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ON RESULTS OF HARDY-ROGERS AND REICH IN CONE B-METRIC SPACE OVER BANACH ALGEBRA AND APPLICATIONS

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In this paper, we establish certain recent results of Miculescu and Mihail [J. Fixed Point Theory Appl. 19, 2153-2163 (2017)] and of Suzuki [J. Inequal. Appl. 2017, 256, 11 p.] in cone b-metric spaces over Banach algebra. Also, we prove Reich contraction theorem in such spaces. Our results generalize, improve and complement several ones in the existing literature.

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1. Introduction and Preliminaries

The concept of b-metric space was introduced of Bakhtin [3] and Czerwik [4]. Since then, many fixed point theorems for various contractions on the b-metric space and generalizations of such spaces have appeared (see [1, 2, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]). Following definitions and results will be needed in the sequel.

Definition 1.1. Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A triplet (X, d, s), is called a b-metric space.

Note that class of metric spaces is included in the class of b-metric spaces. In fact, the notions of convergent sequence, Cauchy sequence and complete space are defined as in metric spaces.

Definition 1.2. Let (X, d, s) b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.
(a) The sequence $\{x_n\}$ is said to be convergent in (X, d, s) and converges to x, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(b) The sequence $\{x_n\}$ is said to be Cauchy sequence in (X, d, s) if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0, p > 0$.
(c) (X, d, s) is said to be a complete b-metric space if every Cauchy sequence in X converges to some $x \in X$.

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In the paper [25] Singh et al obtained the following result (see also Lemma 3. 1. in [12]).

Lemma 1.3. *(Lemma 3. 1. in* [25]*) Let* (X, d, s) *be a b-metric space and let* $\{x_n\}$ *be a sequence in* X*. Assume that there exists* $k \in [0, 1/s)$ *satisfying* $d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1})$ *for any* $n \in \mathbb{N}$*. Then* $\{x_n\}$ *is Cauchy.*

From the previous Lemma, the next result immediately followed.

Lemma 1.4. *Every sequence* $\{x_n\}$ *of elements from a b-metric space* (X, d, s)*, having the property that there exist* $k \in [0, 1/s)$ *and* $C > 0$ *such that*

$$d(x_{n+1}, x_n) \leq Ck^n,$$

for any $n \in N$*, is Cauchy.*

Miculescu and Mihail [15] (Lemma 2. 2.) and Suzuki [26] (Lemma 6) proved that in Lemma 1.3, one may extend the range of k to the case $0 < k < 1$.

In the paper [16] Mitrović very recently presented a short proof of results of Suzuki, Miculescu and Mihail. In this paper we establish these results in a cone b-metric space over Banach algebra and obtain certain new fixed point results.

We recall some well-known definitions which will be needed in the sequel.

Definition 1.5. Let $\mathcal{A}$ be a real Banach algebra, i.e., $\mathcal{A}$ is a real Banach space with a product that satisfies
1. $x(yz) = (xy)z$,
2. $x(y + z) = xy + xz$,
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$,
4. $||xy|| \leq ||x||||y||$,
for all $x, y, z \in \mathcal{A}, \alpha \in \mathbb{R}$.

The Banach algebra $\mathcal{A}$ is said to be unital if there exists an element $e \in \mathcal{A}$ such that $ex = xe = x$ for all $x \in \mathcal{A}$. The element e is called the unit. An element $x \in \mathcal{A}$ is said to be invertible if there is a $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of x, if it exists, is unique and will be denoted by x^{-1} (see [23]).

Let $\mathcal{A}$ be a unital Banach algebra. A non-empty closed set $P \subset \mathcal{A}$ is said to be a cone if
1. $e \in P$,
2. $P + P \subset P$,
3. $\lambda P \subset P$ for all $\lambda \geq 0$,
4. $P \cdot P \subset P$,
5. $P \cap (-P) = \{\theta\}$,
where θ is the zero of the unital Banach algebra $\mathcal{A}$. For a given cone $P \subseteq A$, we can define a a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$ and we write $x \prec y$ if $x \preceq y$ and $x \neq y$ while $x \ll y$ will stands for $y - x \in intP$, where $intP$ denotes the interior of P. If $intP \neq \emptyset$ then P is called a solid cone. The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in A$,

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leq M\|y\|.$$

Cone b-metric space over Banach algebra with constant $s \geq 1$ is introduced in [8] as generalization of a metric space and many of its generalizations (b-metric space, cone metric space). We introduce here cone b-metric space over Banach algebra with constant $s \succeq e$.

