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NUMERICAL CALCULATION OF EIGEN FUNCTIONS AND ENERGY LEVELS OF QUANTUM WELLS, BASED ON BOUNDARY ELEMENT METHOD

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In this article, a numerical method based on Boundary Element Method (BEM) has been applied for the calculation of the eigenvalues and eigenfunctions of 2D and 3D quantum wells. First, the eigenvalue equation has been converted into boundary integral equation, using the Green's theorem. Then, a numerical method to calculate the eigenvalues of the differential equations is illustrated; and the eigenvalues of some kind of potential well such as a circular, elliptical and a 3D potential well are calculated. Then, the mutual effects of the two circular well have been analyzed. Afterwards, to obtain the eigenfunctions, we solved such integral equations, using BEM; and the corresponding eigenfunctions to an energy level of an elliptical well are calculated. A very good agreement was observed among the theoretical and numerical methods. The short performing time of calculations and high accuracy are the dominant characteristics of BEM in comparison with the other methods.

Keyword: Eigenvalue, Eigenfunction, Quantum Well, Quantum Dot, Boundary Element Method.

1. Introduction

Eigenvalue equations are among the group of equations, which have the variety of applications in sciences. Schrödinger equation in quantum mechanics, radiation in electrodynamics, propagation of light trough waveguides and photonic crystals, eigenfaces in image processing, vibration analysis and also studying stress tensor in solid mechanics are just few examples from the applications of these kinds of equations. Accordingly, solving this problem and finding its corresponding eigenvalues and eigenfunctions always are very important for scientists; Furthermore, they proposed lot's of methods to solve this kind of equations[1-16]. In this article, first, we report a method to solve the eigenvalue and eigenfunction of some differential equations base on Boundary Element Method. Then, using this method, some 2D and 3D quantum well such as single and double circular, elliptical and spherical potential well are solved and its energy levels and eigenfunctions are obtained.

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2. Theory

In this section, a method to solve the eigenvalue of some differential operators base on Boundary Element Method is illustrated. Consider $\psi(X)$ satisfy below differential equation:

$$\hat{L}\left(\psi(X)\right) = -f(X) \tag{1}$$

where $\hat{L}$ is a differential operator in the form bellow:

$$\hat{L} = \hat{L}' - \lambda = \vec{\nabla}.\left(p(X)\vec{\nabla}\right) + q(X) - \lambda \tag{2}$$

In this expression $p(X)$ and $q(X)$ are arbitrary functions and λ is a constant parameter, which can be considered as an eigenvalue of the operator $\hat{L}' = \hat{L} + \lambda$ if $f(X) = 0$. The corresponding Green equation to Eq. (1) is

$$\hat{L}\left(G_\lambda(X, X')\right) = -\delta(X - X') \tag{3}$$

Statement of this point is necessary that the operator $\hat{L}$ just apply on X components. The second form of modified Green theory expresses as[17]:

$$\begin{aligned}&\int_\Omega \Phi(X)\hat{L}\left(\psi(X)\right) - \psi(X)\hat{L}\left(\Phi(X)\right) d\Omega = \\ &\oint_\Gamma p(X)\left[\Phi(X)\frac{\partial\psi(X)}{\partial n} - \psi(X)\frac{\partial\Phi(X)}{\partial n}\right] d\Gamma\end{aligned} \tag{4}$$

Ω is a domain which is surrounded by closed boundary, Γ, and $\partial/\partial n$ is a normal derivative perpendicular to the boundary Γ. By considering Eqs. (1) and (3) and also considering $\Phi(X) = G_\lambda(X, X')$, the Eq. (4) can be rewritten as follows:

$$\begin{aligned}&\int_\Omega -G_\lambda(X, X')f(X) + \psi(X)\delta(X - X') d\Omega = \\ &\oint_\Gamma p(X)\left[G_\lambda(X, X')\frac{\partial\psi(X)}{\partial n} - \psi(X)\frac{\partial G_\lambda(X, X')}{\partial n}\right] d\Gamma\end{aligned} \tag{5}$$

