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TZITZEICA 2ND ORDER LAGRANGIAN DYNAMICS

Constantin Udrişte[1], Corina Cipu[2], Vasile Arsinte[3]

Ecuaţiile neliniare joacă un rol major în diverse aplicaţii ale matematicii şi fizicii moderne. Scopul lucrării este de a realiza un studiu de stabilitate a schemelor numerice obţinute dintr-un Lagrangian de ordinul al doilea, asociat ecuaţiei cu derivate parţiale a lui Ţiţeica. Se face o analiza a stabilitaţii von Neumann. Ultima secţiune precizează diferenţele dintre noua ecuaţie discretă a lui Ţiţeica şi cea liniarizată.

Nonlinear equations play a major role in many applications of modern mathematics and physics. The goal of the paper is to make a stability study of numerical schemes derived from the second order Lagrangian associated to Tzitzeica PDE. A discrete Tzitzeica Euler-Lagrange equation is written using this second order Lagrangian. Von Neumann stability analysis for this equation is made. Final section underlines the differences between the new discrete Tzitzeica equation and the linearized one.

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1. Tzitzeica hyperbolic PDE

The *Tzitzeica hyperbolic nonlinear PDE* is

$$(\ln h)_{uv} = h - \frac{1}{h^2}.$$

With a change of function $\ln h = \omega$, this equation rewrites as

$$\omega_{uv} = e^{\omega} - e^{-2\omega}. \tag{1}$$

The great Romanian geometer Tzitzeica arrived at his equation from the viewpoint of the geometry of surfaces [1, 2], obtaining an associated linear representation and a Backlund transformation [4, 5]. Now, the PDE (1) is known under various names, and has been studied from several perspectives, [6]-[8], including geometry, [3], and non-linear mechanics [10].

[1]Professor, Faculty of Applied Sciences, University “Politehnica” of Bucharest, Romania, e-mail: udriste@mathem.pub.ro

[2]Lecturer, Faculty of Applied Sciences, University “Politehnica” of Bucharest, Romania

[3]Teacher, High School Callatis of Mangalia, Constanţa County, Romania

The PDE (1) is in fact the Euler-Lagrange PDE associated to the second order Lagrangian:

$$L = \frac{1}{2}\omega\,\omega_{uv} - e^{\omega} - \frac{1}{2}e^{-2\omega}. \tag{2}$$

2. Discrete two-parameter second order Lagrangian dynamics

A two-parameter second order Lagrangian

$$L(u, v, \omega, \omega_u, \omega_v, \omega_{uv})$$

produces the Euler-Lagrange PDE

$$L_\omega - D_u L_{\omega_u} - D_v L_{\omega_v} + D_{uv} L_{\omega_{uv}} = 0$$

with the unknown function $\omega(u, v)$. The theory of integrators for multi-parameter Lagrangian dynamics shows that instead of discretization of Euler-Lagrange PDEs we must use a discrete Lagrangian, a discrete action, and then discrete Euler-Lagrange equations. The discrete Euler-Lagrange equations associated to multi-time discrete Lagrangian can be solved successfully by the Newton method if it is convergent for a convenient step.

The discretization of a two-parameter second order Lagrangian can be performed by using the *centroid rule* (see [13], [14]) which consists in the substitution of the point (u, v) with the fixed step (k_1, k_2), of the point $\omega(u, v)$ with the fraction

$$\frac{\omega_{kl} + \omega_{k+1l} + \omega_{kl+1} + \omega_{k+1l+1}}{4},$$

of the partial velocities ω_α, $\alpha = 1, 2$, by the fractions

$$\frac{\omega_{k+1l} - \omega_{kl}}{k_1}, \; \frac{\omega_{kl+1} - \omega_{kl}}{k_2}$$

and of the mixed second order derivative ω_{uv}, by the fraction

$$\frac{\omega_{k+1l+1} - \omega_{kl+1} - \omega_{k+1l} + \omega_{kl}}{k_1 k_2}.$$