Definition 1.6. Let X be a nonempty set and the mapping $d : X \times X \to \mathcal{A}$ satisfies:
(CbM1) $d(x, y) = \theta$ if and only if $x = y$;
(CbM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(CbM3) there exists $s \in P$, $e \preceq s$ such that $d(x, y) \preceq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$. Then d is called a cone b-metric on X and (X, d) is called a cone b-metric space over Banach algebra (in short CbMS-BA) with coefficient s. If $s = e$ we say that (X, d) is a cone metric space over Banach algebra (in short CMS-BA).

Definition 1.7. Let $\{x_n\}$ be a sequence in Banach algebra $\mathcal{A}$.
(i) A sequence $\{x_n\}$ said to be a c-sequence, if for each $c \gg \theta$, there exists a natural number n_0 such that $x_n \ll c$ for all $n \geq n_0$.
(ii) A sequence $\{x_n\}$ in a is called a θ-sequence if $x_n \to \theta$ as $n \to \infty$.

Definition 1.8. Let (X, d) be a CbMS-BA with coefficient s and $\{x_n\}$ a sequence in X,
(i) $\{x_n\}$ b-converges to $x \in X$, if $\{d(x_n, x)\}$ is a c-sequence;
(ii) $\{x_n\}$ is a Cauchy sequence whenever for each $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m > N$;
(iii) (X, d) is b-complete, if every b-Cauchy sequence in X is b-convergent.

Let is notice that if $\{x_n\}$ and $\{y_n\}$ be two c-sequences in a solid cone P and $a, b \in P$ are two arbitrarily given vectors, then $ax_n + by_n$ is a c-sequence. Also, if $x \preceq y$ and $y \ll z$, then $x \ll z$.

Lemma 1.9. [23]*Let $\mathcal{A}$ be a Banach algebra with a unit e and $k \in \mathcal{A}$, then* $\lim_{n\to\infty} ||k^n||^{\frac{1}{n}}$ *exists and the spectral radius $r(k)$ satisfies*

$$r(k) = \lim_{n\to\infty} ||k^n||^{\frac{1}{n}} = \inf_{n\geq 1} ||k^n||^{\frac{1}{n}}.$$

If $r(k) < |\lambda|$, then $\lambda e - k$ is invertible in $\mathcal{A}$, moreover,

$$(\lambda e - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{\lambda^{i+1}},$$

where λ is a constant.

Lemma 1.10. [24]*Let $P \subset \mathcal{A}$ be a cone.*
(a) If $a, b \in \mathcal{A}, c \in P$ and $a \preceq b$, then $ca \preceq cb$,
(b) If $a, k \in P$ are such that $r(k) < 1$ and $a \preceq ka$, then $a = \theta$,
(c) If $k \in P$ and $r(k) < 1$, then k^n is a c-sequence and for any fixed $n \in \mathbb{N}$ we have $r(k^n) < 1$.

Lemma 1.11. [6] *Let $\mathcal{A}$ be a Banach algebra and P a solid cone in $\mathcal{A}$. Then each c-sequence in P is a θ-sequence if and only if P is a normal cone.*

Lemma 1.12. [23]*Let $\mathcal{A}$ be a Banach algebra with a unit e and $a, b \in \mathcal{A}$. If a commutes with b, then*

$$r(a + b) \leq r(a) + r(b),\ r(ab) \leq r(a)r(b).$$

Lemma 1.13. [7]*Let $\mathcal{A}$ be a Banach algebra with a unit e and $k \in \mathcal{A}$. If λ is a constant and $r(k) < |\lambda|$, then*

$$r((\lambda e - k)^{-1}) \leq \frac{1}{|\lambda| - r(k)}.$$

2. **Main Results**

In this section, we suppose that (X, d) is a cone b-metric space over Banach algebra $\mathcal{A}$ with coefficient s.

Lemma 2.1. *Let $\{x_n\}$ be a sequence in X. Assume that there exists $k \in P$ such that k and s commutes and $r(k) < \frac{1}{r(s)}$ satisfying $d(x_{n+1}, x_n) \preceq kd(x_n, x_{n-1})$ for any $n \in \mathbb{N}$. Then $\{x_n\}$ is Cauchy.*

Proof. Thus for any $n > m$, it follows that

$$\begin{aligned} d(x_m, x_n) &\preceq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_n)] \\ &\preceq sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_n)] \\ &\preceq sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) \\ &+ s^3[d(x_{m+2}, x_{m+3}) + d(x_{m+3}, x_n)] \\ &\vdots \\ &\preceq sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + \cdots + s^{n-m} d(x_{n-1}, x_n) \\ &\preceq s[k^m + sk^{m+1} + \cdots + s^{n-m-1}k^{n-1}]d(x_0, x_1) \\ &\preceq sk^m(e - sk)^{-1} d(x_0, x_1). \end{aligned}$$

Using Lemma 1.10 we obtain that $\{x_n\}$ is a b-Cauchy sequence. □

From the Lemma 2.1 we obtain the following result.

