}{\alpha}$. Furthermore, assume that for each $m_1, m_2 \in M$, l_1 satisfies the following condition.*

$$q_\xi(l_1 m_1, l_1 m_2) \leq \alpha^2\xi(m_1, m_2)q_\xi(m_1, m_2). \tag{2.3}$$

Then l_1 has a unique fixed point in M.

Proof. Starting with $m_0 \in M$, we construct the sequence (m_t) in M inductively by putting $m_{t+1} = l_1 m_t$, for $t \in \mathbf{N} \cup \{0\}$. To show that (m_t) is a right Cauchy sequence, given $t, s \in \mathbf{N}$ with $s > t$. Let $s = t + j$ for some $j \in \mathbf{N}$. Then

$$\begin{aligned}q_\xi(m_t, m_s) &= q_\xi(l_1 m_{t-1}, l_1 m_{s-1})\\ &\leq \alpha^{2t}\left(\prod_{r=1}^{t}\xi(m_{r-1}, m_r)\right)q_\xi(m_0, m_j)\\ &< \alpha^t q_\xi(m_0, m_j).\end{aligned} \tag{2.4}$$

By letting $t, s \to \infty$, we get

$$\lim_{t,s\to\infty} q_\xi(m_t, m_s) = 0. \tag{2.5}$$

Thus (m_t) is a right Cauchy sequence. On a similar manner, we can show that (m_t) is a left Cauchy sequence. Consequently, (m_t) is a Cauchy sequence. The completeness of q_ξ informs us that there is $\varsigma^* \in M$ such that $m_t \to \varsigma^*$. Now, we show that $l_1\varsigma^* = \varsigma^*$. Using (EQ2), we have

$$q_\xi(l_1\varsigma^*, \varsigma^*) \leq \xi(l_1\varsigma^*, \varsigma^*)[q_\xi(l_1\varsigma^*, m_{t+1}) + q_\xi(m_{t+1}, \varsigma^*). \tag{2.6}$$

Also, we have

$$\begin{aligned}q_\xi(l_1\varsigma^*, m_{t+1}) &= q_\xi(l_1\varsigma^*, l_1 m_t)\\ &\leq \alpha^2\xi(\varsigma^*, m_t)[q_\xi(\varsigma^*, m_t)].\end{aligned} \tag{2.7}$$

Using the equations (**2**.6) and (**2**.7) and letting $t \to \infty$, we get $l_1\varsigma^* = \varsigma^*$.
To prove the uniqueness of ς^*, assume $\exists\ m_* \in M$ with $l_1 m_* = m_*$. Then,

$$q_\xi(\varsigma^*, m_*) = q_\xi(l_1\varsigma^*, l_1 m_*) \leq \alpha^2\xi(\varsigma^*, m_*)q_\xi(\varsigma^*, m_*) \leq \alpha q_\xi(\varsigma^*, m_*). \tag{2.8}$$

Thus, $q_\xi(\varsigma^*, m_*) = 0$, and so, $\varsigma^* = m_*$. □

We illustrate Theorem **2**.2, by the following example.

Example 2.1. *Let $M = [0,1]$. Let $q_\xi : M \times M \to [0,\infty)$ and $\xi : M \times M \to [1,\infty)$ be defined by $q_\xi(m_1, m_2) = |m_1 - m_2|$ and $\xi(m_1, m_2) = 1 + |m_1 - m_2|$. Let $l_1 : M \to M$ be defined by*

$$l_1(m) = \begin{cases}\frac{1-\frac{m}{4}}{2-m^2}, & m \in [0, \frac{1}{2}],\\ \frac{1}{2}, & m \in (\frac{1}{2}, 1].\end{cases}$$

Then, we have the following:

(1) *(q_ξ, M) is a complete extended quasi b-metric space,*
(2) *l_1 satisfies condition **2**.3, with $\alpha = \frac{1}{\sqrt{8}}$.*

