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A NOTION OF APPROXIMATE BIPROJECTIVITY FOR BANACH ALGEBRAS WITH RESPECT TO A CLOSED IDEAL

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In this paper, we present a notion of approximate biprojectivity with respect to a closed ideal for Banach algebras, say approximate I-biprojectivity. The relation between this new notion and left ϕ-contractibility is investigated. Also we study group algebras and Fourier algebras under this new notion. In the final, we give some examples which shows the differences of this new concept and the classical ones.

Keywords: Approximate I-biprojectivity, Banach algebras, Biprojectivity.

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1. Introduction and preliminaries

The concept of biprojectivity has a significant role in studying the structure of Banach algebras. In fact, a Banach algebra A is said to be biprojective if there exists a continuous A-bimodule morphism $\rho : A \to A \otimes_p A$ such that $\pi_A \circ \rho(a) = a$ for all $a \in A$, where $A \otimes_p A$ is denoted for the projective tensor product of A by A and $\pi_A : A \otimes_p A \to A$ is given by $\pi_A(a \otimes b) = ab$. It is shown that for a locally compact group G the group algebra $L^1(G)$ is biprojective if and only if G is a compact group. Also the measure algebra $M(G)$ is biprojective if and only if G is finite. For the history of homological Banach algebras and biprojectivity, see [10].

By studying some sequence algebras, Zhang introduced the notion of approximately biprojective Banach algebras. In fact a Banach algebra A is approximately biprojective if there exists a net of continuous A-bimodule morphisms $\rho_\alpha : A \to A \otimes_p A$ such that $\pi_A \circ \rho_\alpha(a) \to a$ for all $a \in A$. He investigated nilpotent ideals in some Banach algebras, see [14]. Recently, the approximate biprojectivity of semigroup algebras and triangular Banach algebras have been studied. For more information about this results reader see [11].

Sahami et. al. in [12] defined a notion of biprojectivity with respect to a closed ideal. Indeed a Banach algebra A is called I-biprojective, if there exists a bounded A-bimodule morphism $\rho : I \to A \otimes_p A$ such that $\pi_A \circ \rho(i) = i$ for all $i \in I$. They studied the structure of some Banach algebras under this notion.

In this paper, motivated by Zhang's paper, we define a new concept of approximately I-biprojective Banach algebras. We study the group algebras, Fourier algebras, triangular algebras and Segal algebras with respect to this new notion. Some properties of I-approximately biprojective Banach algebras are given. Also we give some examples which demonstrate the differences of our new notion and the classical ones. Here is the definition of our new notion:

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Definition 1.1. *Let A be a Banach algebra and I be a closed ideal of A. Then A is called approximately I-biprojective if there exists a net of A-bimodule morphisms $\rho_\alpha : I \to A \otimes_p A$ such that $\pi_A \circ \rho_\alpha(i) \to i$ for all $i \in I$.*

We should remind some notations and definitions from Banach algebras theory. We recall that if X is a Banach A-bimodule, then with the following actions X^* is also a Banach A-bimodule:

$$a \cdot f(x) = f(x \cdot a), \quad f \cdot a(x) = f(a \cdot x) \qquad (a \in A, x \in X, f \in X^*).$$

The Banach algebra $A \otimes_p A$ is a Banach A-bimodule with the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \ (b \otimes c) \cdot a = b \otimes ca \ (a, b, c \in A).$$

Throughout this paper, $\Delta(A)$ denotes the character space of A, that is, all non-zero multiplicative linear functionals on A. Let $\phi \in \Delta(A)$. Then ϕ has a unique extension on A^{**} denoted by $\tilde{\phi}$ and defined by $\tilde{\phi}(F) = F(\phi)$ for every $F \in A^{**}$. Clearly this extension remains to be a character on A^{**}. Let X and Y be Banach left A-modules. Then the map $T : X \to Y$ is called left A-module morphism if $T(a \cdot x) = a \cdot T(x)$ for every $a \in A$ and $x \in X$. Similarly the right case can be defined. T is called A-bimodule morphism, if T is both a left A-module morphism and a right A-module morphism. A net $(e_\alpha) \subseteq A$ is a left approximate identity for A, if $e_\alpha a \xrightarrow{||\cdot||} a$, for all $a \in A$.