One obtains a *discrete Lagrangian*

$$L_{2d} : R^2 \times R^4 \to R,$$

$$L_{2d} = L\left(k_1, k_2, \frac{u_1 + u_2 + u_3 + u_4}{4}, \frac{u_2 - u_1}{k_1}, \frac{u_3 - u_1}{k_2}, \frac{u_4 - u_3 - u_2 + u_1}{k_1 k_2}\right).$$

The second order Lagrangian (2) determines the 2-dimensional *discrete action*

$$S : R^2 \times R^{(N_1+1)(N_2+1)} \to R,$$

$$S(k_1, k_2, A) = \sum_{k=0}^{N_1-1} \sum_{l=0}^{N_2-1} L_{2d}(k_1, k_2, \omega_{kl}, \omega_{k+1l}, \omega_{kl+1}, \omega_{k+1\,l+1}),$$

where

$$A = (\omega_{kl}), k = 0, ..., N_1,\; l = 0, ..., N_2.$$

The *discrete variational principle*, see [14], consists in the characterization of the matrix A for which the action S is stationary, for any family $\omega_{kl}(\epsilon) \in R$ with

$$k = 0, ..., N_1 - 1,\ l = 0, ..., N_2 - 1,$$
$$\epsilon \in I \subset R, 0 \in I,\ \omega_{kl}(0) = \omega_{kl}$$

and fixed elements

$$\omega_{0l},\ \omega_{N_1 l},\ \omega_{k0},\ \omega_{kN_2}.$$

The discrete variational principle is obtained using the first order variation of S. In other words the matrix (point) $A = (\omega_{kl})$ is stationary for the action S if and only if (*discrete Euler-Lagrange equation*)

$$\sum_{\xi} \frac{\partial L_{2d}}{\partial \omega_{kl}}(\xi) = 0, \tag{3}$$

where ξ runs over the following four points,

$$\xi_1 = (\omega_{kl}, \omega_{k+1l}, \omega_{kl+1}, \omega_{k+1\,l+1}),\ \xi_2 = (\omega_{k-1l}, \omega_{kl}, \omega_{k-1l+1}, \omega_{kl+1})$$
$$\xi_3 = (\omega_{kl-1}, \omega_{k+1l-1}, \omega_{kl}, \omega_{k+1l}),\ \xi_4 = (\omega_{k-1l-1}, \omega_{kl-1}, \omega_{k-1l}, \omega_{kl})$$

and

$$k = 1, ..., N_1 - 1,\ l = 1, ..., N_2 - 1,$$

in which the first two variables k_1, k_2 are omitted being fixed for all points.

The *variational integrator* described by a discrete Euler-Lagrange equation works as follows:

- Step 1: define the lines $(\omega_{00}, \omega_{01}, ..., \omega_{0N})$, $(\omega_{10}, \omega_{11}, ..., \omega_{1N})$;
- Step 2: denote by

$$u = \omega_{k+1l+1}$$
$$A(kl) = \frac{\partial L_{2d}}{\partial \omega_{kl}}(\omega_{k-1l}, \omega_{kl}, \omega_{k-1l+1}, \omega_{kl+1})$$
$$B(kl) = \frac{\partial L_{2d}}{\partial \omega_{kl}}(\omega_{kl-1}, \omega_{k+1l-1}, \omega_{kl}, \omega_{k+1l})$$
$$C(kl) = \frac{\partial L_{2d}}{\partial \omega_{kl}}(\omega_{k-1l-1}, \omega_{kl-1}, \omega_{k-1l}, \omega_{kl});$$
$$f(u) = \frac{\partial L_{2d}}{\partial \omega_{kl}}(\omega_{kl}, \omega_{k+1l}, \omega_{kl+1}, u) + A(kl) + B(kl) + C(kl);$$

- Step 3: solve the nonlinear equation $f(u) = 0$ at each step (k_1, k_2) using eight points of starting as shown a part of the grid

$$\begin{array}{ccc} \clubsuit\omega_{k-1l-1} & \clubsuit\omega_{k-1l} & \clubsuit\omega_{k-1l+1} \\ \clubsuit\omega_{kl-1} & \clubsuit\omega_{kl} & \clubsuit\omega_{kl+1} \\ \clubsuit\omega_{k+1l-1} & \clubsuit\omega_{k+1l} & * \, u = \omega_{k+1l+1} \end{array}$$

Giving the boundary elements ω_{0l}, $\omega_{N_1 l}$, ω_{k0}, ω_{kN_2}, the discrete Euler-Lagrange equation is solved by the Newton method if it is contractive for a small step (k_1, k_2) (see [13], [14]).