Proof. First, observe that ξ is bounded by $\frac{1}{\alpha}=\sqrt{8}$.
Now, we just show (2):
We consider the following cases:
Case (1): If $m_1, m_2 \in [0, \frac{1}{2}]$, then

$$\begin{aligned} q_\xi(l_1m_1, l_1m_2) &= \left|\frac{1-\frac{m_1}{4}}{2-m_1^2} - \frac{1-\frac{m_2}{4}}{2-m_2^2}\right| \\ &= \tfrac{1}{(2-m_1^2)(2-m_2^2)}\left|(x+y-\tfrac{1}{4}xy-\tfrac{1}{2})(m_1-m_2)\right| \\ &\leq \tfrac{1}{(2-m_1^2)(2-m_2^2)}\tfrac{1}{2}\left|m_1-m_2\right| \\ &\leq \tfrac{1}{8}\left|m_1-m_2\right| \\ &\leq \alpha^2\xi(m_1,m_2)q_\xi(m_1,m_2). \end{aligned}$$

Case (2): If $m_1 \in [0, \frac{1}{2}]$ and $m_2 \in (\frac{1}{2}, 1]$, then

$$\begin{aligned} q_\xi(l_1m_1, l_1m_2) &= \left|\frac{1-\frac{m_1}{4}}{2-m_1^2} - \frac{1}{2}\right| \\ &= \frac{1}{2(2-m_1^2)}\left|m_1(m_1-\frac{1}{2})\right| \\ &\leq \frac{1}{8}\left|m_1-\frac{1}{2}\right| \\ &\leq \frac{1}{8}\left|m_1-m_2\right| \\ &\leq \alpha^2\xi(m_1,m_2)q_\xi(m_1,m_2). \end{aligned}$$

The other cases are similar to the previous cases. Hence by Theorem **2**.2, l_1 has a unique fixed point in M. □

Corollary 2.1. *Suppose that $(M, \check{q}_\xi)$ is complete. Assume that there is $\alpha \in (0,1)$ and $s \leq \frac{1}{\alpha}$ such that for all $m_1, m_2 \in M$, l_1 satisfies*

$$\check{q}_\xi(l_1m_1, l_1m_2) \leq \alpha^2 s\check{q}_\xi(m_1,m_2).$$

Then l_1 has a unique fixed point in M.

Next, we introduce the second main results:

Theorem 2.3. *Suppose that (M, q_ξ) is complete and l_1 is continuous on M. Assume that there is $\alpha \in (0,1)$ such that l_1 is a generalized extended (α,ξ)-contraction, where ξ is bounded by $\frac{1}{\alpha}$. Then l_1 has a unique fixed point in M.*

Proof. Let $m_0 \in M$ and construct the sequence (m_t) by putting $m_t = l_1(l_1m_{t-1}) = l_1(l_1^{t-1}m_0)$ $\forall\ t \in \mathbf{N}$. To show that (m_t) is a right Cauchy sequence, let $t, s \in \mathbf{N}$ with $t < s$. Then $s = t+j$ for some $j \in \mathbf{N}$. So

$$\begin{aligned} (1-\alpha)q_\xi(m_{t-1}, m_{s-1}) &= (1-\alpha)q_\xi(m_{t-1}, l_1^j m_{t-1}) \\ &\leq q_\xi(m_{t-1}, m_{s-1}). \end{aligned}$$

Thus,

$$\begin{aligned} q_\xi(m_t, m_s) &= q_\xi(l_1m_{t-1}, l_1m_{s-1}) \\ &\leq \alpha^2\xi(m_{t-1},m_{s-1})\max\{q_\xi(m_{t-1}, l_1m_{t-1}), q_\xi(m_{s-1}, l_1m_{s-1})\} \\ &= \alpha^2\xi(m_{t-1},m_{s-1})\max\{q_\xi(m_{t-1}, m_t), q_\xi(m_{s-1}, m_s)\}. \end{aligned} \quad (\mathbf{2}.9)$$

