# FROM POPULATION DYNAMICS IN WARFARE TO LAGRANGE-HAMILTON GEOMETRICAL OBJECTS

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*The aim of this paper is to develop, via the least squares methods, the Lagrange-Hamilton geometries (in the sense of nonlinear connections, d-torsions and Lagrangian Yang-Mills electromagnetic-like energy) produced by the population dynamics in warfare. From a geometrical point of view, the Jacobi stability of this population dynamical system in warfare is investigated.*

**Keywords:** population dynamical system; (co)tangent bundles; least squares Lagrangian and Hamiltonian; Lagrange-Hamilton geometries.

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## 1. Introduction

The mathematical model of internal warfare in stateless societies for population dynamics was introduced by P. Turchin and A.V. Korotayev in the papers [10], [9]:

$$\begin{cases} \dfrac{dN}{dt} = r_0 N\left(1 - \dfrac{N}{k_{\max} - cW}\right) - \delta NW \\ \dfrac{dS}{dt} = \rho_0 N\left(1 - \dfrac{N}{k_{\max} - cW}\right) - \beta N \\ \dfrac{dW}{dt} = aN^2 - bW - \alpha S, \end{cases} \tag{1}$$

where

- $N(t) \geq 0$ is the *population density* (the number of inhabitants);
- $S(t) \geq 0$ is the *accumulated state resources* (*state strength*), which can be measured in some real terms, e.g., tons of grain;
- $W(t) \geq 0$ is the *warfare intensity* (measured, for example, by extra mortality from conflict);
- $r_0$ is the *intrinsic rate of population growth*;
- $\rho_0$ is the *per capita taxation rate at low population density*;
- $k_{\max}$ is the *upper limit of the carrying capacity* (population size);
- $\delta$ is the *rate of declining of predator population*;
- $\beta$ is the *per capita state expenditure rate*;
- $b$ is the *exponential rate of declining of intensity of warfare*;
- $a, c$ and $\alpha$ are some *nonnegative proportionality constants.*

## 2. From population dynamical system in warfare to Lagrange geometry

Obviously, we can regard the Turchin-Korotayev DEs system (1) as being constructed on the particular 3-dimensional manifold $M = \mathbb{R}^3$, whose coordinates are

$$\left(x^1 = N, x^2 = S, x^3 = W\right).$$

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Let us also consider its corresponding tangent bundle $TM$ and cotangent bundle $T^*M$, which have the coordinates $\left(x^i, y^i\right)_{i=\overline{1,3}}$, respectively $\left(x^i, p_i\right)_{i=\overline{1,3}}$. It follows that, if we take the vector field $X = \left(X^i(x^1, x^2, x^3)\right)_{i=\overline{1,3}}$ on $M = \mathbb{R}^3$, which is defined by

$$\begin{aligned} X^1(N,S,W) &= r_0 N\left(1 - \frac{N}{k_{\max} - cW}\right) - \delta NW, \\ X^2(N,S,W) &= \rho_0 N\left(1 - \frac{N}{k_{\max} - cW}\right) - \beta N, \\ X^3(N,S,W) &= aN^2 - bW - \alpha S, \end{aligned} \tag{2}$$

the Turchin-Korotayev system (1) can be regarded as the dynamical system

$$\frac{dx^i}{dt} = X^i(x(t)), \quad i = \overline{1,3}. \tag{3}$$

Obviusly, the solutions of class $C^2$ of the dynamical system (3) are the global minimum points for the *least squares Lagrangian* (Udrişte's terminology from [8], [11]) $L : TM \to \mathbb{R}$ which is defined by

$$L(x,y) = \left(y^1 - X^1(x)\right)^2 + \left(y^2 - X^2(x)\right)^2 + \left(y^3 - X^3(x)\right)^2. \tag{4}$$

