A COMMON FIXED POINT OF MULTIVALUED MAPS IN EXTENDED b-METRIC SPACE WITH APPLICATION VOLTERRA-TYPE INTEGRAL INCLUSION

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In this work we are interested to prove a general fixed point theorem for a pair of multivalued mappings in extended $b-$metric spaces. The results in this paper generalize the results obtained in [7], [12],[17] *and to obtain other particular results with application Volterra-type integral inclusion.*

Keywords: Metric space, $b-$metric space, extended $b-$metric space, multivalued maps, fixed point.

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1. Introduction and Preliminary

Since the famous Banach fixed point theorem, the study of fixed point theory in metric spaces has several applications in mathematics, especially in solving differential and functional equations. Many authors have introduced a new class of generalized metric space, in particular those called b-metric spaces, and obtained several results in fixed point theory,(see [1],[3]-[16]).

The one due to I. A. Bakhtin [2] and S. Czerwik [4], [5] who, motivated by the problem of the convergence of measurable functions with respect to measure, introduced b-metric spaces (a generalization of metric spaces) and proved the contraction principle in this framework.

Kamran et al. [13] introduced the concept of extended b-metric space by further weakening the triangle inequality.

In this work we are interested to prove a general fixed point theorem for a pair of multivalued mappings in extended $b-$metric spaces with application Volterra-type integral inclusion.

Definition 1.1 ([13]). *Let X be a nonempty set and $\theta : X \times X \to [1,\infty)$. A function $d_\theta : X \times X \to \mathbb{R}^+$ is said to be a extended b-metric on X if the following conditions hold:*

(i) $d_\theta(x,y) = 0$ *if and only if* $x = y$,

(ii) $d_\theta(x,y) = d_\theta(y,x)$ *for all* $x,y \in X$,

(iii) $d_\theta(x,y) \leq \theta(x,y)[d_\theta(x,z) + d_\theta(z,y)]$ *for all* $x,y,z \in X$.

Note that every b-metric space is a extended $b-$metric space with $\theta = s \geq 1$.

Example 1.1. *Let $X = \{1,2,3\}$ We define $\theta : X \times X \longrightarrow [1,\infty)$ and $d_\theta : X \times X \longrightarrow \mathbb{R}^+$ such that:*

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$$\theta(x,y)=1+x+y$$
$$d_\theta(1,1)=d_\theta(2,2)=d_\theta(3,3)=0$$
$$d_\theta(1,2)=d_\theta(2,1)=8,\ d_\theta(1,3)=d_\theta(3,1)=10,\ d_\theta(2,3)=d_\theta(3,2)=6$$

Proof *(i) and (ii) trivially hold. For (iii) we have:*

$$d_\theta(1,2)=8,\ \theta(1,2)[d_\theta(1,3)+d_\theta(3,2)]=4(10+6)=64$$
$$d_\theta(1,3)=10,\ \theta(1,3)[d_\theta(1,2)+d_\theta(2,3)]=5(8+6)=70$$
$$d_\theta(2,3)=6,\ \theta(2,3)[d_\theta(2,1)+d_\theta(1,3)]=6(8+10)=108.$$

So, for all $x,y,z\in X$ we have: $d_\theta(x,y)\le\theta(x,y)[d_\theta(x,z)+d_\theta(z,y)]$.

Then (X,d_θ) is a extended b-metric space.

Definition 1.2 ([13])**.** *Let (X,d_θ) be a extended $b-$metric space, $x\in X$ and (x_n) be a sequence in X. Then*

(i) *(x_n) converges to x if and only if $\lim_{n\to\infty} d_\theta(x,x_n)=0$. We denote this by $x_n\to x$ $(n\to\infty)$ or $\lim_{n\to\infty} x_n=x$.*
(ii) *(x_n) is Cauchy if and only if $\lim_{n,m\to\infty} d_\theta(x_n,x_m)=0$.*
(iii) *(X,d_θ) is complete if and only if every Cauchy sequence in X is convergent.*
(iv) *A subset $A\subset X$ is said to be closed if for every sequence $x_n\in A$ such that $x_n\longrightarrow x$ we have $x\in A$.*
(v) *A subset $A\subset X$ is said to be bounded if $\sup_{x,y\in A} d_\theta(x,y)<+\infty$.*
(vi) *A subset $A\subset X$ is said to be compact if every sequence $x_n\in A$ has a convergent subsequence.*

We denote by $B(X)$ the set of nonempty closed bounded subsets of X provided with the Hausdorff-Pompeiu metric H_θ defined by $H_\theta(A,B)=\max\left(\sup_{x\in A} d_\theta(x,B),\sup_{y\in B} d_\theta(y,A)\right)$, we define also $\theta(A,B)$ by $\theta(A,B)=\max\left(\sup_{x\in A}\theta(x,B),\sup_{y\in B}\theta(y,A)\right)$, where $\theta(x,B)=\inf_{y\in B}\theta(x,y)$.