2. Approximately I-biprojective Banach algebras

Theorem 2.1. *Let A be a Banach algebra and I be a closed ideal of A. Suppose that I posses a left and a right approximate identity. Then A is approximately I-biprojective if and only if I is approximately biprojective.*

Proof. Let A be approximately I-biprojective. Then there exists a net of bounded A-bimodule morphisms $\rho_\lambda : I \to A \otimes_p A$ such that $\pi_A \circ \rho_\lambda(i) \to i$ for all $i \in I$. Let (e_α) and (e'_β) be left and right approximate identities of I, respectively. Suppose that $i \in I$ is an arbitrary element. Then

$$\rho_\lambda(i) = \rho_\lambda(\lim_\alpha e_\alpha i) = \lim_\alpha e_\alpha \cdot \rho_\lambda(i) = \lim_\alpha e_\alpha \cdot \rho_\lambda(\lim_\beta i e'_\beta) = \lim_\alpha \lim_\beta e_\alpha \cdot \rho_\lambda(i) \cdot e'_\beta.$$

It follows that $\rho_\lambda(i) \in I \otimes_p I$. Thus ρ_λ is a net of bounded I-bimodule morphisms from I into $I \otimes_p I$ such that $\pi_I \circ \rho_\lambda(i) \to i$, for all $i \in I$. So I is approximately biprojective.
For converse, suppose that I is approximately biprojective. Then there exists a net of bounded I-bimodule morphisms $\rho_\lambda : I \to I \otimes_p I$ such that $\pi_I \circ \rho_\lambda(i) \to i$, for all $i \in I$. Since I has a left approximate identity, $\overline{I^2} = I$. Let $i \in I$. Then there exist nets (a_α) and (b_α) in I such that $i = \lim_\alpha a_\alpha b_\alpha$. Thus

$$x \cdot \rho_\lambda(i) = x \cdot \rho_\lambda(\lim_\alpha a_\alpha b_\alpha) = x \cdot \lim_\alpha a_\alpha \rho_\lambda(b_\alpha) = \lim_\alpha x a_\alpha \rho_\lambda(b_\alpha) = \rho_\lambda(xi), \quad (x \in A, i \in I).$$

It follows that ρ_λ is a net of left A-module morphisms. Similarly we can see that ρ_λ is a net of right A-module morphisms. It follows that A is approximately I-biprojective. □

Let A be a Banach algebra. We denote the set of all non-zero multiplicative linear functionals on A with $\Delta(A)$. A Banach algebra A is called left ϕ-contractible if there exists an element $m \in A$ such that $am = \phi(a)m$ and $\phi(m) = 1$ for all $a \in A$. For more information see [8]. In the next two theorems we expose the relation between approximately biprojectivity with left ϕ-contractibility.

Theorem 2.2. *Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that I is a closed ideal of A which posses a left approximate identity. If A is approximately I-biprojective, then I is left $\phi|_I$-contractible, provided that $\phi|_I \neq 0$.*

Proof. Suppose that A is approximately I-biprojective. Then there exists a net (ρ_α) of A-bimodule morphisms from I into $A \otimes_p A$ such that $\pi_A \circ \rho_\alpha(i) \to i$ for all $i \in I$. Put $L = \ker \phi \cap I$, as a closed ideal of A. Let $i_0 \in I$ be an element such that $\phi(i_0) = 1$. Define $R_{i_0} : A \to I$ by $R_{i_0}(a) = ai_0$ for all $a \in A$. Set

$$\eta_\alpha := (Id_I \otimes q) \circ (R_{i_0} \otimes R_{i_0}) \circ \rho_\alpha : I \to I \otimes_p \frac{I}{L},$$

where $Id_I : I \to I$ and $q : I \to \frac{I}{L}$ are denoted for the identity map and the quotient map, respectively. One can see that (η_α) is a net of left I-module morphisms. Suppose that $l \in L$. We claim that for each α we have $\eta_\alpha(l) = 0$. To see this, since I has a left approximate identity, we have $\overline{IL} = L$. So for each $l \in L$ we may choose $i' \in I$ and $l' \in L$ such that $l = i'l'$. Applying the fact $q(l) = 0$ for each $l \in L$, follows that

$$\begin{aligned}\eta_\alpha(l) &= (Id_I \otimes q) \circ (R_{i_0} \otimes R_{i_0}) \circ \rho_\alpha(l) = (Id_I \otimes q) \circ (R_{i_0} \otimes R_{i_0}) \circ \rho_\alpha(i'l') \\ &= (Id_I \otimes q) \circ (R_{i_0} \otimes R_{i_0})(\rho_\alpha(i') \cdot l') = (Id_I \otimes q)(i_0 \cdot \rho_\alpha(i') \cdot l'i_0) = 0.\end{aligned}$$