Where, the points X and X' are known as the integration and observation points, respectively. By using the characteristics of Delta function, the second term in the left hand side of Eq. (5) will simplify to:

$$\int_\Omega \psi(X)\delta(X - X') d\Omega = C(X')\psi(X') \tag{6}$$

By choosing $f(X) = 0$ and putting $\psi(X) = 1$, Coefficient $C(X')$ will obtain as [18]:

$$C(X') = -\oint_\Gamma p(X)\frac{\partial G_\lambda(X, X')}{\partial n} d\Gamma \tag{7}$$

If the observation point, X', lies outside the boundary Γ, the numerical amount of $C(X')$ will vanish; and if it places inside, its amount will be equal to one; but, if it locates on the boundary, its amount should obtain by Eq. (7).

Substituting Eq. (6) into Eq. (5) yield:

$$C(X')\psi(X') = \int_{\Omega} f(X) G_{\lambda}(X, X') d\Omega + \oint_{\Gamma} p(X) \left[G_{\lambda}(X, X') \frac{\partial \psi(X)}{\partial n} - \psi(X) \frac{\partial G_{\lambda}(X, X')}{\partial n} \right] d\Gamma \tag{8}$$

Considering Eq. (8), it is obvious, providing that the amount of $\psi(X)$ and $\partial\psi(X)/\partial n$ be definite on the boundary Γ and also by choosing the appropriate Green function which satisfy Eq. (3), we will be able to calculate the amount of $\psi(X)$ on any arbitrary observation points. Therefore, solving Eq. (1) will shorten just to find the amount of $\psi(X)$ and $\partial\psi(X)/\partial n$ on the boundary Γ. As stated in introduction, in the most of the sciences such as mathematics, physics and engineering as well as in the most of the problems, finding the eigenvalues and eigenfunctions of a differential operator like $\hat{L}' = \hat{L} + \lambda$ are in demand. To reach this aim, we will investigate just equations, in which, $f(X) = 0$. These equations are well-known as homogeneous equations and express as follows:

$$\hat{L}(\psi(X)) = \hat{L}'(\psi(X)) - \lambda\psi(X) = 0 \tag{9}$$

In fact, this equation is an eigenvalue equation of operator $\hat{L}'$, and represent in this way:

$$\hat{L}'(\psi(X)) = \lambda\psi(X) \tag{10}$$

$\psi(X)$ and λ are eigenfunction and eigenvalue of operator $\hat{L}'$, respectively.

Choosing the appropriate $p(X)$ and $q(X)$ will lead to convert Eq. (10) to the set of well-known equation in physics, which is stated in table 1. By considering Eq. (8), the corresponding integral equation to Eq. (10) yields as follows:

$$C(X')\psi(X') = \oint_{\Gamma} p(X) \left[G_{\lambda}(X, X') \frac{\partial \psi(X)}{\partial n} - \psi(X) \frac{\partial G_{\lambda}(X, X')}{\partial n} \right] d\Gamma \tag{11}$$

Eq. (11) is an integral equation and should solve by considering the boundary conditions. In this equation, in addition to $\psi(X)$ and $\partial\psi(X)/\partial n$ on the boundary Γ, the amount of λ is undetermined as well.

Follows, we will investigate a problem that is applicable in photonic crystals, fiber optic photonic crystals, quantum potential wells and in micro optic elements.

Consider two domains, Ω_1 and Ω_2, which are separated from each other by the boundary Γ. The govern differential equation on $\psi(X)$ in these two regions are:

$$\hat{L}'\psi_1(X) = \lambda\psi_1(X) \ , \ X \in \Omega_1 \tag{12-a}$$

$$\hat{L}'\psi_2(X) = \lambda\psi_2(X) \ , \ X \in \Omega_2 \tag{12-b}$$

By considering the below boundary condition on the boundary Γ:

$$\psi_2(X) = \alpha\psi_1(X) \tag{13-a}$$

$$\frac{\partial\psi_2(X)}{\partial n} = \beta\frac{\partial\psi_1(X)}{\partial n} \tag{13-b}$$

and using the Eq. (11), the corresponding integral equations to the eigenvalue equations (Eq. (12)) will yield as follows:

$$C(X')\psi_1(X') = \oint_\Gamma p(X)\left[G_{1,\lambda}(X,X')\frac{\partial\psi_1(X)}{\partial n} - \psi_1(X)\frac{\partial G_{1,\lambda}(X,X')}{\partial n}\right]d\Gamma \tag{14-a}$$

$$C(X')\psi_2(X') = \oint_\Gamma p(X)\left[G_{2,\lambda}(X,X')\frac{\partial\psi_2(X)}{\partial n_2} - \psi_2(X)\frac{\partial G_{2,\lambda}(X,X')}{\partial n_2}\right]d\Gamma$$