Given $F,G:X\longrightarrow B(X)$, for $c,d\in[0,1]$ and $x,y\in X$, we shall use the following notation:

$N_\theta(x,y)=max\{d_\theta(x,y),cd_\theta(x,Fx),cd_\theta(y,Gy),\frac{d}{2}(d_\theta(x,Gy)+d_\theta(y,Fx))\}$

for a sequence (x_n), of elements from X, sometimes, for the sake of brevity, we shall use the notation: $d_n=d_\theta(x_n,x_{n+1})$, where $n\in\mathbb{N}$.

Definition 1.3 ([17])**.** *A function $F:X\longrightarrow B(X)$, where (X,d_θ) is a extended b-metric space, is called closed if for all sequences (x_n) and (y_n) of elements from X and $x,y\in X$ such that $\lim_{n\to\infty} x_n=x$, $\lim_{n\to\infty} y_n=y$ and $y_n\in F(x_n)$ for every $n\in\mathbb{N}$, we have $y\in F(x)$.*

Definition 1.4 ([17])**.** *Given a extended b-metric space (X,d_θ), the b-metric d_θ is called $*$-continnuous if for every $A\in B(X)$, every $x\in X$ and every sequence (x_n) of elements from X such that $\lim_{n\to\infty} x_n=x$, we have $\lim_{n\to\infty} d_\theta(x_n,A)=d_\theta(x,A)$.*

Lemma 1.1 ([17])**.** *Every sequence (x_n) of elements from a extended b-metric space (X,d) having the property that there exists $\gamma\in[0,1)$ such that*

$$d_\theta(x_{n+1},x_n)\le\gamma d_\theta(x_n,x_{n-1}),$$

for every $n\in\mathbb{N}$. If

$$\limsup_{n,m\to\infty}\theta(x_n,x_m)<\infty, \tag{1}$$

then (x_n) is Cauchy.

2. **Main results**

Theorem 2.1. *Let (X, d_θ) be a extended b-metric space and $F, G : X \to B(X)$ having the property that there exist $c, d \in [0, 1]$ and $k \in [0, 1)$ such that:*

$$H_\theta(Fx, Gy) \leq kN_\theta(x, y) \text{ for all } x, y \in X. \quad (2)$$

Then for every $x_0 \in X$, there exists $(x_n) \subset X$, $x_{2n+1} \in Fx_{2n}$ and $x_{2n} \in Gx_{2n-1}$ such that

$$d_\theta(x_{2n}, x_{2n+1}) \leq \beta N_\theta(x_{2n}, x_{2n-1}) \text{ and } d_\theta(x_{2n-1}, x_{2n}) \leq \beta N_\theta(x_{2n-2}, x_{2n-1}) \text{ where } \beta = \frac{2k}{1+k}.$$

If $\limsup_{n,m\to\infty} k\theta(x_n, x_m) < \frac{1}{d}$ then:

(a) $d_\theta(x_{n+1}, x_n) \leq \gamma d_\theta(x_n, x_{n-1})$, $\gamma \in [0, 1)$, for every $n \in \mathbb{N}^*$,
(b) (x_n) is Cauchy.

Proof. Let $x_0 \in X$ and $x_1 \in Fx_0$, us consider $\beta = \frac{2k}{1+k}$, then using (2), we have

$$d_\theta(x_1, Gx_1) \leq H_\theta(Fx_0, Gx_1) \leq kN_\theta(x_0, x_1).$$

According to the characterization of the lower bound we have for $\varepsilon = \frac{1-k}{1+k} H_\theta(Fx_0, Gx_1)$, there exists $x_2 \in Gx_1$ such that

$$\begin{aligned} d_\theta(x_1, x_2) &\leq H_\theta(Fx_0, Gx_1) + \frac{1-k}{1+k} H_\theta(Fx_0, Gx_1) \\ &= \frac{2}{1+k} H_\theta(Fx_0, Gx_1) \\ &\leq \frac{2k}{1+k} N_\theta(x_0, x_1) \\ &= \beta N_\theta(x_0, x_1) \end{aligned}$$

Since

$$d_\theta(x_2, Fx_2) \leq H_\theta(Fx_2, Gx_1) \leq kN_\theta(x_2, x_1).$$

According to the characterization of the lower bound we have for $\varepsilon = \frac{1-k}{1+k} H_\theta(Fx_2, Gx_1)$, there exists $x_3 \in Fx_2$ such that

$$\begin{aligned} d_\theta(x_2, x_3) &\leq H_\theta(Fx_2, Gx_1) + \frac{1-k}{1+k} H_\theta(Fx_2, Gx_1) \\ &= \frac{2}{1+k} H_\theta(Fx_2, Gx_1) \\ &\leq \frac{2k}{1+k} N_\theta(x_2, x_1) \\ &= \beta N_\theta(x_2, x_1) \end{aligned}$$

In the same there exists $x_4 \in Gx_3$ such that

$$d_\theta(x_3, x_4) \leq \beta N_\theta(x_2, x_3).$$

By recurrence, we construct a sequence (x_n) such that $x_{2n+1} \in Fx_{2n}$, and $x_{2n} \in Gx_{2n-1}$ which satisfies:

$$d_\theta(x_{2n}, x_{2n+1}) \leq \beta N_\theta(x_{2n}, x_{2n-1}) \text{ and } d_\theta(x_{2n-1}, x_{2n}) \leq \beta N_\theta(x_{2n-2}, x_{2n-1}), \quad n = 1, 2, 3, ... \quad (3)$$

We put $s = \sup \{\theta(x_n, x_m), \quad n, m \in \mathbb{N}\}$.