So we can drop η_α on $\frac{I}{L}$ (which we denote it again by η_α). Thus $\eta_\alpha : \frac{I}{L} \to I \otimes_p \frac{I}{L}$ is a net of left I-module morphisms. Since $\phi|_I \neq 0$, clearly $\overline{\phi|_I} : \frac{I}{L} \to \mathbb{C}$ given by $\overline{\phi|_I}(i + L) = \phi(i)$ is also a character. Define

$$\xi_\alpha := (id_I \otimes \overline{\phi|_I}) \circ \eta_\alpha : \frac{I}{L} \to I.$$

Using the fact that η_α is an I-module morphism implies that

$$\begin{aligned}\xi_\alpha(i_1 \cdot (i_2 + L)) &= (id_I \otimes \overline{\phi|_I}) \circ \eta_\alpha(i_1 \cdot (i_2 + L)) \\ &= (id_I \otimes \overline{\phi|_I}) \circ \eta_\alpha(i_1 i_2 + L) \\ &= i_1(id_I \otimes \overline{\phi|_I}) \circ \eta_\alpha(i_2 + L) = i_1 \cdot \xi_\alpha(i_2 + L),\end{aligned}$$

for all $i_1, i_2 \in I$. It follows that ξ_α is a net of left I-module morphisms. We claim that the net (ξ_α) is not zero. To see this consider

$$\begin{aligned}\phi|_I(\xi_\alpha(i_0 + L)) &= (\phi|_I \otimes \overline{\phi|_I}) \circ \eta_\alpha(i_0 + L) \\ &= (\phi|_I \otimes \overline{\phi|_I}) \circ \eta_\alpha(i_0) = (\phi \otimes \phi) \circ \rho_\alpha(i_0) \\ &= \phi \circ \pi_A \circ \rho_\alpha(i_0) \to \phi(i_0) = 1\end{aligned}$$

Thus there is an α_0 such that $\phi|_I(\xi_{\alpha_0}(i_0 + L)) \neq 0$. Now put $m = \xi_{\alpha_0}(i_0 + L)$. Then

$$im = i\xi_{\alpha_0}(i_0 + L) = \xi_{\alpha_0}(ii_0 + L) = \xi_{\alpha_0}(\phi|_I(i)i_0 + L) = \phi|_I(i)\xi_{\alpha_0}(i_0 + L) = \phi|_I(i)m.$$

Replacing m with $\frac{m}{\phi|_I(m)}$, gives that I is left $\phi|_I$-contractible. □

Let G be a locally compact group and $L^1(G)$ be the group algebra associated with G. It is known that $L^1(G)$ is a closed ideal of the measure algebra $M(G)$ which posses a bounded approximate identity.

Corollary 2.1. *Let G be a locally compact group. Then $M(G)$ is approximately $L^1(G)$-biprojective if and only if G is compact.*

Proof. Let $M(G)$ be approximately $L^1(G)$-biprojective. We know that $L^1(G)$ has a bounded approximate identity and there exists at leat one non-zero character ϕ on $M(G)$ which its restriction to $L^1(G)$ is not zero(for instance the augmentation character). Thus by the previous Theorem $L^1(G)$ is left ϕ-contractible. So by [8, Theorem 6.1] G is compact.
For converse, let G be a compact group. Then $L^1(G)$ is biprojective. By Theorem 2.1 the proof is complete. □

Remark 2.1. *Suppose that the net (ρ_α) in the Definition 1.1 is bounded. Clearly $(\rho_\alpha) \subseteq B(I, A \otimes_p A) \subseteq B(I, (A \otimes_p A)^{**}) \cong (I \otimes_p (A \otimes_p A)^*)^*$. It follows that the bounded net (ρ_α) has a limit-point in the w^*-topology, say $\rho \in (I \otimes_p (A \otimes_p A)^*)^* \cong B(I, (A \otimes_p A)^{**})$. It is easy to see that ρ is a bounded A-bimodule morphism from I into $(A \otimes_p A)^{**}$ which $\pi_A \circ \rho(i) = i$ for all $i \in I$. This property is called I-biflatness, for more information see* [12].
(*i*) *Now we present an example which shows that approximate I-biprojectivity does not implies I-biflatness necessarily.*