Now,

$$\begin{aligned} (1-\alpha)q_\xi(m_{t-2}, m_{t-1}) &= (1-\alpha)q_\xi(m_{t-2}, l_1m_{t-2}) \\ &\leq q_\xi(m_{t-2}, m_{t-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} q_\xi(m_{t-1},m_t) &= q_\xi(l_1m_{t-2},l_1m_{t-1}) \\ &\leq \alpha^2\xi(m_{t-2},m_{t-1})\max\{q_\xi(m_{t-2},l_1m_{t-2}),q_\xi(m_{t-1},l_1m_{t-1})\} \\ &= \alpha^2\xi(m_{t-2},m_{t-1})\max\{q_\xi(m_{t-2},m_{t-1}),q_\xi(m_{t-1},m_t)\}. \end{aligned}$$

If $\max\{q_\xi(m_{t-2},m_{t-1}),q_\xi(m_{t-1},m_t)\} = q_\xi(m_{t-1},m_t)$, then we get that $q_\xi(m_{t-1},m_t) < q_\xi(m_{t-1},m_t)$, which is a contradiction. So,

$$q_\xi(m_{t-1},m_t) \leq \alpha^2\xi(m_{t-2},m_{t-1})q_\xi(m_{t-2},m_{t-1}). \quad (\mathbf{2}.10)$$

Therefore, inductively we get that

$$q_\xi(m_{t-1},m_t) \leq \alpha^{2t-2}\left(\prod_{r=1}^{t-1}\xi(m_{r-1},m_r)\right)q_\xi(m_0,m_1).$$

So,

$$q_\xi(m_{t-1},m_t) < \alpha^{t-1}q_\xi(m_0,m_1). \quad (\mathbf{2}.11)$$

By a similar argument, we get

$$q_\xi(m_{s-1},m_s) < \alpha^{s-1}q_\xi(m_0,m_1). \quad (\mathbf{2}.12)$$

From Equations (**2**.11),(**2**.12), we obtain

$$\begin{aligned} q_\xi(m_t,m_s) &< \alpha^2\xi(m_{t-1},m_{s-1})\max\left\{\alpha^{t-1}q_\xi(m_0,m_1),\alpha^{s-1}q_\xi(m_0,m_1)\right\} \\ &< \alpha\max\left\{\alpha^{t-1}q_\xi(m_0,m_1),\alpha^{s-1}q_\xi(m_0,m_1)\right\}. \end{aligned} \quad (\mathbf{2}.13)$$

Letting $t,s\to\infty$, we get that

$$\lim_{t,s\to\infty} q_\xi(m_t,m_s) = 0. \quad (\mathbf{2}.14)$$

Hence (m_t) is a right Cauchy sequence. Similarly, we can prove that (m_t) is a left Cauchy sequence. So, we conclude that (m_t) is a Cauchy sequence.
The completeness of q_ξ informs us that $\exists\ \beta^* \in M$ such that $m_t \to \beta^*$.
Since l_1 is a continuous function, then we have $m_{t+1} = l_1m_t \to l_1\beta^*$. The uniqueness of the limit ensures that $l_1\beta^* = \beta^*$.
To prove that β^* is unique, assume $\exists\ m_* \in M$ with $l_1m_* = m_*$. Then,

$$\begin{aligned} (1-\alpha)q_\xi(\beta^*,m_*) &= (1-\alpha)q_\xi(\beta^*,l_1^i m_*) \\ &\leq q_\xi(\beta^*,m_*). \end{aligned}$$

So, we have

$$\begin{aligned} q_\xi(\beta^*,m_*) &= q_\xi(l_1\beta^*,l_1m_*) \\ &\leq \alpha^2\xi(\beta^*,m_*)\max\{q_\xi(\beta^*,l_1\beta^*),q_\xi(m_*,l_1m_*)\} \\ &= \alpha^2\xi(\beta^*,m_*)\max\{q_\xi(\beta^*,\beta^*),q_\xi(m_*,m_*)\} \\ &= 0. \end{aligned} \quad (\mathbf{2}.15)$$

Hence the result. □

Theorem 2.4. *Suppose (M,q_ξ) is complete, l_1 is continuous on M and ξ is bounded by $\frac{1}{\alpha}$, where $\alpha\in(0,1)$. Assume that for all $m_1,m_2\in M$, l_1 satisfies*

$$q_\xi(l_1m_1,l_1m_2) \leq \alpha^2\xi(m_1,m_2)\max\{q_\xi(m_1,l_1m_1),q_\xi(m_2,l_1m_2)\}.$$

Then l_1 has a unique fixed point in M.