According to (3) we have:

$$\begin{array}{rcl} d_{2n} & \leq & \beta N_\theta(x_{2n}, x_{2n-1}) \\ & = & \beta \max\left\{ \begin{array}{c} d_\theta(x_{2n}, x_{2n-1}), cd_\theta(x_{2n}, Fx_{2n}), cd_\theta(x_{2n-1}, Gx_{2n-1}), \\ \frac{d}{2}(d_\theta(x_{2n}, Gx_{2n-1}) + d_\theta(x_{2n-1}, Fx_{2n})) \end{array} \right\} \\ & \leq & \beta \max\left\{ d_{2n-1}, cd_{2n}, cd_{2n-1}, \frac{d}{2} d_\theta(x_{2n-1}, x_{2n+1}) \right\} \\ & \leq & \beta \max\left\{ d_{2n-1}, cd_{2n}, cd_{2n-1}, \frac{ds}{2}(d_{2n-1} + d_{2n}) \right\} \text{ because } (\theta(x_{2n-1}, x_{2n+1}) \leq s) \\ & \leq & \beta \max\left\{ d_{2n-1}, \frac{ds}{2}(d_{2n-1} + d_{2n}) \right\}, \end{array}$$

for every $n \in \mathbb{N}^*$, where the justification of the last inequality is as follow :
if $\max\{d_{2n-1}, cd_{2n}, cd_{2n-1}, \frac{ds}{2}(d_{2n-1} + d_{2n})\} = cd_{2n}$, then we get that
$d_{2n} \leq \beta cd_{2n} \leq \beta d_{2n} < d_{2n}$,which is a contradiction.

Consequently, $d_{2n} \leq \beta d_{2n-1}$ or $d_{2n} \leq \beta \frac{ds}{2}(d_{2n-1} + d_{2n})$, i.e $d_{2n} \leq \beta d_{2n-1}$ or $d_{2n} \leq \frac{ds\beta}{2-ds\beta} d_{2n-1}$ for every $n \in \mathbb{N}^*$, thus $d_{2n} \leq \max\{\beta, \frac{ds\beta}{2-ds\beta}\} d_{2n-1}$, i.e

$$d_\theta(x_{2n+1}, x_{2n}) \leq \gamma d_\theta(x_{2n}, x_{2n-1}) \quad \forall n \in \mathbb{N}^*, \text{ where } \gamma = \max\{\beta, \frac{ds\beta}{2 - ds\beta}\} < 1. \tag{4}$$

Similarly we find:

$$d_\theta(x_{2n}, x_{2n-1}) \leq \gamma d_\theta(x_{2n-2}, x_{2n-1}) \quad \forall n \in \mathbb{N}^*. \tag{5}$$

According to (4) and (5) we have for everything $n \in \mathbb{N}^*$ $d_\theta(x_{n+1}, x_n) \leq \gamma d_\theta(x_n, x_{n-1})$.

Hence the sequence (x_n) satisfies (a). From Lemma 1.1 we deduce that it also satisfies (b).

Theorem 2.2. *Let (X, d_θ) be a complete extended $b-$metric and $F, G : X \longrightarrow B(X)$, such that:*

(i) *there exist $c, d \in [0, 1]$ and $k \in [0, 1)$ such that $H_\theta(Fx, Gy) \leq kN_\theta(x, y)$ for all $x, y \in X$,*

(ii) *for every $x_0 \in X$, there exists $(x_n) \subset X$, $x_{2n+1} \in Fx_{2n}$ and $x_{2n} \in Gx_{2n-1}$ for every $n \in \mathbb{N}$.*

If $\limsup_{n,m\to\infty} k\theta(x_n, x_m) < \frac{1}{d}$, and any of the following conditions are satisfied:

(iii) *F and G are closed,*

(iv) *d_θ is $*$-continuous.*

Then F and G have a common fixed point $x \in X$.
Moreover, if x is absolutely fixed for F or G (which means that $F(x) = \{x\}$ or $G(x) = \{x\}$), then the fixed point is unique.

proof.

Existence.

Based on (i) and (ii), according to Theorem 2.1, there exists a Cauchy sequence (x_n) of elements of X such that:

$$x_{2n+1} \in Fx_{2n} \;\; and \;\; x_{2n} \in Gx_{2n-1} \qquad for\ every\ n \in \mathbb{N}. \tag{6}$$

As the extended b-metric space (X, d_θ) is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

(iii) Suppose that F and G are closed, according to (6) we see that $x \in Fx$ and $x \in Gx$, i.e F and G have a common fixed point $x \in X$.