Let $\mathbb{H}$ be the group of all upper triangular 3×3 matrices over $\mathbb{Z}$ with ones on the diagonal, called the integer Heisenberg group

$$\mathbb{H} = \left\{ \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{Z} \right\}.$$

Then $\mathbb{H}$ is amenable and so the Fourier algebra $A(\mathbb{H})$ has a bounded approximate identity. We implies that $A(\mathbb{H})$ is not $A(\mathbb{H})$-biflat, since $\mathbb{H}$ does not have an abelian subgroup of finite index. Applying [3, Proposition 3.8] *and* [13, Theorem 4.2] *we conclude that $A(\mathbb{H})$ is $A(\mathbb{H})$-approximately biprojective.*
(*ii*) *Here we give an example which shows that I-biflatness does not implies approximate I-biprojectivity.*

Let G be a locally compact, amenable, noncompact group. So by [12, Corollary 3.5] *$M(G)$ is $L^1(G)$-biflat but using Corollary 2.1 $M(G)$ is not approximately $L^1(G)$-biprojective.*

Proposition 2.1. *Let A be a commutative Banach algebra and $\phi \in \Delta(A)$. Suppose that I is a closed ideal of A such that $\phi|_I \neq 0$. If A is approximately I-biprojective, then I is left ϕ-contractible.*

Proof. Suppose that A is approximately I-biprojective. Then there exists a net (ρ_α) of A-bimodule morphisms from I into $A \otimes_p A$ such that $\pi_A \circ \rho_\alpha(i) \to i$, for each $i \in I$. Define $T : A \otimes_p A \to A$ by $T(a \otimes b) = \phi(b)a$ for each $a, b \in A$. Clearly T is a bounded linear map which

$$\phi \circ T = \phi \circ \pi_A, \quad T(x \cdot a) = \phi(a)T(x), \quad aT(x) = T(a \cdot x), \quad (a \in A, x \in A \otimes_p A).$$

Pick i_0 in I such that $\phi(i_0) = 1$. Put $m_\alpha = T \circ \rho_\alpha(i_0)$. So

$$\phi(m_\alpha) = \phi \circ T(\rho_\alpha(i_0)) = \phi \circ \pi_A \circ \rho_\alpha(i_0) \to \phi(i_0) = 1.$$

We may choose α_0 such that $\phi(m_{\alpha_0}) \neq 0$. We have

$$im_{\alpha_0} = iT \circ \rho_{\alpha_0}(i_0) = T \circ \rho_{\alpha_0}(ii_0) = T \circ \rho_{\alpha_0}(i_0 i) = \phi(i)T \circ \rho_{\alpha_0}(i_0) = \phi(i)m_{\alpha_0}, \quad (i \in I).$$

Replacing m_{α_0} with $\frac{m_{\alpha_0}}{\phi(m_{\alpha_0})}$, gives that I is left ϕ-contractible. □

Theorem 2.3. *Let G be a locally compact group and $A(G)$ be the Fourier algebra. Suppose that I is a nontrivial ideal of $A(G)$. If $A(G)$ is approximately I-biprojective, then G is discrete.*

Proof. Let $\rho_\alpha : I \to A(G) \otimes_p A(G)$ be a net of $A(G)$-bimodule morphism such that $\pi_{A(G)} \circ \rho_\alpha(i) \to i$ for all $i \in I$. Since $\Delta(A(G)) = \{\phi_t : t \in G\}$, where $\phi_t(f) = f(t)$ for $f \in A(G)$, it follows that

$$\bigcap_{t \in G} \ker \phi_t = \{0\}.$$

So for some $t_0 \in G$ we have $\phi_{t_0}|_I \neq 0$. Pick $i_0 \in I$ such that $\phi_{t_0}(i_0) = 1$. It is known that with the pointwise multiplication $A(G)$ is a commutative Banach algebra. Now by Proposition 2.1 I is left ϕ_{t_0}-contractible. Apply [8, Propositin 3.8] follows that $A(G)$ is left ϕ_{t_0}-contractible. So G is discrete. □

