ON SOME PROPERTIES OF SYLVESTER MATRIX RANK FUNCTIONS

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The aim of this short note is to obtain, using elementary methods, some properties of Sylvester matrix rank functions [3, 5.1]. *In the main result we show that the sum of Sylvester matrix rank functions of two matrix polynomials is the same as the sum of the Sylvester matrix rank function of the matrix obtained by applying greatest common divisor, with the Sylvester matrix rank function of the matrix obtained by applying lowest common multiple.*

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1. Introduction

Let k be a commutative ring and A be an associative unital k-algebra, with identity element denoted 1_A and zero element denoted 0_A. A Sylvester matrix rank function rk is a function that assigns a non-negative real number to each matrix over A and satisfies some conditions. We will follow the approach adopted in [3, Section 5]. Historically, the first appearance of Sylvester matrix rank functions is given by Malcolmson in [5, Section 3], under the name of algebraic rank functions. The motivation of Malcolmson was to describe an alternative way of determining homomorphisms from associative unital rings to skew fields (division rings). For our first goal we will weaken the definition of Sylvester matrix rank function by removing conditions (SMat1) and (SMat2) of [3, 5.1]. We will present *weak Sylvester matrix rank functions* in Section 2 and we continue with some probably well-known properties that are satisfied by Sylvester matrix rank functions but, in order to obtain the proof, it is enough to consider weak Sylvester matrix rank functions. Let $k[X]$ be the algebra of polynomials with coefficients in k (which is a field) and f, g be two polynomials in $k[X]$. For shortness, we denote by d the greatest common divisor (f, g) and by m we denote the lowest common multiple $[f, g]$ of the polynomials. We give now the announced main result of this paper about the relation between the weak Sylvester matrix rank functions of two matrix polynomials and the weak Sylvester matrix rank functions of the matrix obtained by applying d, respectively m.

Theorem 1.1. *Let A be a k-algebra and* rk *be a weak Sylvester matrix rank function on A. Let $f, g \in k[X]$ with d and m as above and assume that k is a field. Then for any positive integer n and any matrix $M \in \mathcal{M}_n(A)$ the following relation holds*

$$\mathrm{rk}(f(M)) + \mathrm{rk}(g(M)) = \mathrm{rk}(d(M)) + \mathrm{rk}(m(M)).$$

The proof of this theorem is given in Section 3. It is well known that for any matrix $M \in \mathcal{M}_{n,p}(k)$ (n, p are any positive integers) the rank of M, denoted $\mathrm{rank}(M)$, is dimension of the vector space generated (or spanned) by its columns. Equivalently, it largest order of any non-zero minor in M. Obviously, if we take $A := k$ as a k-algebra then rank is a

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(weak) Sylvester matrix rank function on k. In this case, Theorem 1.1 was proved by the first author [7] and, as consequences he obtained a multitude of results see [7, Corollary 1-8]) that are more or less known. The proof of the main result is given in Section 3. Note that we cannot use any of the new three proofs presented in [6] for our Theorem 1.1. We are using the approach of [7].

2. Reminder on Sylvester matrix rank function

Definition 2.1. A *weak Sylvester matrix rank function* rk is a function that assigns to each matrix M (i.e $M \in \mathcal{M}_{n,p}(A)$, for some positive integers n, p) a non-negative real number and satisfies:

- (wSM1) : $\operatorname{rk}(M_1 \cdot M_2) \leq \min\{\operatorname{rk}(M_1), \operatorname{rk}(M_2)\}$, for any matrices M_1, M_2 that can be multiplied;
- (wSM2) : $\operatorname{rk}\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} = \operatorname{rk}(M_1) + \operatorname{rk}(M_2)$, for any matrices M_1, M_2;

By adding the conditions $\operatorname{rk}(1_A) = 1, \operatorname{rk}(0_A) = 0$ and $\operatorname{rk}\begin{bmatrix} M_1 & M_3 \\ 0 & M_2 \end{bmatrix} \geq \operatorname{rk}(M_1) + \operatorname{rk}(M_2)$, for any matrices M_1, M_2, M_3 of appropriate sizes, we recover exactly the concept of Sylvester matrix rank function [3, 5.1].

Next, we give some properties of weak Sylvester matrix rank functions, see also [4, Section 1] for other properties of Sylvester matrix rank functions. These properties are easily deduced from Definition 2.1.

Proposition 2.1. *Let A be a k-algebra and* rk *be a weak Sylvester matrix rank function on A. Let n, p be two positive integers and $M \in \mathcal{M}_{n,p}(A)$.*

(i) *If $X \in \mathcal{M}_n(A), Y \in \mathcal{M}_p(A)$ are invertible matrices then* $\operatorname{rk}(X \cdot M) = \operatorname{rk}(M) = \operatorname{rk}(M \cdot Y)$;

(ii) $\operatorname{rk}\begin{bmatrix} 0 & M_1 \\ M_2 & 0 \end{bmatrix} = \operatorname{rk}(M_1) + \operatorname{rk}(M_2)$, *for any matrices M_1, M_2;*

(ii) *If $a \in A$ is invertible then* $\operatorname{rk}(aM) = \operatorname{rk}(M)$.

Proof. (i) We only show the first part of the equality. By using Definition 2.1, (wSM1) we obtain

$$\operatorname{rk}(M) = \operatorname{rk}(X^{-1} \cdot (X \cdot M)) \leq \operatorname{rk}(X \cdot M) \leq \operatorname{rk}(M),$$

hence the conclusion.