$G_{1,\lambda}$ and $G_{2,\lambda}$ are Green functions of operator $\hat{L}$ in regions Ω_1 and Ω_2 respectively. Substituting boundary conditions (13) in recent equation and using $\partial/\partial n_2 = -\partial/\partial n$ will simplify it:

$$C(X')\psi_2(X') = \oint_\Gamma p(X)\left[-\beta G_{2,\lambda}(X,X')\frac{\partial\psi_1}{\partial n} + \alpha\psi_1(X)\frac{\partial G_{2,\lambda}(X,X')}{\partial n}\right]d\Gamma \tag{14-b}$$

To solve the coupled integral Eqs. (14-a) and (14-b), first, should specify the amount of the eigenvalue λ; then, calculate the numerical amount of $\psi_1(X)$ and $\partial\psi_1(X)/\partial n$ on the boundary Γ. To reach this goal, we will convert these equations to algebraic equations, using an ordinary approximation in BEM. In this method, the boundary Γ will divide to some finite boundary elements and also the integral over the boundary Γ will change to the integral on the each boundary elements by considering this equation:

$$\oint_\Gamma (.)d\Gamma = \sum_{i=1}^{N}\int_{\Gamma_i}(.)d\Gamma \tag{15}$$

N is the number of boundary elements on the boundary Γ. To use the Constant Element Approximation (CEA) [18], we consider one point on every boundary element as a node and show their positions by X_i. Also, we ignore the variations of $\psi_1(X)$ and $\partial\psi_1(X)/\partial n$ on a boundary element according to CEA. These approximations express as follows:

$$\psi_1(X)\Big|_{\Gamma_i} \approx \psi_1(X_i) = R_i \tag{16-a}$$

$$\left.\frac{\partial\psi_1(X)}{\partial n}\right|_{\Gamma_i} \approx \frac{\partial\psi_1(X_i)}{\partial n} = S_i \tag{16-b}$$

After meshing Γ and choosing one node on every element and using Eqs. (15) and (16), the Eqs. (14-a) and (14-b) will approximated to the forms below:

$$C(X')\psi_1(X') = \sum_{i=1}^{N}\left[S_i\int_{\Gamma_i} p(X)G_{1,\lambda}(X,X')d\Gamma - R_i\int_{\Gamma_i} p(X)\frac{\partial G_{1,\lambda}(X,X')}{\partial n}d\Gamma\right] \tag*{17-a)}$$

$$C(X')\psi_2(X') = \sum_{i=1}^{N}\left[-S_i\int_{\Gamma_i} \beta p(X)G_{2,\lambda}(X,X')d\Gamma + R_i\int_{\Gamma_i} \alpha p(X)\frac{\partial G_{2,\lambda}(X,X')}{\partial n}d\Gamma\right] \tag{17-b}$$

By describing follows integral relations:

$$I_R(i;X') = -\int_{\Gamma_i} p(X)\frac{\partial G_{1,\lambda}(X,X')}{\partial n}d\Gamma \tag{18-a}$$

$$I_S(i;X') = +\int_{\Gamma_i} p(X)G_{1,\lambda}(X,X')d\Gamma \tag{18-b}$$

$$J_R(i;X') = +\int_{\Gamma_i} \alpha p(X)\frac{\partial G_{2,\lambda}(X,X')}{\partial n}d\Gamma \tag{18-c}$$

$$J_S(i;X') = -\int_{\Gamma_i} \beta p(X)G_{2,\lambda}(X,X')d\Gamma \tag{18-d}$$

the Eq. (17) will converted to these forms:

$$C(X')\psi_1(X') = \sum_{i=1}^{N} R_i I_R(i;X') + S_i I_S(i;X') \tag{19-a}$$

$$C(X')\psi_2(X') = \sum_{i=1}^{N} R_i J_R(i;X') + S_i J_S(i;X') \tag{19-b}$$

In Eqs. (19-a) and (19-b), the observation point X' is taken inside the regions Ω_1 and Ω_2, respectively. In order to solve the Eq. (19), consider two points, one inside the region Ω_1 and the next one inside the region Ω_2. Then move these two points toward the common node X_j on the common boundary of two regions. By Doing this action the Eq. (19) will convert to these forms:

$$C(X_j)\psi_1(X_j) = \sum_{i=1}^{N} R_i I_R(i;X_j) + S_i I_S(i;X_j) \tag{20-a}$$

$$C(X_j)\psi_2(X_j) = \sum_{i=1}^{N} R_i J_R(i;X_j) + S_i J_S(i;X_j) \tag{20-b}$$

Using Eq. (13-a) and (16-a), the Eq. (20) will convert to the set of homogenous equations:

$$\sum_{i=1}^{N} R_i \left(I_R \left(i; X_j \right) - \delta_{ij} C \left(X_j \right) \right) + S_i I_S \left(i; X_j \right) = 0 \tag{21-a}$$

$$\sum_{i=1}^{N} R_i \left(J_R \left(i; X_j \right) - \delta_{ij} \alpha C \left(X_j \right) \right) + S_i J_S \left(i; X_j \right) = 0 \tag{21-b}$$

By Choosing $j \in (1,...,N)$, these equations will construct 2N sets of homogenous equations:

$$A^{(\lambda)} Y = 0 \tag{22}$$

$A^{(\lambda)}$ is the coefficient matrix of these set of equations and its elements state as:

$$A^{(\lambda)}_{j,i} = I_R \left(i; X_j \right) - \delta_{ij} C \left(X_j \right) \tag{23-a}$$

$$A^{(\lambda)}_{j,i+N} = I_S \left(i; X_j \right) \tag{23-b}$$

$$A^{(\lambda)}_{j+N,i} = J_R \left(i; X_j \right) - \delta_{ij} \alpha C \left(X_j \right) \tag{23-c}$$

$$A^{(\lambda)}_{j+N,i+N} = J_S \left(i; X_j \right) \tag{23-d}$$

$$i,\ j \in (1,...,N)$$

Upper index, λ, denote that the numerical amount of each element in this matrix is depend on the eigenvalue λ. Dependence of these equations to λ is arising from the dependence of Green function to this value. The column matrix Y is considered in the form of below:

$$\begin{aligned} Y_i &= R_i \\ Y_{i+N} &= S_i \end{aligned} \tag{24}$$

In order to have the non-zero answer for these set of equations, below condition should satisfy:

$$\det\left(A^{(\lambda)} \right) = 0 \tag{25}$$

Note that Eq. (25), will not result in characteristic equation; also, the complicated dependence of each element of matrix $A^{(\lambda)}$ to the parameter λ arise from dependence of Green's function to λ. Therefore, to solve the Eq. (25) and determine the amount of λ we used a method which is so similar to the bisection method which uses to find an algebraic equation roots. In this method, first, to obtain the Eq. (25) roots, we should determine the appropriate intervals that Eq. (25) just has a single root in each one. To specify these intervals, we give some values to λ and calculate the $\det\left(A^{(\lambda)} \right)$. Determining the sign of $\det\left(A^{(\lambda)} \right)$, enable us to specify the allowable intervals. Then, we can find the roots of Eq. (25) in these intervals using bisection method. We show the roots of Eq. (25) by λ_k. Note that whenever each amount of λ determined, the corresponding elements of $A^{(\lambda)}$