(iv) Suppose that d_θ is $*$-continuous, according to (6), with the notation $d_\theta(x_n, x) = \delta_n$ we have

$$\begin{aligned} d_\theta(x_{2n+1}, Gx) &\leq H_\theta(Fx_{2n}, Gx) \leq kN_\theta(x_{2n}, x) \\ &\leq k \ \max\{\delta_{2n}, cd_\theta(x_{2n}, x_{2n+1}), cd_\theta(x, Gx), \frac{d}{2}(d_\theta(x_{2n}, Gx) + d_\theta(x, x_{2n+1}))\} \\ &\quad \text{because } x_{2n+1} \in Fx_{2n} \end{aligned}$$

$$= k \ \max\{\delta_{2n}, cd_{2n}, cd_\theta(x, Gx), \frac{d}{2}(d_\theta(x_{2n}, Gx) + \delta_{2n+1})\}, \tag{7}$$

for every $n \in \mathbb{N}$.

Since $\lim\limits_{n\to\infty} \delta_{2n+1} = \lim\limits_{n\to\infty} \delta_{2n} = \lim\limits_{n\to\infty} d_{2n} = 0$ and $\lim\limits_{n\to\infty} d_\theta(x_{2n}, Gx) = \lim\limits_{n\to\infty} d_\theta(x_{2n+1}, Gx) = d_\theta(x, Gx)$ (as d is $*$-continuous and $d_{2n} \leq \theta(x_{2n}, x_{2n+1})(\delta_{2n} + \delta_{2n+1})$ and $\lim\limits_{n\to\infty} x_n = x$), letting $n \longrightarrow \infty$ in (7), we get $d_\theta(x, Gx) = 0$, because if $d_\theta(x, Gx) > 0$, then

$$\begin{aligned} d_\theta(x, Gx) &\leq k \ max\{cd_\theta(x, Gx), \frac{d}{2} d_\theta(x, Gx)\} \\ &\leq max\{kc, \frac{kd}{2}\} d_\theta(x, Gx) \\ &< d_\theta(x, Gx) \quad because \quad max\{kc, \frac{kd}{2}\} < 1, \end{aligned}$$

which is a contradiction, hence $x \in Gx$ and G has a fixed point.
In the same way we find: $x \in Fx$ and consequently F and G have a common fixed point $x \in X$.

Unicity.
Suppose that $F(x) = \{x\}$ and $y \in X$ is another common fixed point of F and G, then by (i) we have

$$\begin{aligned} d_\theta(x, y) \leq H_\theta(Fx, Gy) &\leq kN_\theta(x, y) \\ &\leq k \ \max\{d_\theta(x, y), \frac{d}{2}(d_\theta(x, y) + d_\theta(y, x))\}, \ \ because \ y \in Gy \\ &\leq k \ \max\{d_\theta(x, y), d_\theta(x, y)\} \ \ because \ d \leq 1 \\ &= k \ d_\theta(x, y) < d_\theta(x, y). \end{aligned}$$

which is a contradiction. Hence $d_\theta(x, y) = 0$ then $x = y$.

So x is the unique common fixed point of F and G.

Example 2.1. *Let* $(X = [0, 1], d_\theta)$ *be a complete extended* $b-$*metric space with* $\theta(x, y) = x + y + 2$ *and* $d_\theta(x, y) = |x - y|^2$. *We define* $F, G : X \longrightarrow B(X)$, *by* $Fx = \left[0, \frac{x}{4}\right]$, $Gx = \left[0, \frac{x}{8}\right]$
and
$d_\theta(x, Fx) = |x - \frac{x}{4}|^2$ $\quad d_\theta(y, Gy) = |y - \frac{y}{8}|^2$ $\quad H_\theta(Fx, Gy) = |\frac{x}{4} - \frac{y}{8}|^2$.
(i) *It is easy to see that* F *and* G *are closed.*
(ii) *We prove that* F *and* G *check*

$$\begin{aligned} H_\theta(Fx, Gy) &\leq \frac{1}{8} \max\left\{d_\theta(x, y), d_\theta(x, Fx), d_\theta(y, Gy), \frac{1}{2}(d_\theta(x, Gy) + d_\theta(y, Fx))\right\}. \\ &\leq \frac{1}{8} N_{1,1}(x, y). \end{aligned}$$

Indeed, we have the following situations:

1) If $x \leq \frac{y}{2}$, then $|\frac{x}{4} - \frac{y}{8}| = \frac{y}{8} - \frac{x}{4} = \frac{1}{4}(\frac{y}{2} - x) \leq \frac{1}{4}|y - x|$, from where

$$|\frac{x}{4} - \frac{y}{8}|^2 \leq \frac{1}{16} d_\theta(x,y) \leq \frac{1}{8} \max\left\{d_\theta(x,y), d_\theta(x,Fx), d_\theta(y,Gy), \frac{1}{2}(d_\theta(x,Gy) + d_\theta(y,Fx))\right\}.$$