We introduce the *discrete momenta* via a *discrete Legendre transformation*

$$p^{kl} = \frac{\partial L_{2d}}{\partial \omega_{kl}}(\omega_{kl}, \omega_{k+1l}, \omega_{kl+1}, \omega_{k+1l+1}). \tag{4}$$

Then (3) becomes a linear initial value problem with constant coefficients

$$p^{kl} + p^{k-1l} + p^{kl-1} + p^{k-1l-1} = 0, \tag{5}$$

called *dual variational integrator* equation.

3. Discrete Tzitzeica equation

Schief [4] have obtained an integrable discrete version of the Tzitzeica equation as the compatibility condition of the discrete Gauss equation, and transformed the discrete Tzitzeica equation into the trilinear form. R. Hirota [9] shows that the Tzitzeica equation is equivalent to the Toda molecule equation with the special boundary condition, hence he studies discrete Toda molecule equation with a special boundary condition.

The associated *discrete second order Tzitzeica Lagrangian* is

$$L_{2d} = \frac{\omega_{kl} + \omega_{k+1l} + \omega_{kl+1} + \omega_{k+1l+1}}{8} \cdot \frac{\omega_{k+1l+1} - \omega_{kl+1} - \omega_{k+l} + \omega_{kl}}{k_1 k_2}$$

$$-e^{(\omega_{kl} + \omega_{k+1l} + \omega_{kl+1} + \omega_{k+1l+1})/4} - \frac{1}{2} e^{-(\omega_{kl} + \omega_{k+1l} + \omega_{kl+1} + \omega_{k+1l+1})/2}.$$

It produces the *discrete Tzitzeica equation* (discrete Euler-Lagrange equation)

$$\frac{1}{k_1 k_2}(\omega_{k-1l-1} + 2\omega_{kl} + \omega_{k+1l+1}) + \frac{1}{4k_1 k_2}(\omega_{k-1l} + \omega_{kl-1} + \omega_{kl+1} + \omega_{k+1l})$$

$$-e^{(\omega_{kl} + \omega_{k+1l} + \omega_{kl+1} + \omega_{k+1l+1})/4} - e^{(\omega_{k-1l} + \omega_{kl} + \omega_{k-1l+1} + \omega_{kl+1})/4}$$

$$-e^{(\omega_{kl-1} + \omega_{k+1l-1} + \omega_{kl} + \omega_{k+1l})/4} - e^{(\omega_{k-1l-1} + \omega_{kl-1} + \omega_{k-1l} + \omega_{kl})/4}$$

$$+e^{-(\omega_{kl} + \omega_{k+1l} + \omega_{kl+1} + \omega_{k+1l+1})/2} + e^{-(\omega_{k-1l} + \omega_{kl} + \omega_{k-1l+1} + \omega_{kl+1})/2}$$

$$+e^{-(\omega_{kl-1} + \omega_{k+1l-1} + \omega_{kl} + \omega_{k+1l})/2} + e^{-(\omega_{k-1l-1} + \omega_{kl-1} + \omega_{k-1l} + \omega_{kl})/2} = 0.$$

This is a second order *nonlinear implicit finite difference equation.* The *singularity set* with respect to $u = \omega_{k+1l+1}$ is defined by the equation

$$e^{(\omega_{kl} + \omega_{k+1l} + \omega_{kl+1} + \omega_{k+1l+1})/4} + 2e^{-(\omega_{kl} + \omega_{k+1l} + \omega_{kl+1} + \omega_{k+1l+1})/2} = \frac{4}{k_1 k_2}.$$

If we denote $Y = e^{(\omega_{kl} + \omega_{k+1l} + \omega_{kl+1} + u)/4}$, the singularity set with respect to u is given by positive solutions of third degree algebraic equation,

$$Y^3 - \frac{4}{k_1 k_2} Y^2 + 2 = 0.$$

For $k_1 k_2 > \frac{4\sqrt[3]{2}}{3}$, the singularity set is empty, [12].