We remined that for Banach algebras X and Y the *weak* operator topology* (W^*OT) on $B(X,Y^*)$ (the set of all bounded linear operators from X into Y^*) is the topology determined by the seminorms $\{p_{x,f} : x \in X, f \in Y\}$, that $p_{x,f}(T) = |T(x)(f)|$, where $T \in B(X,Y^*)$. In the other word $T_\alpha \xrightarrow{W^*OT} T$ if and only if for every $x \in X$; $T_\alpha(x) \xrightarrow{w^*} T(x)$. Note that, since $B(X,Y^*) \cong (X \otimes_p Y)^*$, every bounded set in $B(X,Y^*)$ has a w^*-limit point, with respect to w^*-topology on $(X \otimes_p Y)^*$. A Banach algebra A is called approximately biflat if there is a net of bounded A-bimodule morphism $\rho_\alpha : (A \otimes_p A)^* \to A^*$ such that $\rho_\alpha \circ \pi_A^* \xrightarrow{W^*OT} id_{A^*}$, see [13].

Lemma 2.1. *Let A be a Banach algebra which is a closed ideal of A^{**}. If A^{**} is approximately A-biprojective, then A is approximately biflat Banach algebra.*

Proof. Suppose that $\rho_\alpha : A \to A^{**} \otimes_p A^{**}$ is a net of bounded A^{**}-bimodule morphisms such that $\pi_{A^{**}} \circ \rho_\alpha(a) - a \to 0$, for every $a \in A$. So we can view ρ_α as a net of bounded A-bimodule morphisms. It is known that, there exists a bounded linear map $\psi : A^{**} \otimes_p A^{**} \to (A \otimes_p A)^{**}$ such that for $a, b \in A$ and $m \in A^{**} \otimes_p A^{**}$, satisfies the following;

(i) $\psi(a \otimes b) = a \otimes b$,
(ii) $\psi(m) \cdot a = \psi(m \cdot a), \qquad a \cdot \psi(m) = \psi(a \cdot m)$,
(iii) $\pi_A^{**}(\psi(m)) = \pi_{A^{**}}(m)$,

see [2, Lemma 1.7]. Set

$$\eta_\alpha = \psi \circ \rho_\alpha : A \to (A \otimes_p A)^{**}.$$

Put $\tilde{\eta}_\alpha = \eta_\alpha^*|_{(A \otimes_p A)^*}$. It is easy to see that η_α is a net of A-bimodule morphisms. Consider

$$\langle a, \tilde{\eta}_\alpha(\pi_A^*(f))\rangle = \langle \pi_A^*(f), \eta_\alpha(a)\rangle = \langle f, \pi_A^{**} \circ \eta_\alpha(a)\rangle \to \langle a, f\rangle \quad (a \in A, f \in A^*).$$

It follows that $\tilde{\eta}_\alpha \circ \pi_A^* \xrightarrow{W^*OT} Id_{A^*}$. Thus A is approximately biflat □

Let A be a Banach algebra and $\phi \in \Delta(A)$. Then A is called ϕ-pseudo-amenable, if there exists a net (m_α) in $A \otimes_p A$ such that $a \cdot m_\alpha - \phi(a)m_\alpha \to 0$ and $\phi \circ \pi_A(m_\alpha) \to 1$, for each $a \in A$ [7].

For any locally compact group G, it is well-known that $L^1(G)^{**}$ is a closed ideal of $M(G)^{**}$ [1, Proposition 1.3].

Proposition 2.2. *Let G be a locally compact group. If $M(G)^{**}$ is approximately $L^1(G)^{**}$-biprojective, then G is amenable.*

Proof. Suppose that $M(G)^{**}$ is approximately $L^1(G)^{**}$-biprojective. Then there exists a net of bounded $M(G)^{**}$-bimodule morphisms $(\rho_\alpha)_{\alpha \in I}$ from $L^1(G)^{**}$ into $M(G)^{**} \otimes_p M(G)^{**}$ such that $\pi_{M(G)^{**}} \circ \rho_\alpha(a) - a \to 0$ for every $a \in L^1(G)^{**}$. Suppose that $\phi \in \Delta(L^1(G))$ and pick $i_0 \in L^1(G)$ such that $\phi(i_0) = 1$. We denote $\tilde{\phi}$ for unique extension ϕ to $L^1(G)^{**}$(it can be extended to $M(G)^{**}$ which we denote it again with $\tilde{\phi}$). Suppose that R_{i_0} and L_{i_0} are given for the maps of right and left multiplication by i_0, respectively. We know that $L^1(G)^{**}$ is a closed ideal in $M(G)^{**}$, so map $R_{i_0} \otimes L_{i_0} : M(G)^{**} \otimes_p M(G)^{**} \to L^1(G)^{**} \otimes_p L^1(G)^{**}$ is a bounded $M(G)^{**}$-bimodule morphism. Also one can easily see that $(R_{i_0} \otimes L_{i_0})^{**}$ is a bounded $M(G)^{**}$-bimodule morphism. On the other hand, there exists

$$\psi : L^1(G)^{**} \otimes_p L^1(G)^{**} \to (L^1(G) \otimes_p L^1(G))^{**}$$

such that for $a, b \in L^1(G)$ and $m \in L^1(G)^{**} \otimes_p L^1(G)^{**}$, the following holds;