Further, the Euler-Lagrange equations of the least squares Lagrangian (4), which are described by

$$\frac{\partial L}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^k}\right) = 0, \quad y^k = \frac{dx^k}{dt}, \quad k = \overline{1,3},$$

can be rewritten in the following geometrical form:

$$\frac{d^2x^k}{dt^2} + 2G^k(x,y) = 0, \tag{5}$$

where the geometrical object[1]

$$G^k = \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^j \partial y^k} y^j - \frac{\partial L}{\partial x^k}\right) = -\frac{1}{2}\left[\left(\frac{\partial X^k}{\partial x^j} - \frac{\partial X^j}{\partial x^k}\right) y^j + \frac{\partial X^j}{\partial x^k} X^j\right]$$

has the meaning of *semispray* of $L$ (here we have summation by $j$ from 1 to 3). Consequently, if we follow the Lagrangian geometrical ideas developed in the research works Miron and Anastasiei [4], Udrişte and Neagu [11], [8], and Balan-Neagu [1], we can construct a whole natural collection of nonzero Lagrangian geometrical objects (such as canonical nonlinear connection, d-torsions and Yang-Mills electromagnetic-like energy) that characterize the dynamical system (3) and, implicitly, the initial Turchin-Korotayev system (1).

In order to expose these Lagrangian geometrical ideas, we need the Jacobian matrix of $X$:

$$J = \left(\frac{\partial X^i}{\partial x^j}\right)_{i,j=\overline{1,3}} = \begin{pmatrix} \mathcal{U} & 0 & \mathcal{V} \\ \mathcal{T} & 0 & \mathcal{Y} \\ 2aN & -\alpha & -b \end{pmatrix},$$

where

$$\mathcal{U} = r_0 - \delta W - \frac{2r_0 N}{k_{\max} - cW}, \quad \mathcal{V} = -\delta N - \frac{c r_0 N^2}{\left(k_{\max} - cW\right)^2},$$

$$\mathcal{T} = \rho_0 - \beta - \frac{2\rho_0 N}{k_{\max} - cW}, \quad \mathcal{Y} = -\frac{c\rho_0 N^2}{\left(k_{\max} - cW\right)^2}.$$

[1] The Latin indices $i, j, k, \ldots$ run from 1 to 3. Moreover, the Einstein convention of summation is adopted all over this paper.

**Proposition 2.1.** *The* ***Lagrangian canonical nonlinear connection*** *on the tangent bundle $TM$, produced by the Turchin-Korotayev system (1), is given by the skew-symmetric matrix*

$$\mathcal{N} = \left(\mathcal{N}^i_j\right)_{i,j=\overline{1,3}} = \begin{pmatrix} 0 & \mathcal{A} & \mathcal{B} \\ -\mathcal{A} & 0 & \mathcal{C} \\ -\mathcal{B} & -\mathcal{C} & 0 \end{pmatrix},$$

*where*

$$\mathcal{A} = -\frac{\rho_0 N}{k_{\max} - cW} - \frac{\beta - \rho_0}{2}, \quad \mathcal{B} = \frac{(\delta + 2a)\, N}{2} + \frac{c r_0 N^2}{2\,(k_{\max} - cW)^2},$$

$$\mathcal{C} = -\frac{\alpha}{2} + \frac{c\rho_0 N^2}{2\,(k_{\max} - cW)^2}.$$

*Proof.* The entries of the canonical nonlinear connection matrix are given by the formulas $\mathcal{N}^i_j = \partial G^i / \partial y^j$ (see [4]). By direct computations, it follows that we have (see [1])

$$\mathcal{N} = -\frac{1}{2}\left[J - J^t\right].$$

□

The canonical Cartan linear connection on the tangent bundle $TM$, produced by the least squares Lagrangian (4), has all adapted components equal to zero. Further, we get the result:

**Proposition 2.2.** *The Lagrangian canonical Cartan linear connection, produced by the Turchin-Korotayev system (1), has the following* d*−****torsion matrices****:*

$$R_1 = \begin{pmatrix} 0 & A & B \\ -A & 0 & C \\ -B & -C & 0 \end{pmatrix}, \quad R_2 = 0, \quad R_3 = \begin{pmatrix} 0 & -C & \mathbb{A} \\ C & 0 & \mathbb{B} \\ -\mathbb{A} & -\mathbb{B} & 0 \end{pmatrix},$$