For $k_1k_2 < \dfrac{4\sqrt[3]{2}}{3}$, the previous implicit equation gives three real solutions but only one is a positive solution,

$$u = U - (\omega_{kl} + \omega_{k+1l} + \omega_{kl+1}),$$

and

$$U = \log(-\frac{8}{3k_1k_2}\cos(\pi/3 + \delta/3)), \delta = \arccos(\frac{27}{64}(k_1k_2)^3 - 1).$$

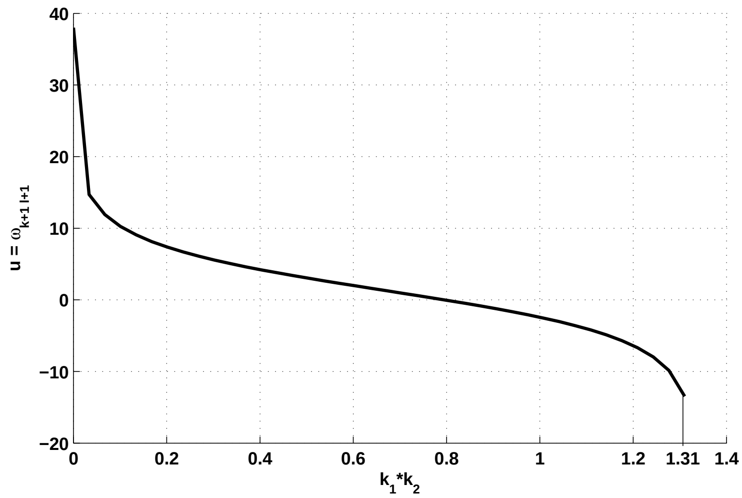

FIGURE 1. Singularity set with respect to u

4. **Von Neumann analysis of dual variational integrator equation**

To verify the stability of the dual variational equation (5), we pass to the frequency domain, accepting that u is a *spatial coordinate* and v is a *temporal coordinate*. Using a 1D *discrete spatial Fourier transform* with respect to variable k, which can be obtained via the substitutions

$$p^{kl} \to P^l(\alpha)e^{j\alpha h},$$

where α denotes the *radian wave scalar*, and h the new variable. We find a second order linear difference equation (*digital filter*)

$$P^l + P^l e^{-j\alpha h} + P^{l-1} + P^{l-1} l e^{-j\alpha h} = 0$$

that need its stability checked. For this purpose we introduce the *z-transform* $E(z, \alpha)$ and we must impose that the poles of the recursion do not lie outside the unit circle in the z-plane. To simplify, we accept the initial conditions $P^0 = 0$. One obtains the homogeneous linear equation

$$(1 + e^{-j\alpha h})(1 + z^{-1})E = 0.$$

The pole $z=-1$ satisfies the condition

$$|z| \leq 1$$

that ensures the stability of the scheme over the region that not contains the singularity set, whatever will be relation between the grid spacing h and the wave number α.

5. **Von Neumann analysis of linearized discrete Tzitzeica equation**

The linearization of discrete Tzitzeica equation is

$$\frac{1}{k_1k_2}(\omega_{k-1l-1}+2\omega_{kl}+\omega_{k+1l+1})+\frac{1}{k_1k_2}(\omega_{k-1l}+\omega_{kl-1}+\omega_{kl+1}+\omega_{k+1l})-\frac{3}{4}\omega_{kl}$$
$$-\frac{3}{2}(\omega_{k+1l}+\omega_{kl+1}+\omega_{kl-1}+\omega_{k-1l})-\frac{3}{4}(\omega_{k+11l+1}+\omega_{k+1l-1}+\omega_{k-1l+1}+\omega_{k-1l-1})=0.$$

To verify the stability of this finite difference scheme, we pass to the frequency domain, through what is called von Neumann analysis. For that

(1) accept that u is a *spatial coordinate* and v is a *temporal coordinate*;

(2) consider a uniform grid spacing in u, i.e., $h=k_1$ is constant, and an unbounded domain R;

(3) denote by $\tau = k_2$ the "time" step regarding v and we define the constant level sets $\frac{4}{3k_1k_2}=\rho$.