(i) $\psi(a \otimes b) = a \otimes b$,
(ii) $\psi(m) \cdot a = \psi(m \cdot a), \qquad a \cdot \psi(m) = \psi(a \cdot m)$,
(iii) $\pi_{L^1(G)}^{**}(\psi(m)) = \pi_{L^1(G)^{**}}(m)$,

see [2, Lemma 1.7]. Set $\eta_\alpha =: \psi \circ (R_{i_0} \otimes L_{i_0}) \circ \rho_\alpha|_{L^1(G)} : L^1(G) \to (L^1(G) \otimes_p L^1(G))^{**}$. It is easy to see that $(\eta_\alpha)_{\alpha \in I}$ is a net of bounded $L^1(G)$-bimodule morphisms. Let $(e_\beta)_{\beta \in \Theta}$ be a bounded approximate identity for $L^1(G)$. Set $m^\alpha_\beta = \eta_\alpha(e_\beta)$ which is a net in $(L^1(G) \otimes_p L^1(G))^{**}$. So for each $a \in L^1(G)$, we have

$$\lim_\beta \lim_\alpha a \cdot m^\alpha_\beta - m^\alpha_\beta \cdot a = \lim_\beta \lim_\alpha a \cdot \eta_\alpha(e_\beta) - \eta_\alpha(e_\beta) \cdot a = 0.$$

Also we have

$$\lim_\beta \lim_\alpha \tilde{\phi} \circ \pi^{**}_{L^1(G)}(m^\alpha_\beta) = \lim_\beta \lim_\alpha \tilde{\phi} \circ \pi^{**}_{L^1(G)} \circ \psi \circ R_{i_0} \otimes L_{i_0} \circ \rho_\alpha(e_\beta) = 1,$$

to see this, consider

$$\begin{aligned} \tilde{\phi} \circ \pi^{**}_{L^1(G)} \circ \psi \circ R_{i_0} \otimes L_{i_0} \circ \rho_\alpha(a) &= \tilde{\phi} \circ \pi_{L^1(G)^{**}} \circ (R_{i_0} \otimes L_{i_0}) \circ \rho_\alpha(a) \\ &= \tilde{\phi} \circ \pi_{M(G)^{**}} \circ \rho_\alpha(a) \\ &\to \phi(a), \end{aligned}$$

where $a \in L^1(G)$. Let $E = I \times \Theta^I$ be a directed set with product ordering which is defined by

$$(\alpha, \beta) \leq_E (\alpha', \beta') \Leftrightarrow \alpha \leq_I \alpha', \beta \leq_{\Theta^I} \beta' \qquad (\alpha, \alpha' \in I, \quad \beta, \beta' \in \Theta^I),$$

where Θ^I is the set of all functions from I into Θ and $\beta \leq_{\Theta^I} \beta'$ means that $\beta(d) \leq_\Theta \beta'(d)$ for every $d \in I$. Suppose that $\gamma = (\alpha, \beta_\alpha)$ and $m_\gamma = \eta_\alpha(e_{\beta_\alpha}) \in (L^1(G) \otimes_p L^1(G))^{**}$. By iterated limit theorem [5, Page 69], one can see that

$$a \cdot m_\gamma - m_\gamma \cdot a \to 0, \quad \tilde{\phi} \circ \pi^{**}_{L^1(G)}(m_\gamma) \to 1, \quad (a \in L^1(G)).$$