2) If $x \geq \frac{y}{2}$, we have $d_\theta(x,Gy) = |x - \frac{y}{8}|^2$. Then

$$\begin{aligned} |\frac{x}{4} - \frac{y}{8}| &= \frac{x}{4} - \frac{y}{8} = \frac{1}{4}(x - \frac{y}{2}) \\ &\leq \frac{1}{4}|x - \frac{y}{8}|, \end{aligned}$$

from where

$$\begin{aligned} |\frac{x}{4} - \frac{y}{8}|^2 &\leq \frac{1}{16} d_\theta(x,Gy) \leq \frac{1}{8}(\frac{1}{2}(d_\theta(x,Gy) + d_\theta(y,Fx))) \\ &\leq \frac{1}{8} \max\left\{d_\theta(x,y), d_\theta(x,Fx), d_\theta(y,Gy), \frac{1}{2}(d_\theta(x,Gy) + d_\theta(y,Fx))\right\}. \end{aligned}$$

This implies

$$\begin{aligned} H_\theta(Fx,Gy) &\leq \frac{1}{8} \max\left\{d_\theta(x,y), d_\theta(x,Fx), d_\theta(y,Gy), \frac{1}{2}(d_\theta(x,Gy) + d_\theta(y,Fx))\right\}, \\ &\leq \frac{1}{8} N_{1,1}(x,y) \qquad \text{for all } x,y \in X. \end{aligned}$$

(iii) For every $x_0 \in X$ we have $\lim_{n,m\to\infty} \frac{1}{8}\theta(x_n,x_m) = \lim_{n,m\to\infty} \frac{1}{8}(x_n + x_m + 2) \leq \frac{4}{8} < 1$.

So all the conditions of Theorem 2.2 are satisfied, then 0 is the unique common absolutely fixed point of F and G.

Corollary 2.1. *Let (X,d_θ) be a complete extended $b-$metric space where the condition (1) is fulfilled and $F,G : X \longrightarrow B(X)$, such that:*

$$H_\theta(Fx,Gy) \leq k d_\theta(x,y) \text{ for all } x,y \in X, \tag{8}$$

where $k \in [0,1)$. Then there exists a sequence $(x_n) \subset X$ converges to some point $x \in X$ such that $x_{2n+1} \in Fx_{2n}$ and $x_{2n} \in Gx_{2n-1}$ for every $n \in \mathbb{N}$. Also, x is a common fixed point of F and G if any of the following conditions are satisfied:

(i) *F and G are closed,*

(ii) *d_θ is $*$-continuous.*

Moreover, if x is absolutely fixed for F or G (which means that $F(x) = \{x\}$ or $G(x) = \{x\}$), then the fixed point is unique.

proof. (8) $\Rightarrow$ (2), with $c = d = 0$. Let $x_0 \in X$ and $x_1 \in Fx_0$, let us consider $\beta = \frac{2k}{1+k} = \gamma$, then there exists a sequence $(x_n) \subset X$ such that for every $n \in \mathbb{N}^*$ $\quad d_\theta(x_{n+1},x_n) \leq \gamma d_\theta(x_n,x_{n-1})$. According to lemma 1.1 we have (x_n) is a Cauchy sequence, so using the same argument as in theorem 2.2, we deduce that $x \in Fx \cap Gx$. Similarly from theorem 2.2, if x is absolutely fixed for F or G, then the fixed point is unique.

3. **Consequences of the main result**

From theorem 2.1, with $\theta = s$ constant we obtain theorem 3.1 [7]
From theorem 2.2, with $\theta = s$ constant we obtain theorem 3.3 and theorem 3.6 [7]
From theorem 2.1, if $F = G = T$, with $\theta = s$ constant we obtain theorem 2.1 [12]
From theorem 2.2, if $F = G = T$, with $\theta = s$ constant we obtain theorem 3.2 [12]
From corollary 2.1, if $F = G = T$, then we obtain theorem 4.8 [17]

4. Application

In this section, we give existence theorem for Volterra-type integral inclusion. Let $X = C([0,1],\mathbb{R})$ be the set of real continuous functions defined on $[0,1]$. For $x, y \in X$, take $d_\theta : X \times X \to \mathbb{R}^+$ and $\theta : X \times X \to [1,\infty)$ given by

$$d_\theta(x,y) = \sup_{t\in[0,1]} |x(t) - y(t)|^2 \text{ and } \theta(x,y) = |x(t)| + |y(t)| + 2. \tag{9}$$

Then (X, d_θ) is a complete extended b-metric space.

Consider the Volterra-type integral inclusion as

$$x(t) \in \int_0^t F(t,u,x(u))du + g(t), \quad t \in [0,1], \tag{10}$$

where $F : [0,1] \times [0,1] \times \mathbb{R} \longrightarrow CV(\mathbb{R})$, such that $F_x(t,u) = F(t,u,x(u))$ is continuous function for all $(t,u) \in [0,1] \times [0,1]$, $x \in X$ and $CV(\mathbb{R})$ denotes the family of nonempty compact and convex subsets of $\mathbb{R}$ and $g : [0,1] \to \mathbb{R}$ is continuous.