Lemma 2.2. *Let $\{x_n\}$ be a sequence in X. Assume that there exists $k \in P$ such that $r(k) < \frac{1}{r(s)}$ and $C \in P$ such that*

$$d(x_{n+1}, x_n) \preceq Ck^n,$$

for any $n \in N$. Then $\{x_n\}$ is Cauchy.

Lemma 2.3. *Let $\{x_n\}$ be a sequence in X. Then for all $n, p \in \mathbb{N}$,*

$$d(x_n, x_{n+p}) \preceq s^p[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p})]$$

holds.

Proof. Obvious. □

Lemma 2.4. *Let $\{x_n\}$ be a sequence in X. Assume that there exists $k \in \mathcal{A}$ such that $0 < r(k) < 1$ and s and k commutes and satisfying*

$$d(x_{n+1}, x_n) \preceq kd(x_n, x_{n-1}), \tag{2.1}$$

for any $n \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ such that $n_0 > -\frac{\log r(s)}{\log r(k)}$, then

(1) *$\{x_{nn_0}\}$ is Cauchy,*
(2) *$\{d(x_n, x_{n_0\lfloor \frac{n}{n_0} \rfloor})\}$ is a c-sequence.*

Proof. 1. Using Lemma 2.3 and condition (2.1) we get the following

$$\begin{aligned} d(x_{(n+1)n_0}, x_{nn_0}) &\preceq s^{n_0}[d(x_{(n+1)n_0}, x_{(n+1)n_0-1}) + \cdots + d(x_{nn_0+1}, x_{nn_0})] \\ &\preceq s^{n_0}(k^{(n+1)n_0-1} + \cdots + k^{nn_0}) d(x_1, x_0) \\ &\preceq s^{n_0} k^{nn_0} d(x_1, x_0)(e - k)^{-1} \\ &\preceq C\mu^n, \end{aligned}$$

where $C = s^{n_0} d(x_1, x_0)(e - k)^{-1}$ and $\mu = k^{n_0}$. Since $r(\mu) = r(k^{n_0}) \leq r(k)^{n_0}$ (because of Lemma 1.12) and $n_0 > -\frac{\log r(s)}{\log r(k)}$, we have that $r(\mu) < \frac{1}{r(s)}$. So, from Lemma 2.2 we conclude that $\{x_{nn_0}\}$ is Cauchy.
2.

$$\begin{aligned} d(x_n, x_{n_0\lfloor \frac{n}{n_0} \rfloor}) &\preceq s^{n_0}[d(x_n, x_{n-1}) + \cdots + d(x_{n_0\lfloor \frac{n}{n_0} \rfloor+1}, x_{n_0\lfloor \frac{n}{n_0} \rfloor})] \\ &\preceq s^{n_0}(k^{n-1} + \cdots + k^{n_0\lfloor \frac{n}{n_0} \rfloor}) d(x_1, x_0) \\ &\preceq s^{n_0} k^{n_0\lfloor \frac{n}{n_0} \rfloor} d(x_1, x_0)(e - k)^{-1}. \end{aligned}$$

So, $\{d(x_n, x_{n_0\lfloor \frac{n}{n_0} \rfloor})\}$ is a c-sequence. □

Lemma 2.5. *Let $\{x_n\}$ be a sequence in X. Assume that there exists $k \in \mathcal{A}$ such that $r(k) \in [0,1)$ satisfying $d(x_{n+1}, x_n) \preceq kd(x_n, x_{n-1})$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is Cauchy.*

Proof. If $r(k) = 0$ proof is obvious. Let $0 < r(k) < 1$ and $n_0 \in \mathbb{N}$ such that $n_0 > -\frac{\log r(s)}{\log r(k)}$, then the proof follows from Lemma 2.4 and the following inequality

$$d(x_n, x_m) \preceq s^2[d(x_n, x_{n_0\lfloor \frac{n}{n_0} \rfloor}) + d(x_{n_0\lfloor \frac{n}{n_0} \rfloor}, x_{n_0\lfloor \frac{m}{n_0} \rfloor}) + d(x_{n_0\lfloor \frac{m}{n_0} \rfloor}, x_m)],$$

for all $n, m \in \mathbb{N}$. □

Remark 2.6. Lemma 2.5 is a generalization of the Lemma 11 in [1], Lemma 2.2. in [10] and Lemma 3.1. in [12].