Using Goldestine's theorem, we can assume that $m_\gamma \in L^1(G) \otimes_p L^1(G)$ and two above limits hold in the weak topology of, respectively, $L^1(G) \otimes_p L^1(G)$ and $\mathbb{C}$(with $\pi^{**}_{L^1(G)}$ replaced by $\pi_{L^1(G)}$ and $\tilde{\phi}$ replaced by ϕ). Now by Mazur's lemma we may assume that (m_γ) is a net in $L^1(G) \otimes_p L^1(G)$ which

$$a \cdot m_\gamma - m_\gamma \cdot a \to 0, \quad \phi \circ \pi_{L^1(G)}(m_\gamma) \to 1, \quad (a \in L^1(G)).$$

Let $\phi \in \Delta(L^1(G))$ and a_0 be an element in $L^1(G)$ which $\phi(a_0) = 1$. Define $T : L^1(G) \otimes_p L^1(G) \to L^1(G) \otimes_p L^1(G)$ by $T(a \otimes b) = \phi(b) a \otimes a_0$ for all $a, b \in A$. Clearly T is a bounded linear map which satisfies

$$aT(b) = T(ab) = \phi(b)T(a), \quad \phi \circ \pi_{L^1(G)} \circ T(x) = \phi \circ \pi_{L^1(G)}(x) \quad (a, b \in A, x \in A \otimes_p A).$$

Put $n_\gamma = T(m_\gamma) \in L^1(G) \otimes_p L^1(G)$. It is easy to see that

$$a \cdot n_\gamma - \phi(a) \cdot n_\gamma \to 0, \quad \phi \circ \pi_{L^1(G)}(n_\gamma) = \phi \circ \pi_{L^1(G)}(m_\gamma) \to 1, \quad (a \in L^1(G)).$$

It follows that $L^1(G)$ is ϕ-pseudo-amenable. Thus by [7, Theorem 3.1] G is amenable. □

3. **Examples and applications**

In this section we give some examples among matrix algebras and semigroup algebras which show the differences between our new notion and the classical ones.
Note that although I-biprojectivity implies approximate I-biprojectivity but the converse is not valid, as the following example shows.

Example 3.1. *Let ℓ^2 be the Banach sequence algebra with pointwise multiplication. Set $I = \{(a_n)_{n=1}^\infty \in \ell^2 : a_{2n} = 0, \quad \forall n \in \mathbb{N}\}$. It is easy to see that I is a closed ideal in ℓ^2. We claim that I is not biprojective. We assume conversely that I is biprojective. Thus there exists a bounded I-bimodule morphism $\rho : I \to I \otimes_p I$ such that $\pi_I \circ \rho(x) = x$ for all*

$x \in I$. Set δ_{2n+1} for an element of I which is 1 at $2n+1$ and 0 elsewhere. Clearly we have $\rho(\delta_{2n+1}) = \delta_{2n+1}\rho(\delta_{2n+1})\delta_{2n+1}$. So for each $x = \sum_{n=1}^{\infty} \alpha_{2n+1}\delta_{2n+1}$ in I it follows that

$$\rho(x) = \sum_{n=1}^{\infty} \alpha_{2n+1}\delta_{2n+1} \otimes \delta_{2n+1}.$$

We can identify $BL(I, I)$(the set of all bounded linear operator from I into I) with $(I \otimes_p I)^*$. So $Id_I \in (I \otimes_p I)^*$. Now we have

$$|Id_I(\rho(x))| \leq ||Id_I||||\rho||||x|| \leq ||\rho||||x||.$$

On the other hand $Id_I(\delta_{2n+1} \otimes \delta_{2n+1}) = 1$. It gives that $Id_I(\rho(x)) = \sum_{n=1}^{\infty} \alpha_{2n+1}$. It means that for each $x = \sum_{n=1}^{\infty} \alpha_{2n+1}\delta_{2n+1} \in I$, $\sum_{n=1}^{\infty} \alpha_{2n+1}$ converges which is impossible.
Similar to the above arguments one can show that ℓ^2 is not I-biprojective.
We claim that ℓ^2 is approximately I-biprojective. To see this, it is known that ℓ^2 is pseudo-contractible, that is, there exists a net (m_α) in $\ell^2 \otimes_p \ell^2$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_{\ell^2}(m_\alpha)a \to a$, for all $a \in \ell^2$. Define $\rho_\alpha : I \to \ell^2 \otimes_p \ell^2$ by $\rho_\alpha(x) = x \cdot m_\alpha$ for all $x \in I$. Clearly ρ_α is a net of ℓ^2-bimodule morphisms and $\pi_{\ell^2} \circ \rho_\alpha(x) \to x$ for all $x \in \ell^2$. So ℓ^2 is approximatly I-biprojective.