Introducing a 1D *discrete spatial Fourier transform* which can be obtained via the substitutions

$$\omega_{kl} \to \Omega_l(\alpha)e^{j\alpha h},$$

where α denotes the *radian wave scalar*. We find a second order linear difference equation (*digital filter*)

$$\rho(\Omega_{l-1}+\Omega_l)e^{-j\alpha h}+\rho(\Omega_l+\Omega_{l+1})e^{j\alpha h}$$
$$-(\Omega_{l+1}+2\Omega_l+\Omega_{l-1})(e^{j\alpha h}+e^{-j\alpha h}+2-\rho)=0.$$

In order to check the stability of the digital filter we introduce the *z-transform* $F(z,\alpha)$ and we must impose that the poles of the recursion do not lie outside the unit circle in the z-plane. To simplify, we accept the initial conditions $\Omega_0=0$. One obtains the homogeneous linear equation

$$\rho(z^{-1}+1)e^{-j\alpha h}F+\rho(1+z)e^{j\alpha h}F$$
$$-(z+2+z^{-1})(e^{j\alpha h}+e^{-j\alpha h}+2-\rho)F=0.$$

The poles are the roots of the characteristic equation

$$\rho(z^{-1}+1)e^{-j\alpha h}+\rho(1+z)e^{j\alpha h}$$
$$-(z+2+z^{-1})(e^{j\alpha h}+e^{-j\alpha h}+2-\rho)=0.$$

with the unknown z. Explicitly, we have

$$(z+1)(a_1z+a_0)=0,$$

where

$$a_1 = (\rho - 2)(1 + \cos(\alpha h)) + j\rho \sin(\alpha h), a_0 = \bar{a}_1.$$

As long as $a_1 \neq 0$ the stability is verified, since $|z| = 1$.
There are two cases in which a_1 becomes zero, namely for:
1. $h = \pi/\alpha$;
2. $h = 2\pi/\alpha$ and $\rho = 2$, i.e., $hk_2 = 2/3$.
In all other cases our scheme is marginally stable.

6. **Conclusions**

The von Neumann analysis was used to prove the stability of the finite difference scheme regarding the linearized discrete Tzitzeica equation. Comparing the foregoing results with those in paper [16], we can formulate the following statements.

1. The singularity set is empty for $k_1k_2 > \frac{3}{\sqrt[3]{4}}$, in case of the Lagrangian of order one, and for $k_1k_2 > \frac{4\sqrt[3]{2}}{3}$, in case of the Lagrangian of second order.

2. When the singularity set is not empty, the cubic equation that leads to the singularity set for $u = \omega_{k+1l+1}$, implies two positive solutions

$$u = U_1 - (\omega_{kl} + \omega_{k+1l}); u = U_2 - (\omega_{kl} + \omega_{k+1l})$$

where

$$U_{1,2} = 3\log((3/2 - 3cos(\pi/3 \pm \delta/3))/(k_1k_2)), \delta = \arccos(-(6k_1k_2{+}8(k_1k_2)^3)/27),$$

in case of the Lagrangian of order one, and a single solutio,:

$$u = U - (\omega_{kl} + \omega_{k+1l} + \omega_{kl+1}),$$

with

$$U = \log(-\frac{8}{3k_1k_2}\cos(\pi/3 + \delta/3)), \delta = \arccos(\frac{27}{64}(k_1k_2)^3 - 1).$$

in case of the Lagrangian of second order.

3. The conditions of stability implies a critical surface $F(\rho, \alpha h)$ for which the positiveness ensure the stability of the scheme:
$F_{1,2} \geq 0$ with $F_{1,2} = 4(1-\rho)\sqrt{2 - 2\cos(\alpha h)} - | - a_1 \pm \sqrt{\Delta}|$,
in case of a Lagrangian of order one. For the second order Lagrangian there is no critical surface.

The results are more general in case of Lagrangian of order two.

Future works: We shall consider two directions of our research. One is related to numerical simulations by scattering methods (see [11]), for which stability verification properties could be made. On the other hand we shall use the second order Lagrangian form, [15], in order to improve the scheme stability.

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