3. **Some Applications**

Using Lemma 2.5 we can improve and generalize the series results in the literature that were obtained recently.
We first give a result of Hardy-Rogers [5] in CbMS-BA with coefficient s.

Theorem 3.1. *Let (X, d) be a CbMS-BA with coefficient $s, (e \preceq s)$ and $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \preceq \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)] \tag{3.1}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \in P$ commutes such that $r(\alpha) + 2r(\beta) + 2r(\gamma)r(s) < 1$ and $r(s)(r(\beta) + r(s)r(\gamma)) < 1$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. From condition (3.1) we have that

$$\begin{aligned} d(x_{n+1}, x_n) &\preceq \alpha d(x_n, x_{n-1}) + \beta[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\ &+ +\gamma[d(x_n, x_n) + d(x_{n-1}, x_{n+1})]. \end{aligned}$$

So,

$$\begin{aligned} (e - \beta)d(x_{n+1}, x_n) &\preceq (\alpha + \beta)d(x_{n-1}, x_n) + \gamma d(x_{n-1}, x_{n+1}) \\ &\preceq (\alpha + \beta)d(x_{n-1}, x_n) + \gamma s[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]. \end{aligned}$$

Thus,

$$(e - \beta - \gamma s)d(x_{n+1}, x_n) \preceq (\alpha + \beta + \gamma s)d(x_{n-1}, x_n),$$

how is it $r(\beta) + r(\gamma)r(s) < 1$ from Lemma 1.9, we have,

$$d(x_{n+1}, x_n) \preceq [e - (\beta + \gamma s)^{-1}](\alpha + \beta + \gamma s)d(x_n, x_{n-1}). \tag{3.2}$$

Put $\lambda = [e - (\beta + \gamma s)^{-1}](\alpha + \beta + \gamma s)$. From Lemma 1.12 and Lemma 1.13, we have that $r(\lambda) \leq \frac{r(\alpha)+r(\beta)+r(\gamma)r(s)}{1-r(\beta)-r(\gamma)r(s)}$. So, $r(\lambda) \in [0,1)$. From Lemma 2.5 follows that $\{x_n\}$ is a Cauchy sequence in (X, d). By completeness of (X, d) there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*. \tag{3.3}$$

Now we obtain that x^* is the unique fixed point of T. Namely, we have

$$\begin{aligned} d(x^*, Tx^*) &\preceq sd(x^*, x_{n+1}) + sd(x_{n+1}, Tx^*) \\ &= sd(x^*, x_{n+1}) + sd(Tx_n, Tx^*) \\ &\preceq sd(x^*, x_{n+1}) + s\alpha d(x_n, x^*) + s\beta[d(x_n, x_{n+1}) + d(x^*, Tx^*)] \\ &+ s\gamma[d(x_n, Tx^*) + d(x^*, x_{n+1})] \\ &\preceq sd(x^*, x_{n+1}) + s\alpha d(x_n, x^*) + s\beta[d(x_n, x_{n+1}) + d(x^*, Tx^*)] \\ &+ s\gamma[s(d(x_n, x^*) + d(x^*, Tx^*)) + d(x^*, x_{n+1})]. \end{aligned}$$

Since $\lim_{n\to\infty} d(x^*, x_n) = \theta$, $\lim_{n\to\infty} d(x_n, x_{n+1}) = \theta$, we obtain

$$d(Tx^*, x^*) \preceq (s\beta + s^2\gamma)d(Tx^*, x^*).$$

Since, $r(s)(r(\beta) + r(s)r(\gamma)) < 1$, from Lemma 1.10 we claim that $d(x^*, Tx^*) = \theta$, that is, $Tx^* = x^*$.

For uniqueness, let y^* be another fixed point of T. Then it follows from (3.1) that

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \preceq \alpha d(x^*, y^*) + \beta[d(x^*, Tx^*) + d(y^*, Ty^*)] \\ &+ \gamma[d(x^*, Ty^*) + d(y^*, Tx^*)] \\ &\preceq (\alpha + 2\gamma)d(x^*, y^*). \end{aligned}$$

Now again from Lemma 1.10 we obtain $d(x^*, y^*) = \theta$, i.e., $x^* = y^*$. □

From the previous theorem we obtain the Reich type theorem [22] in CbMS-BA with coefficient s.