should find from Eq. (23) simultaneously. After determining λ_k, we are able to obtain the direction of its corresponding vector $Y^{(k)}$ in 2N-dimension complex space, using Eq. (22). Afterwards, by normalizing the vector $Y^{(k)}$, we can calculate it completely. Of course, in spite of finding $Y^{(k)}$, the eigenfunctions of differential equations of Eq. (12) are still remained to determine. We are able to calculate numerically these eigenfunctions using Eq. (19). Then, numerical amount of $\psi_1(X')$ or $\psi_2(X')$ can be calculated by choosing X' in any arbitrary point inside the regions of Ω_1 (Eq. (19-a)) or Ω_2 (Eq. (19-b)). One of the dominate characteristics of BEM is that, after determining the amount of desire functions on the boundaries, calculating the amount of that function in every arbitrary point inside the domain is possible. The other characteristic of this method is that, in necessary cases we are able to derivate analytically from functions $\psi_1(X')$ and $\psi_2(X')$ with respect to X', using Eq. (19). Since in BEM, meshing is being done just on the boundary of the domain, then, performing time of the calculation is too short and also the accuracy is too high, in comparison with the other methods.

3. Numerical Results

In this part, due to investigate the correctness and preciseness of the mentioned method we examine some numerical examples. First, we consider a particle with mass m and electrical charge q which is lain in a potential well with circular cross section:

$$V(\rho) = \begin{cases} V_1 & \rho \le a \\ V_2 & \rho > a \end{cases} \tag{27}$$

that $V_1 < V_2$ and a is the radius of the potential well. By solving the time independent Schrodinger equation, eigenfunctions of the particle obtained as follows:

$$\psi_m(\rho,\varphi) = \begin{cases} A_m J_m(K_1\rho)\exp(im\varphi) & \rho \le a \\ B_m K_m(K_2\rho)\exp(im\varphi) & \rho > a \end{cases} \tag{28}$$

which J_m and K_m are mth_order Bessel function of first kind and mth_order modified Bessel function of second kind, respectively; and $K_1 = \sqrt{2m(E - V_1)/\hbar^2}$ and $K_2 = \sqrt{2m(V_2 - E)/\hbar^2}$. Applying the continuity boundary conditions to the state function and its normal derivative on the boundary of the potential well, the amount of E and B_m / A_m ratio are calculable. As an example, this problem is solved by the method which is mentioned in this article and energy spectrum of the bounded particle is calculated. The numerical answers are compared for different values of V_1 and V_2 and a very good accommodation observed between two results. Table 2 shows theoretically and numerically calculated results.

In table 2, numerical results of the energy levels are calculated by $10^{-2}eV$ accuracy; increasing the accuracy to $10^{-3}eV$ will vanish the slight difference between the numerical and theoretical values. In Fig. 1, the behavior of $\det(A)$ with respect to energy of electron in a circular potential well is shown for the values expressed in table 2. As is clear, the $\det(A)$ vanishes in the position of each energy levels.

To investigate the energy level degeneracy and also investigate the perturbation effect on energy level gaps, we change the cross section shape of potential well to an ellipse with radii of $a = 2.2A^{\circ}$ and $b = 1.8A^{\circ}$. Analytically calculating the energy levels of potential well with arbitrary cross section is too hard. However, using the mentioned method in this article, the time duration and accuracy of calculating the energy levels are independent to the shape of the cross section of the well. In table 3, the Energy levels of an electron in the elliptical well (for the given parameter stated in description of table 2) are expressed.

The displacement and splitting of the energy levels of a particle in an elliptical potential well in comparison with a circular potential well are shown in Fig. 2. Behavior of $\det(A)$ is shown in Fig. 1 with respect to energy of the particle as well. In Fig. 1, the splitting of energy levels is completely clear.

Afterwards, the probability density of existence of an electron in the elliptical potential well for the forth level (13.943 eV) is calculated (Fig. 3). The dimensions of the elliptical potential well considered 1.5 and 2.5 A° and the amount of the potential inside and outside it are chosen zero and 20 eV respectively.

As a next problem, the energy levels of a particle in two circular potential well adjacent to each other, with characteristics that expressed in the description of table 2, are calculated. Radius of each well and distance between their center are considered as $a = 2A^{\circ}$ and $d = 4.4A^{\circ}$, respectively. Energy eigenvalues of the particle through the double potential well are given in table 4. To investigate the mutual effect of two wells on each other, we increased the distance between the centers of two wells gradually and we calculated the energy spectrum of the particle in each level. We saw that by increasing the distance between the centers of two wells, the number of energy levels are decreased from 12 to 4 levels that the numerical amount of these four levels are completely match with energy levels of the sole circular potential well which expressed in table 2. In the last stage, the distance between the centers is considered as $d = 14A^{\circ}$. As we expected, by increasing the distance of two wells, not only the well’s wave function haven’t had any overlap anymore but also each well behave as an isolated one.