We can define a multivalued operator T: $X \longrightarrow CV(X)$ by

$$Tx(t) = \left\{ v \in X : v \in \int_0^t F(t,u,x(u))du + g(t), \quad t \in [0,1] \right\}. \tag{11}$$

Suppose that the following condition is satisfied :
(H) For all $x(.), y(.) \in X$, $t, u \in [0,1]$, we have

$$H_\theta\left(F_x(t,u), F_y(t,u)\right) \le e^{-\tau}|x(u) - y(u)|^2 \text{ where } \tau > 0.$$

Then the integral inclusion (10) has a solution in X.

proof. We have to show that the operator T satisfies all conditions of corollary 2.1.

(i) Let $x, y \in X$ and $v \in Tx$ then there exists $f_x(t,u) \in F_x(t,u)$, for $t, u \in [0,1]$ such that:

$$v(t) = g(t) + \int_0^t f_x(t,u)du \in Tx(t), \quad t \in [0,1].$$

Also by hypothesis (H),

$$H_\theta\left(F_x(t,u), F_y(t,u)\right) \le e^{-\tau}|x(u) - y(u)|^2, \quad \forall t, u \in [0,1].$$

Then

$$d_\theta\left(f_x(t,u), F_y(t,u)\right) \le H_\theta\left(F_x(t,u), F_y(t,u)\right) \le e^{-\tau}|x(u) - y(u)|^2, \quad \forall t, u \in [0,1].$$

Since d_θ is continuous and F_y is compact then there exists $f(t,u) \in F_y(t,u)$ such that:

$$d_\theta\left(f_x(t,u), f(t,u)\right) \le e^{-\tau}|x(u) - y(u)|^2, \quad \forall t, u \in [0,1],$$

from where

$$|f_x(t,u) - f(t,u)|^2 \le e^{-\tau}|x(u) - y(u)|^2, \quad \forall t, u \in [0,1].$$

Define a multivalued operator R by

$$R(t,u) = F_y(t,u) \cap \left\{ w \in \mathbb{R}, |f_x(t,u) - w|^2 \le e^{-\tau}|x(u) - y(u)|^2 \right\},$$

for all $t, u \in [0,1]$. Since R is continuous operator with compact convex values, there exists a continuous operator $f_y : [0,1] \times [0,1] \to \mathbb{R}$ such that $f_y(t,u) \in R(t,u)$ for all $t, u \in [0,1]$. Thus we get

$$h(t) = g(t) + \int_0^t f_y(t,u)du \in Ty(t), \quad t \in [0,1],$$

and for each $t \in [0,1]$, we have

$$\begin{aligned} d_\theta(v(t), Ty(t)) &\leq d_\theta(v(t), h(t)) = \sup_{t\in[0,1]} |v(t) - h(t)|^2 \\ &\leq \sup_{t\in[0,1]} \left(\int_0^t |f_x(t,u) - f_y(t,u)| du \right)^2 \\ &\leq \sup_{t\in[0,1]} \left[\left(\int_0^t du \right)^{\frac{1}{2}} \left(\int_0^t |f_x(t,u) - f_y(t,u)|^2 du \right)^{\frac{1}{2}} \right]^2 \\ &= \sup_{t\in[0,1]} t. \int_0^t |f_x(t,u) - f_y(t,u)|^2 du \leq e^{-\tau} \sup_{t\in[0,1]} \int_0^t |x(u) - y(u)|^2 du \\ &\leq kd_\theta(x(t), y(t)), \text{ where } k = e^{-\tau} \in (0,1). \end{aligned}$$

Since $v(t)$ is arbitrary, we have

$$\sup_{v\in Tx} d_\theta(v, Ty) \leq kd_\theta(x,y), \quad \forall x, y \in X. \tag{12}$$

Similarly, we get

$$\sup_{h\in Ty} d_\theta(h, Tx) \leq kd_\theta(x,y), \quad \forall x, y \in X. \tag{13}$$

From (12) and (13), we get $H_\theta(Tx, Ty) \leq kd_\theta(x,y), \quad \forall x, y \in X$.

(ii) Let's show that T is closed.

Let $x, y \in X$ and (x_n), (y_n) two sequences of element X with $y_n \in Tx_n$ such that $\lim_{n\to\infty} x_n = x$ et $\lim_{n\to\infty} y_n = y$. We prove that $y \in Tx$.

We have $y_n \in Tx_n$ then there exists $f_{x_n}(t,u) \in F_{x_n}(t,u)$ for everything $t, u \in [0,1]$ such that

$$y_n(t) = g(t) + \int_0^t f_{x_n}(t,u) du \in Tx_n(t), \quad t \in [0,1].$$

According to the hypothesis (H), $\forall t, u \in [0,1]$ we get:

$$\begin{aligned} d_\theta\left(f_{x_n}(t,u), F_x(t,u)\right) &\leq H_\theta\left(F_{x_n}(t,u), F_x(t,u)\right) \leq e^{-\tau}|x_n(u) - x(u)|^2, \\ &\leq e^{-\tau} d_\theta\left(x_n, x\right). \end{aligned}$$

Since d_θ is continuous and F_x is compact then there exists $f(t,u) \in F_x(t,u)$ such that:

$$d_\theta\left(f_{x_n}(t,u), f(t,u)\right) \leq e^{-\tau} d_\theta\left(x_n, x\right), \quad \forall t, u \in [0,1].$$

So $\lim_{n\to\infty} f_{x_n}(t,u) = f(t,u)$.