Theorem 3.2. *Let (X, d) be a CbMS-BA with coefficient $s, (e \preceq s)$ and $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \preceq \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] \tag{3.4}$$

for all $x, y \in X$, where $\alpha, \beta \in P$ commutes such that $r(\alpha) + 2r(\beta) < 1$ and $r(s)r(\beta) < 1$. Then T has a unique fixed point.

Remark 3.3. We note that if $r(s) < 2$, the condition $r(s)r(\beta) < 1$ in Theorem 3.2 is superfluous.

Example 3.1. *Let $A = \{a = (a_{ij})_{3\times3} : a_{ij} \in \mathbb{R}, 1 \le i, j \le 3\}$ and*

$$\|a\| = \frac{1}{3} \sum_{1\le i,j\le 3} |a_{ij}|.$$

Take a cone $P = \{a \in A : a_{ij} \ge 0, 1 \le i, j \le 3\}$ in $\mathcal{A}$. Let $X = \{1, 2, 3\}$. Define a mapping $d : X \times X \longrightarrow \mathcal{A}$ by $d(1, 1) = d(2, 2) = d(3, 3) = (0)_{3\times3}$ and

$$d(1,2) = d(2,1) = \begin{pmatrix} 0 & 4 & 8 \\ 4 & 8 & 12 \\ 32 & 16 & 28 \end{pmatrix},$$

$$d(3,1) = d(1,3) = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 8 & 4 & 7 \end{pmatrix},$$

$$d(2,3) = d(3,2) = \begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 6 \\ 16 & 8 & 14 \end{pmatrix}.$$

Then (X, d) be a CbMS-BA with coefficient $s = \begin{pmatrix} \frac{4}{3} & 0 & 0 \\ 0 & \frac{4}{3} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$. Let $T : X \to X$ be a mapping define by $T1 = 1, T2 = 3, T3 = 1$ and let $\alpha = \beta = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$. Then a mapping T satisfying:

$$d(Tx, Ty) \preceq \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$, where α and β commutes such that $r(\alpha)+2r(\beta) < 1$ and $r(s)r(\beta) = \frac{4}{3}\cdot\frac{1}{4} < 1$ and T has a unique fixed point $x = 1$.

Note that Theorem 3.1 improve and generalize Theorem 2. 1. in [8].

Theorem 3.4. *(Theorem 2.1, [8]) Let (X, d) be a complete cone b-metric space over Banach algebra $\mathcal{A}$ with the coeficient $s \geq 1$. Let K be a solid not necessarily normal cone of $\mathcal{A}$. Suppose $T : X \to X$ is a mapping and suppose that there exists $k \in K$ such that, for all $x, y \in X$, at least one of the following generalized Lipschitz conditions holds:*
(i) $d(Tx, Ty) \preceq kd(x, y)$ and $r(k) < \frac{1}{s}$;
(ii) $d(Tx, Ty) \preceq k(d(Tx, x) + d(Ty, y))$ and $r(k) < \frac{1}{1+s}$;
(iii) $d(Tx, Ty) \preceq k(d(Tx, y) + d(Ty, x))$ and $r(k) < \frac{1}{s+s^2}$.
Then T has a unique fixed point in X.

Also, Theorem 3.1 improve and generalize Theorem 2. 1. in [9].

Theorem 3.5. *(Theorem 2.1, [9]) Let (X, d) be a complete cone b-metric space over Banach algebra $\mathcal{A}$ with the coeficient $s \geq 1$. Let K be a solid not necessarily normal cone of $\mathcal{A}$. Suppose $T : X \to X$ is a mapping and suppose that there exists $k \in K$ such that, for all $x, y \in X$, the following generalized Lipschitz conditions holds:*

$$d(Tx, Ty) \preceq kd(x, y),$$

and $r(k) < 1$. Then T has a unique fixed point in X and for any $x \in X$, the iterative sequence $\{T^n x\}$ b-converges to the fixed point.

Remark 3.6. In (i) of Theorem 3.4 the condition $r(k) < \frac{1}{s}$ can be replaced by a weaker condition $r(k) < 1$. Similarly, in condition (ii), $r(k) < \frac{1}{1+s}$ we can relax with $r(k) < \min\{\frac{1}{2}, \frac{1}{r(s)}\}$, and in condition (iii) instead of $r(k) < \frac{1}{s+s^2}$ put $r(k) < \min\{\frac{1}{2r(s)}, \frac{1}{r^2(s)}\}$.

Remark 3.7. Using Lemma 2.5 we can improve and generalize the following results: Theorem 12. in [1], Theorem 2.9. in [7], Theorem 2.5 in [8], Theorem 2.3. in [10], Theorem 3.3. in [12], Theorem 3.2. in [21].

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