Finally, the energy levels of a particle in a 3D potential well are calculated. One application of this problem is to find the energy spectrum of electrons and holes in semiconductors with zero-dimension (quantum dots). The valance electrons of atoms in a nano meters semiconductor are bent to just inside the volume of the nano particle. This limitation causes the creation of distinct energy levels that they behave as energy levels of an atom. To appear these levels, these kinds of nano particles are named as artificial atoms. In the simplest approximation, the quantum dot can be considered as a spherical potential well that the electrons are bent through the potential well. We consider this approximation in spherical coordinate as follow:

$$V(r) = \begin{cases} V_1 & r \le a \\ V_2 & r > a \end{cases} \tag{29}$$

Using the mentioned method in this article, we calculate the energy levels of bent electrons. The numerical results are obtained considering $V_1 = 0\,eV$, $V_2 = 50\,eV$ and $a = 2\,A^\circ$. Results are shown in table 5.

4. Conclusion

In this article, based on BEM a numerical method has been applied to calculate the eigenvalues and eigenfunctions of some common differential equations in sciences, especially in physics. First, the energy levels of a particle in a circular potential well are calculated, analytically and numerically; and a very good agreement is observed between the two results. Then, the eigenvalues of the elliptical potential well are calculated. Also, the corresponding eigenfunctions to a specific energy level of an elliptical potential well have been obtained. Then, the mutual effect between two circular potential well are investigated. We show that by increasing the distance between the centers of two wells gradually the number of energy levels start to decrease to the energy levels of a sole circular well. Finally, the energy levels of a quantum dot are calculated.

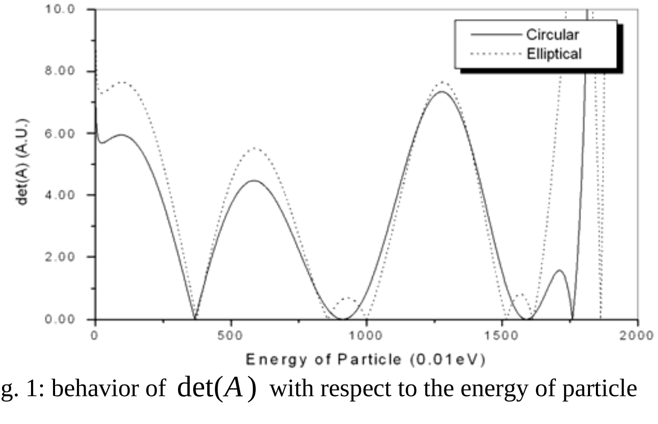

Fig. 1: behavior of $\det(A)$ with respect to the energy of particle

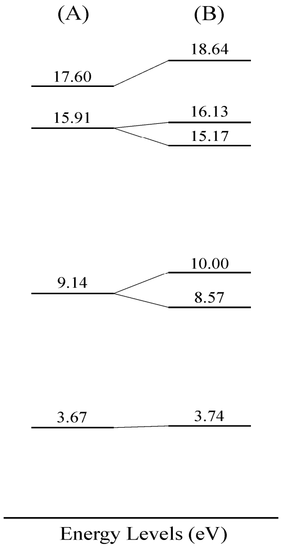

Fig. 2: energy levels of a particle in circular (A) and elliptical (B) quantum wells.