Corollary 2.2. *Suppose* (M, q_ξ) *is complete,* l_1 *is continuous on* M *and* ξ *is bounded by* $\frac{1}{\delta_1+\delta_2}$ *with* $\delta_1, \delta_2 \in (0,1)$ *and* $\delta_1 + \delta_2 < 1$*. Assume that for all* $m_1, m_2 \in M$*,* l_1 *satisfies*

$$q_\xi(l_1m_1, l_1m_2) \leq \delta_1^2\xi(m_1, m_2)q_\xi(m_1, l_1m_1) + \delta_2^2\xi(m_1, m_2)q_\xi(m_2, l_1m_2).$$

Then l_1 *has a unique fixed point in* M*.*

Theorem 2.5. *Suppose* $(M, \check{q}_\xi)$ *is complete and* l_1 *is continuous on* M*. Assume there exist* $\alpha \in (0,1)$ *and* $s \leq \frac{1}{\alpha}$ *such that for all* $m_1, m_1 \in M$*,* l_1 *satisfies*

$$\check{q}_\xi(l_1m_1, l_1m_2) \leq \alpha^2 s \max\{\check{q}_\xi(m_1, l_1m_1), \check{q}_\xi(m_2, l_1m_2)\}.$$

Then l_1 *has a unique fixed point in* M*.*

Corollary 2.3. *Suppose* $(M, \check{q}_\xi)$ *is complete and* l_1 *is continuous on* M*. Assume that* $s \leq \frac{1}{\delta_1+\delta_2}$ *where* $\delta_1, \delta_2 \in (0,1)$ *and* $\delta_1 + \delta_2 < 1$ *such that for all* $m_1, m_1 \in M$*,* l_1 *satisfies*

$$\check{q}_\xi(l_1m_1, l_1m_2) \leq \delta_1^2 s\check{q}_\xi(m_1, l_1m_1) + \delta_2^2 s\check{q}_\xi(m_2, l_1m_2).$$

Then l_1 *has a unique fixed point in* M*.*

3. Application

To highlight the importance of our work, we utilize Theorem **2**.2 to prove that the following nonlinear equation

$$Dx + nx^{n+1} = x^n + 1 \quad n \in \mathbf{N},$$

has a unique solution in $[0,1]$.

Theorem 3.1. *The nonlinear equation*

$$Dx + nx^{n+1} = x^n + 1, \quad n \in \mathbf{N}, \tag{\textbf{3}.1}$$

where $D \geq 2n$ *has a unique solution in* $[0,1]$*.*

Proof. Let $M = [0,1]$ and let $\xi : M \times M \to [1,\infty)$ and $q_\xi : M \times M \to [0,\infty)$ be defined via $\xi(m_1, m_2) = 1 + \frac{1}{D}|m_1 - m_2|$ and $q_\xi(m_1, m_2) = |m_1 - m_2|$. Let $\alpha = \frac{D}{D+1}$. Then it is obvious that (M, q_ξ) is a complete quasi extended b-metric space and ξ is bounded by $\frac{1}{\alpha}$. To show that Equation (**3**.1) has a unique solution it is equivalent to show that the function $l_1 : M \to M$ which defined by $l_1(m) = \dfrac{1+m^n}{D+nm^n}$ where $n \in \mathbf{N}$ has a unique fixed point in M.