(ii) We assume that $M_1 \in \mathcal{M}_{n,p}(A)$ and $M_2 \in \mathcal{M}_{q,r}(A)$, where q, r are positive integers. It is clear that $\begin{bmatrix} 0 & M_1 \\ M_2 & 0 \end{bmatrix} \in \mathcal{M}_{n+q,p+r}(A)$ and that $\begin{bmatrix} 0 & I_r \\ I_p & 0 \end{bmatrix}$ is an invertible matrix of $\mathcal{M}_{r+p}(A)$. By applying statement (i) and Definition 2.1, (wSM2) we obtain

$$\operatorname{rk}\begin{bmatrix} 0 & M_1 \\ M_2 & 0 \end{bmatrix} = \operatorname{rk}\left(\begin{bmatrix} 0 & M_1 \\ M_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & I_r \\ I_p & 0 \end{bmatrix}\right) = \operatorname{rk}\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

$$= \operatorname{rk}(M_1) + \operatorname{rk}(M_2).$$

(iii) Let $a \in A$ be an invertible element of the monoid $(A, \cdot)$. Then $X = aI_n \in \mathcal{M}_n(A)$ is an invertible matrix, hence

$$\operatorname{rk}(aM) = \operatorname{rk}(XM) = \operatorname{rk}(M),$$

where the last part of the equality is true by (i).

□

3. Proof of main result

Proof. (**of Theorem 1.1**) The following relations are well known in $k[X]$. There are $\varphi_1, \varphi_2, \psi_1, \psi_2 \in K[X]$ such that

$$f \cdot \varphi_1 + \varphi_2 \cdot g = d, \quad g = \psi_1 \cdot d, \quad f = d \cdot \psi_2, \quad m \cdot d = f \cdot g \quad \psi_1 \cdot f = m = g \cdot \psi_2.$$

By using the polynomial evaluation homomorphism with respect to M we obtain the next relations

$$f(M) \cdot \varphi_1(M) + \varphi_2(M) \cdot g(M) = d(M), \tag{1}$$

$$g(M) = \psi_1(M) \cdot d(M), \quad f(M) = d(M) \cdot \psi_2(M), \tag{2}$$

$$m(M) \cdot d(M) = f(M) \cdot g(M) \quad \psi_1(M) \cdot f(M) = m(M) = g(M) \cdot \psi_2(M). \tag{3}$$

First we consider the matrix $N \in \mathcal{M}_{2n}(A)$, which is a block matrix given by $N = \begin{bmatrix} f(M) & 0 \\ 0 & g(M) \end{bmatrix}$ and the following invertible matrices

$$C_1 = \begin{bmatrix} I_n & \varphi_1(M) \\ 0 & I_n \end{bmatrix}, \quad L_1 = \begin{bmatrix} I_n & \varphi_2(M) \\ 0 & I_n \end{bmatrix},$$

$$C_2 = \begin{bmatrix} I_n & 0 \\ -\psi_2(M) & I_n \end{bmatrix}, \quad L_2 = \begin{bmatrix} I_n & 0 \\ -\psi_1(M) & I_n \end{bmatrix}.$$

We obtain

$$\begin{aligned}
&L_2 \cdot L_1 \cdot N \cdot C_1 \cdot C_2 \\
&= \begin{bmatrix} I_n & 0 \\ -\psi_1(M) & I_n \end{bmatrix} \cdot \begin{bmatrix} I_n & \varphi_2(M) \\ 0 & I_n \end{bmatrix} \cdot \begin{bmatrix} f(M) & 0 \\ 0 & g(M) \end{bmatrix} \cdot \\
&\cdot \begin{bmatrix} I_n & \varphi_1(M) \\ 0 & I_n \end{bmatrix} \cdot \begin{bmatrix} I_n & 0 \\ -\psi_2(M) & I_n \end{bmatrix} \\
&= \begin{bmatrix} I_n & 0 \\ -\psi_1(M) & I_n \end{bmatrix} \cdot \begin{bmatrix} f(M) & d(M) \\ 0 & g(M) \end{bmatrix} \cdot \begin{bmatrix} I_n & 0 \\ -\psi_2(M) & I_n \end{bmatrix} \\
&= \begin{bmatrix} f(M) - d(M) \cdot \psi_2(M) & d(M) \\ -\psi_1(M) \cdot f(M) + \psi_1(M) \cdot d(M) \cdot \psi_2(M) - g(M) \cdot \psi_2(M) & \psi_1'(M) \end{bmatrix} \\
&= \begin{bmatrix} 0 & d(M) \\ -m(M) & 0 \end{bmatrix}.
\end{aligned}$$

where $\psi_1'(M) := -\psi_1(M) \cdot d(M) + g(M)$ (for shortness), the first part of the equality is true by (1) and the last part follows by applying (2) and (3).

Finally, by using Definition 2.1, (wSM2) and all statements of Proposition 2.1, we obtain

$$\begin{aligned}
\operatorname{rk}(f(M)) + \operatorname{rk}(g(M)) = \operatorname{rk}(N) &= \operatorname{rk}(L_2 \cdot L_1 \cdot N \cdot C_1 \cdot C_2) \\
&= \operatorname{rk} \begin{bmatrix} 0 & d(M) \\ -m(M) & 0 \end{bmatrix} \\
&= \operatorname{rk}(d(M)) + \operatorname{rk}(m(M)).
\end{aligned}$$

□

4. **Conclusions**

Obviously, if we take $A := k$ as a k-algebra then the usual rank of matrices is a (weak) Sylvester matrix rank function on k. In this case, the above Theorem was proved by the first author and, as consequences, he obtained a multitude of other results that are more or less known, see [7]. Note that we cannot use any of the new three proofs obtained by Pop and Negrescu in [6] for the above Theorem. There is now a characterization, given by the first author, of the rank of matrices, through a functional inequation; see [1, p.321].

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