*where*

$$A = -\frac{\rho_0}{k_{\max} - cW}, \quad B = \frac{\delta + 2a}{2} + \frac{c r_0 N}{(k_{\max} - cW)^2},$$

$$C = \frac{c\rho_0 N}{(k_{\max} - cW)^2}, \quad \mathbb{A} = \frac{c^2 r_0 N^2}{(k_{\max} - cW)^3}, \quad \mathbb{B} = \frac{c^2 \rho_0 N^2}{(k_{\max} - cW)^3}.$$

*Proof.* The general formulas which give the Lagrangian d-torsions are expressed by (see [4])

$$R_k = \left(R^i_{jk} := \frac{\delta \mathcal{N}^i_j}{\delta x^k} - \frac{\delta \mathcal{N}^i_k}{\delta x^j}\right)_{i,j=\overline{1,3}}, \ \forall\, k \in \{1,2,3\},$$

where

$$\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - \mathcal{N}^r_k \frac{\partial}{\partial y^r}.$$

By direct calculation, we get (see [1])

$$R_k = \frac{\partial \mathcal{N}}{\partial x^k}, \quad \forall\, k = \overline{1,3}.$$

□

**Proposition 2.3.** *The* ***Lagrangian Yang-Mills electromagnetic-like energy****, produced by the Turchin-Korotayev system (1), has the expression*

$$\mathcal{EYM}(x) = \frac{1}{2} \cdot Trace\left[F \cdot F^t\right] = \left(\frac{\rho_0 N}{k_{\max} - cW} + \frac{\beta - \rho_0}{2}\right)^2 +$$

$$+\frac{1}{4}\left[(\delta+2a)\,N+\frac{cr_0N^2}{(k_{\max}-cW)^2}\right]^2+\frac{1}{4}\left[\alpha-\frac{c\rho_0N^2}{(k_{\max}-cW)^2}\right]^2.$$

*where the electromagnetic-like matrix is expressed by* $F=-\mathcal{N}$. *For more details, see the works* [4] *and* [1].

**Open problem.** From our new geometric-physical point of view, the surfaces of constant level of the Lagrangian Yang-Mills electromagnetic-like energy produced by the Turchin-Korotayev system (1) could have important connotations for the dynamics of population taken in study. Obviously, the surfaces of constant level

$$\Sigma_R:\mathcal{EYM}(x)=R\geq 0$$

are represented by some cylindrical surfaces whose generators are parallel with the axis $OS$. In this direction, we believe that the computer drawn graphics of these surfaces are important for the study of the phenomena involved in the dynamics of population (1). There exists a meaning of these surfaces related to the Turchin-Korotayev's population dynamics?

### 2.1. KCC theory

The *matrix of deviation curvature* from the Kosambi-Cartan-Chern (KCC) geometrical theory is given by the matrix formula (see [2], [3])

$$P=\left(P^i_j\right)_{i,j=\overline{1,3}}=\sum_{r=1}^{3}R_ry^r+\mathcal{E},$$

where

$$\mathcal{E}^i=2G^i-N^i_jy^j=-\frac{1}{2}\left(\frac{\partial X^i}{\partial x^j}-\frac{\partial X^j}{\partial x^i}\right)y^j-\frac{\partial X^j}{\partial x^i}X^j$$

is the *first invariant of the semispray* of the Lagrangian (4), and we have

$$\mathcal{E}=\left(\frac{\delta\mathcal{E}^i}{\delta x^j}\right)_{i,j=\overline{1,3}}.$$

By direct computations, we infer that the entries of the matrix $\mathcal{E}$ are given by [7]

$$\frac{\delta\mathcal{E}^i}{\delta x^j}=-\frac{1}{2}\left(\frac{\partial^2X^i}{\partial x^j\partial x^k}-\frac{\partial^2X^k}{\partial x^i\partial x^j}\right)y^k-\frac{\partial^2X^k}{\partial x^i\partial x^j}X^k-\frac{\partial X^k}{\partial x^i}\frac{\partial X^k}{\partial x^j}-$$

$$-\frac{1}{4}\left(\frac{\partial X^k}{\partial x^j}-\frac{\partial X^j}{\partial x^k}\right)\left(\frac{\partial X^i}{\partial x^k}-\frac{\partial X^k}{\partial x^i}\right),\quad\forall\, i,j\in\{1,2,3\},$$

where we have summation by $k$ from 1 to 3.