Now,

$$\begin{aligned}
q_\xi(l_1m_1, l_1m_2) &= \left|\tfrac{1+m_1^n}{D+nm_1^n} - \tfrac{1+m_2^n}{D+nm_2^n}\right| \\
&= \tfrac{1}{(D+nm_1^n)(D+nm_2^n)}\left|D(m_1^n - m_2^n) - n(m_1^n - m_2^n)\right| \\
&\leq \tfrac{D-n}{D^2}\left|m_1^n - m_2^n\right| \\
&= \tfrac{D-n}{D^2}\left|m_1 - m_2\right|\left(m_1^{n-1} + m_2m_1^{n-2} + \cdots + m_2^{n-2}m_1 + m_1^{n-1}\right) \\
&\leq \tfrac{n(D-n)}{D^2}|m_1 - m_2| \\
&\leq \left(\tfrac{D}{D+1}\right)^2|m_1 - m_2| \\
&\leq \left(\tfrac{D}{D+1}\right)^2\left(1 + \tfrac{1}{D}|m_1 - m_2|\right)|m_1 - m_2| \\
&= \alpha^2\xi(m_1, m_2)q_\xi(m_1, m_2).
\end{aligned}$$

Hence, l_1 satisfies all the properties of Theorem **2**.2 and so l_1 has a unique fixed point in M. Consequently, the above equation has a unique solution in M. □

Example 3.1. *The equation*

$$200x + 100x^{101} = x^{100} + 1$$

has a unique solution in [0,1].

Proof. Follows from Theorem **3**.1 by choosing $n = 100$. □

Now, we move on to give an application to integral equations of Fredholm type. Let $\mathbf{U} = C([0,1], \mathbf{R})$ the set of all continuous real-valued functions on [0,1]. For $x \in \mathbf{U}$, let $\|x\| = \max_{t\in[0,1]}|x(t)|$, and let $K : \mathbf{U} \to [2,3)$ be defined by $K(x) = \frac{2+3\|x\|}{1+\|x\|}$.

Define $\xi : \mathbf{U}\times\mathbf{U} \to [1,\infty)$, and $q_\xi : \mathbf{U}\times\mathbf{U} \to [0,\infty)$ by $\xi(x,y) = K(x)$, and $q_\xi(x,y) = \frac{1}{2}K(x)\|x-y\|^2$. It is clear that $(\mathbf{U}, q_\xi)$ is a complete extended quasi metric space, and ξ is bounded by 3.

Theorem 3.2. *Consider the integral equation*

$$x(t) = g(t) + \int_0^1 H(t,s,x(s))ds \quad t,s \in [0,1], \tag{3.2}$$

where $g : [0,1] \to \mathbf{R}$, $H : [0,1]\times[0,1]\times\mathbf{R} \to \mathbf{R}$ are continuous functions. Suppose H satisfies the following condition for all $x,y \in \mathbf{U}$ and $t,s \in [0,1]$

$$|H(t,s,x(s)) - H(t,s,y(s))| \leq \frac{2}{\sqrt{27}}\|x-y\|.$$

*Then, the integral equation (**3**.2) has a unique solution.*

Proof. Let $f : \mathbf{U} \to \mathbf{U}$ be defined by $fx(t) = g(t) + \int_0^1 H(t,s,x(s))ds$. We show that the operator f satisfies the hypothesis of Theorem **2**.2 with $\alpha = \frac{1}{3}$.
Let $x, y \in \mathbf{U}$. Then, for any $t,s \in [0,1]$, we have

$$\begin{aligned}|fx(t)-fy(t)| &= \left|\int_0^1 \big(H(t,s,x(s))-H(t,s,y(s))\big)\ ds\right| \\ &\leq \int_0^1 |H(t,s,x(s))-H(t,s,y(s))|\ ds \\ &\leq \frac{2}{\sqrt{27}}\|x-y\|.\end{aligned}$$

So, $\|fx-fy\|^2 \leq \frac{4}{27}\|x-y\|^2$.
Now,

$$\begin{aligned}q_\xi(fx,fy) &= \frac{1}{2}K(fx)\ \|fx-fy\|^2 \\ &\leq \frac{3}{2}\|fx-fy\|^2 \\ &\leq \frac{2}{9}\|x-y\|^2 \\ &\leq \alpha^2\xi(x,y)q_\xi(x,y).\end{aligned}$$

Hence, f has a unique fixed point. Consequently, the integral equation (**3**.2) has a unique solution. □

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