We put $h(t) = g(t) + \int_0^t f(t,u) du \in Tx(t), \quad t \in [0,1]$.

for all $t \in [0,1]$, we have

$$\begin{aligned} d_\theta(y_n(t), h(t)) &= \sup_{t\in[0,1]} |y_n(t) - h(t)|^2 \\ &\leq \sup_{t\in[0,1]} \left(\int_0^t |f_{x_n}(t,u) - f(t,u)| du \right)^2 \\ &\leq \sup_{t\in[0,1]} \left[\left(\int_0^t du \right)^{\frac{1}{2}} \left(\int_0^t |f_{x_n}(t,u) - f(t,u)|^2 du \right)^{\frac{1}{2}} \right]^2 \\ &\leq d_\theta\left(f_{x_n}(t,u), f(t,u)\right). \end{aligned}$$

Therefore $\lim_{n\to\infty} y_n(t) = h(t)$, $\forall t \in [0,1]$. By the uniqueness of the limit, we find $y(t) = h(t) \in Tx(t)$. $\forall t \in [0,1]$. From where $y \in Tx$.

(iii) Let $x_0 \in X$ and $x_1 \in Tx_0$ then there exists $f_{x_0}(t,u) \in F_{x_0}(t,u)$, for $t,u \in [0,1]$ such that:

$$x_1(t) = g(t) + \int_0^t f_{x_0}(t,u)du \in Tx_0(t), \quad t \in [0,1].$$

Also by hypothesis (H),

$$H_\theta\left(F_{x_0}(t,u), F_{x_1}(t,u)\right) \leq e^{-\tau}|x_0(u) - x_1(u)|^2, \quad \forall t,u \in [0,1].$$

Then

$$d_\theta\left(f_{x_0}(t,u), F_{x_1}(t,u)\right) \leq H_\theta\left(F_{x_0}(t,u), F_{x_1}(t,u)\right) \leq e^{-\tau}|x_0(u) - x_1(u)|^2, \quad \forall t,u \in [0,1].$$

Since d_θ is continuous and F_{x_1} is compact then there exists $f_1(t,u) \in F_{x_1}(t,u)$ such that:

$$d_\theta\left(f_{x_0}(t,u), f_1(t,u)\right) \leq e^{-\tau}|x_0(u) - x_1(u)|^2, \quad \forall t,u \in [0,1],$$

from where

$$|f_{x_0}(t,u) - f_1(t,u)| \leq e^{\frac{-\tau}{2}}|x(u) - y(u)|, \quad \forall t,u \in [0,1].$$

Define a multivalued operator R_1 by

$$R_1(t,u) = F_{x_1}(t,u) \cap \left\{w_1 \in \mathbb{R}, |f_{x_0}(t,u) - w| \leq e^{\frac{-\tau}{2}}|x_0(u) - x_1(u)|\right\},$$

for all $t,u \in [0,1]$. Since R_1 is continuous operator with compact convex values, there exists a continuous operator $f_{x_1} : [0,1] \times [0,1] \to \mathbb{R}$ such that $f_{x_1}(t,u) \in R_1(t,u)$ for all $t,u \in [0,1]$. Thus we get

$$x_2(t) = g(t) + \int_0^t f_{x_1}(t,u)du \in Tx_1(t), \quad t \in [0,1],$$

and for each $t \in [0,1]$, we have

$$|x_2(t) - x_1(t)| = \left|\int_0^t (f_{x_1}(t,u) - f_{x_0}(t,u))du\right| \leq e^{\frac{-\tau}{2}} \sup_{t\in[0,1]} |x_1(t) - x_0(t)|.$$

From where

$$\sup_{t\in[0,1]} |x_2(t) - x_1(t)| \leq e^{\frac{-\tau}{2}} \sup_{t\in[0,1]} |x_1(t) - x_0(t)|.$$

By recurrence, we construct a sequence (x_n) such that $x_{n+1} \in Tx_n$, which satisfies :

$$\sup_{t\in[0,1]} |x_{n+1}(t) - x_n(t)| \leq e^{\frac{-\tau}{2}} \sup_{t\in[0,1]} |x_n(t) - x_{n-1}(t)| \;\; n = 1,2,3,...$$

from where for everything $n \in \mathbb{N}^*$, we have

$$\sup_{t\in[0,1]} |x_{n+1}(t) - x_n(t)| \leq e^{\frac{-n\tau}{2}} \sup_{t\in[0,1]} |x_1(t) - x_0(t)| \tag{14}$$