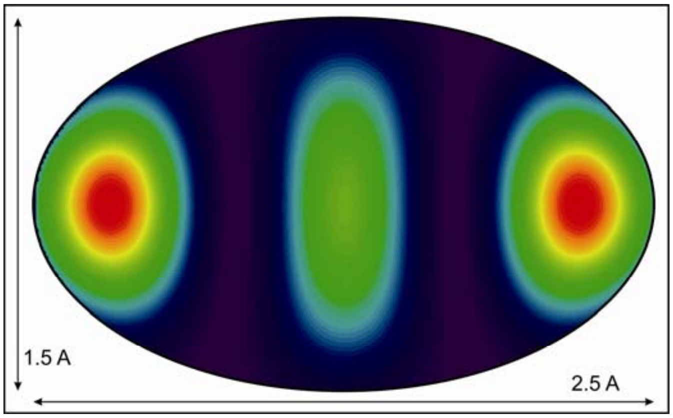

Fig. 3: The probability density of existence of the electron in the elliptical potential well.

Table 1

Values of $p(X)$, $q(X)$, $f(X)$ and λ for some different partial differential equations.

Equation	$p(X)$	$q(X)$	λ	$f(X)$
Laplace	**1**	**0**	**0**	**0**
Poisson	**1**	**0**	**0**	$\neq 0$
Homogeneous Helmholtz	**1**	**0**	$-K^2$	**0**
Inhomogeneous Helmholtz	**1**	**0**	$-K^2$	$\neq 0$
Schrödinger	**1**	$-2mV(X)/\hbar^2$	$-2mE/\hbar^2$	**0**
Heat	$K(X)$	**0**	**0**	$Q(X)$

Table 2

Analytically and numerically calculated amount of energy levels of a particle for $m = m_e$ (m_e is electron mass), $V_1 = 0eV$, $V_2 = 20eV$ and $a = 2A^{\circ}$.

	$\varepsilon_1(eV)$	$\varepsilon_2(eV)$	$\varepsilon_3(eV)$	$\varepsilon_4(eV)$
Analytical results	**3.666**	**9.122**	**15.890**	**17.596**
Numerical Results	**3.674**	**9.14**	**15.91**	**17.6**

Table 3

Energy levels of a particle in an elliptical quantum well with radii $a = 2.2A^{\circ}$ and $b = 1.8A^{\circ}$ and for the given values described in the description of table 2.

$\varepsilon_1(eV)$	$\varepsilon_2(eV)$	$\varepsilon_3(eV)$	$\varepsilon_4(eV)$	$\varepsilon_5(eV)$	$\varepsilon_6(eV)$
3.47	**8.57**	**10.00**	**15.17**	**16.13**	**18.64**

Table 4

Energy levels of the electron in two circular potential well adjacent to each other with respect to eV and for different distance between their centers.

	$d = 4.4A^{\circ}$	$d = 5.0A^{\circ}$	$d = 6.0A^{\circ}$	$d = 7.0A^{\circ}$	$d = 8.0A^{\circ}$	$d = 9.0A^{\circ}$	$d = 14.0A^{\circ}$
1	**3.512**	**3.635**	**3.674**	**3.67**	**3.67**	**3.674**	**3.674**
2	**3.798**	**3.71**	**9.138**	**9.138**	**9.14**	**9.14**	**9.14**
3	**8.478**	**8.928**	**15.825**	**15.9**	**15.91**	**15.91**	**15.91**
4	**9.054**	**9.115**	**15.865**	**15.92**	**17.58**	**17.59**	**17.6**
5	**9.214**	**9.16**	**15.96**	**17.55**	**17.624**	**17.61**	**---**
6	**9.726**	**15.597**	**16**	**17.66**	**---**	**---**	**---**
7	**15.2**	**15.727**	**17.485**	**---**	**---**	**---**	**---**
8	**15.456**	**16.105**	**17.76**	**---**	**---**	**---**	**---**
9	**16.38**	**16.213**	**---**	**---**	**---**	**---**	**---**
10	**16.508**	**17.36**	**---**	**---**	**---**	**---**	**---**
11	**17.252**	**18.09**	**---**	**---**	**---**	**---**	**---**
12	**18.676**	**---**	**---**	**---**	**---**	**---**	**---**

Table 5

Energy levels of the electron in a spherical potential well.

$\varepsilon_1(eV)$	$\varepsilon_2(eV)$	$\varepsilon_3(eV)$	$\varepsilon_4(eV)$	$\varepsilon_5(eV)$
9.5	**19.4**	**31.8**	**37.6**	**46.6**

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