Lemma 3.1. *Let A be a Banach algebra. If A is approximately A-biprojective, then $\overline{A^2} = A$.*

Proof. We assume in contradiction that $\overline{A^2} \neq A$. Applying Hahn-Banach theorem, there exists a functional $f \in A^*$ such that $f(a_0) = 1$ and $f(\overline{A^2}) = \{0\}$. Since A is approximately biprojective, there exists a net of A-bimodule morphisms from A into $A \otimes_p A$ such that $\pi_A \circ \rho_\alpha(a) \to a$, for each $a \in A$. We know that $\pi_A \circ \rho_\alpha(a)$ is a net in A^2. Thus $0 = f(\pi_A \circ \rho_\alpha(a_0)) \to f(a_0) = 1$ which is a contradiction. □

In the sequel, we give a non biprojective Banach algebra A which posses a non biprojective closed ideal I for which A is approximately I-biprojective.

Example 3.2. *Let $A = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$ and $I = \left\{ \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} : z \in \mathbb{C} \right\}$. Clearly with matrix operations and ℓ^1-norm A becomes a Banach algebra and I is a closed ideal in A. Note that A is not biprojective, see* [6]. *Also since $I^2 = \{0\}$, by Lemma 3.1 I is not biprojective. Define $\rho : I \to A \otimes_p A$ by*

$$\rho\left(\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Clearly ρ is a bounded A-bimodule morphism and $\pi_A \circ \rho(i) = i$ for all $i \in I$. So A is approximately I-biprojective.

The above example shows that approximate I-biprojectivity is different from biprojectivity. A semigroup S is called an *inverse semigroup*, if for each $s \in S$ there exists a unique $s^* \in S$ such that $ss^*s = s^*$ and $s^*ss^* = s$. There exists a partial order on each inverse semigroup S, that is, $s \leq t \Leftrightarrow s = ss^*t \quad (s, t \in S)$. Let $(S, \leq)$ be an inverse semigroup. For each $s \in S$, set $(x] = \{y \in S : y \leq x\}$. S is called *uniformly locally finite* if $\sup\{|(x]| : x \in S\} < \infty$. Suppose that S is an inverse semigroup and $e \in E(S)$, where $E(S)$ is the set of all idempotents of S. Then $G_e = \{s \in S : ss^* = s^*s = e\}$ is a maximal subgroup of S with respect to e. See [4] as a main reference of semigroup theory. Ramsden in [9] showed that for a uniformly locally finite inverse semigroup S with the collection of all D-classes, say $\{D_\lambda : \lambda \in \Gamma\}$, $\ell^1(S)$ is isometrically isomorphic with $\ell^1 - \oplus M_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$, where G_{p_λ} is a maximal group and $M_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ is a usual matrix algebra over $\ell^1(G_{p_\lambda})$ which belongs to the class of ℓ^1-Munn algebras. It is easy to see that $M_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ is a closed ideal in $\ell^1(S)$.

Example 3.3. *Let S be a uniformly locally finite inverse semigroup. Then $\ell^1(S)$ is approximately $M_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$-biprojective if and only if G_{p_λ} is finite. To see this, let $\ell^1(S)$ be approximately $M_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$-biprojective. Since $\ell^1(G_{p_\lambda})$ is unital then $M_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ has an approximate identity, namely*

$$\{\sum_{k\in F}\delta_{e_\lambda}E_{kk}\}_{F\subseteq E(D_\lambda)},$$

where e_λ is the identity element of the group G_{p_λ} and E_{kk}'s are the matrix units in $M_{E(D_\lambda)}(\mathbb{C})$ and F is a finite subset of $E(D_\lambda)$. So Theorem 2.1 follows that $M_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ is approximately biprojective. Applying [11, Lemma 3.5] *implies that $\ell^1(G_{p_\lambda})$ is approximately biprojective. So G_{p_λ} is compact (and discrete). Thus it is finite. For converse, let G_{p_λ} be finite. Then by* [9, Propositin 2.4] *and* [9, Propositin 2.7] *$M_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))\cong M_{E(D_\lambda)}(\mathbb{C})\otimes_p\ell^1(G_{p_\lambda})$ is biprojective. So $M_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ is approximately biprojective. Now applying Theorem 2.1, we can see that $\ell^1(S)$ is approximately $M_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$-biprojective.*

The last example shows that approximate I-biprojectivity is far from approximate biprojectivity.

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