Now we have to show that (x_n) is a Cauchy sequence. Let $m,n \in \mathbb{N}^*$, then

$$|x_n(t) - x_{n+m}(t)| \leq |x_n(t) - x_{n+1}(t)| + ... + |x_{n+m-1}(t) - x_{n+m}(t)|, \quad \forall t \in [0,1]. \tag{15}$$

According to (14) and (15) we have

$$\begin{aligned}
|x_n(t) - x_{n+m}(t)| &\leq \left(e^{\frac{-n\tau}{2}} + e^{\frac{-(n+1)\tau}{2}} + ... + e^{\frac{-(n+m-1)\tau}{2}}\right) \sup_{t\in[0,1]} |x_1(t) - x_0(t)| \\
&= e^{\frac{-n\tau}{2}} \left(\frac{1 - e^{\frac{-m\tau}{2}}}{1 - e^{\frac{-\tau}{2}}}\right) \sup_{t\in[0,1]} |x_1(t) - x_0(t)| \\
&\leq \frac{e^{\frac{-n\tau}{2}}}{1 - e^{\frac{-\tau}{2}}} \sup_{t\in[0,1]} |x_1(t) - x_0(t)|.
\end{aligned}$$

from where $\lim_{n\to\infty} |x_n(t) - x_{n+m}(t)| = 0$ for $m \in \mathbb{N}^*$. Then (x_n) is a Cauchy sequence. As (X, d_θ) is complete, there exists $l \in X$ such that $\lim_{n\to\infty} x_n(t) = l(t) \quad \forall t \in [0,1]$.

Consequently $\limsup_{n,m\to\infty} \theta(x_n, x_m) = \limsup_{n,m\to\infty}(|x_n(t)| + |x_m(t)| + 2) = 2|l(t)| + 2 < \infty$.

So all the conditions of corollary 2.1 are satisfied, then the integral inclusion (10) has a solution in X.

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REFERENCES

[1] *M. U. Ali, T. Kamran, M. Postolache*, Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem. Nonlinear Anal. Modelling Control **22**(2017), No. 1, 17-30.

[2] *I. A. Bakhtin*, The contraction mapping principle in almost metric spaces. 30. In Functional Analysis. Ul'yanovsk Gos. Ped. Inst., Ul'yanovsk, 1989:26-37.

[3] *M. Boriceanu, A. Petrusel, I.A. Rus* , Fixed point theorems for some multivalued generalized contractions in b-metric spaces, Int. J. Math. Stat., **6**(2010), 65-76.

[4] *S. Czerwik*, Nonlinear set valued contraction mappings in b-metric spaces, Atti Sem. Mat. Fiz. Univ. Modena, **46**(1998), 263-276.

[5] *S. Czerwik*, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostrav, **1**(1993), 5-11.

[6] *T. Kamran, M. Postolache, MU. Ali, Q. Kiran* : Feng and Liu type F-contraction in b-metric spaces with application to integral equations. J. Math. Anal. **7**(2016), No. 5, 18-27.

[7] *N. Makran, A. El Haddouchi, B. Marzouki*, A Generalized Common Fixed Point of Multi-Valued Maps in b-metric Space, Bol. Soc. Paran. Mat. v. 1–9 (2021).

[8] *N. Makran, A. El Haddouchi, B. Marzouki*, A common fixed point of multi-valued maps in b-Metric space. U.P.B. Sci. Bull., Series A, Vol. 82, Iss. 1, 2020.

[9] *N. Makran, A. El Haddouchi, B. Marzouki*, A generalized common fixed points for multivalued mappings in G_b-metric spaces with an Application . U.P.B. Sci. Bull., Series A, Vol. 83, Iss. 1, 2021.

[10] *B. Marzouki, A. El Haddouchi*, Common fixed point of multivalued maps. Nonliear Analysis and Differential Equations. **4** (2016), 1-7.

[11] *B. Marzouki, A. El Haddouchi*, A generalized fixed point theorem in G-metric space. Journal of Analysis and applications. Vol 17 N2(2019), 89 - 105.

[12] *R. Miculescu, A. Mihail*, New fixed point theorems for set-valued contraction in b-metric spaces, J. Fixed point Theory Appl.(2017).

[13] *T. Kamran, M. Samreen and Q. UL Ain,* A generalization of b-Metric space and some fixed point theorems, Mathematics, 5 (2017), 19.

[14] *M. Samreen, T. Kamran, M. Postolache,* Extended b-metric space, extended b-comparison function and nonlinear contractions, U. Politeh. Buch. Ser. A **80**(2018), No. 4, 21-28.

[15] *W. Shatanawi, A. Pitea, R. Lazovic,* : Contraction conditions using compa-rison functions on b-metric spaces, Fixed Point Theory Appl. Art. No. 135 (2014).

[16] *W. Shatanawi*, Fixed and common fixed point for mappings satisfying some nonlinear contractions in $b-$metric spaces. Journal of Mathematical Analysis, Volume 7 Issue 4(2016), 1-12.

[17] *Zoran D. Mitrović, Huseyin Isk, Stojan Radenović*, The new results in extended $b-$metric spaces and applications. Int. J. Nonlinear Anal. Appl. 11 (2020) No. 1, 473-482.