Consequently, following the geometrical ideas used in the KCC theory and Jacobi stability (see the research papers Böhmer et al. [2], Bucătaru-Miron [3] and Neagu-Ovsiyuk [7]), the neighboring solutions of the Euler-Lagrange equations (5), and, implicitly, of the equations (1), are *Jacobi stable* if and only if the real parts of the eigenvalues of the deviation tensor $P$ are strictly negative everywhere, and *Jacobi unstable*, otherwise. The Jacobi stability or instability means that the trajectories of the Euler-Lagrange equations (5) are bunching together or are dispersing (or even are chaotic).

## 3. From population dynamical system in warfare to Hamilton geometry

Using the general formulas $p_r = \partial L/\partial y^r$ and $H = p_r y^r - L$, we can introduce the *least squares Hamiltonian* $H : T^*M \to \mathbb{R}$, associated with the Lagrangian (4), which is described by the formula

$$H(x,p) = \frac{1}{4}\left(p_1^2 + p_2^2 + p_3^2\right) + X^1(x)p_1 + X^2(x)p_2 + X^3(x)p_3. \tag{6}$$

Now, following the Hamilton geometrical ideas from the works Miron et al. [5] and Neagu-Oană [6], we can again construct a natural and distinct collection of nonzero Hamiltonian geometrical objects (such as nonlinear connection and d-torsions), which also characterize the Turchin-Korotayev system (1).

**Proposition 3.1.** *The* ***Hamiltonian canonical nonlinear connection*** *on the cotangent bundle $T^*M$, produced by the Turchin-Korotayev system (1), is given by the symmetric matrix*

$$\mathbf{N} = (\mathbf{N}_{ij})_{i,j=\overline{1,3}} = \begin{pmatrix} 2\mathfrak{U} & \mathfrak{T} & \mathcal{V} + 2aN \\ \mathfrak{T} & 0 & \mathcal{Y} - \alpha \\ \mathcal{V} + 2aN & \mathcal{Y} - \alpha & -2b \end{pmatrix}.$$

*Proof.* The Hamiltonian nonlinear connection on the cotangent bundle $T^*M$ has the adapted components (see [5])

$$\mathbf{N}_{ij} = \frac{\partial^2 H}{\partial x^j \partial p_i} + \frac{\partial^2 H}{\partial x^i \partial p_j}.$$

It follows that, by direct computations, we find $\mathbf{N} = J + J^t$. For more details, see [6]. □

The canonical Cartan linear connection, produced by the least squares Hamiltonian (6), has all adapted components equal to zero. Further, we infer

**Proposition 3.2.** *The Hamiltonian canonical Cartan linear connection, produced by the Turchin-Korotayev system (1), is characterized by the d−**torsion matrices** $\mathbf{R}_k = -2R_k$, $\forall\, k = \overline{1,3}$.*

*Proof.* The general formulas giving the Hamiltonian d-torsions are (see [5])

$$\mathbf{R}_k = \left(R_{kij} := \frac{\delta \mathbf{N}_{ki}}{\delta x^j} - \frac{\delta \mathbf{N}_{kj}}{\delta x^i}\right)_{i,j=\overline{1,3}}, \ \forall\, k \in \{1,2,3\},$$

where

$$\frac{\delta}{\delta x^j} = \frac{\partial}{\partial x^j} - \mathbf{N}_{rj}\frac{\partial}{\partial p_r}.$$

By direct computations, we obtain what we were looking for (see [6]), that is

$$\mathbf{R}_k = \frac{\partial}{\partial x^k}\left[J - J^t\right], \quad \forall\, k = \overline{1,3}.$$

□

**Open problem.** What is the real meaning in the dynamics of population for our above constructed Lagrange-Hamilton geometrical objects? Could these objects offer new insights for the studied Turchin-Korotayev's population dynamics?

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