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CAYLEY INTERVAL-VALUED FUZZY GRAPHS

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The concept of Cayley fuzzy graphs was introduced by Namboothiri in [9]*. Akram in* [1] *defined interval-valued fuzzy graphs. In this paper, we propose a class of Cayley interval-valued fuzzy graphs and then study its various graph theoretic properties in terms of algebraic properties. Moreover, we define the concepts of α-connectedness, weakly α-connectedness, semi α-connectedness, locally α-connectedness, quasi α-connectedness and strength of connectivity in interval-valued fuzzy graphs and then investigate these concepts in terms of algebraic properties.*

Keywords: Cayley interval-valued fuzzy graphs, α-connectedness, quasi α-connectedness.
MSC2010: 05C 99.

1. Introduction

The major role of graph theory in computer applications is the development of graph algorithms. These algorithms are used to solve problems that are modeled in the form of graphs and the corresponding computer science application problems. A digraph is a graph whose edges have direction and are called arcs. The Cayley graph was first considered for finite groups by Cayley in 1878. Max Dehn in his unpublished lectures on group theory from 1909 to 1910 introduced Cayley graphs under the name Gruppenbid (group diagram), which led to the geometric group theory of today. His most important application was the solution of the word problem for the fundamental group of surfaces with genus, which is equivalent to the topological problem of deciding which closed curves on the surface contract to a point. The notion of fuzzy sets was introduced by Zadeh [24] as a method of representing uncertainty and vagueness. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines. In 1975, Zadeh [25] introduced the notion of interval-valued fuzzy sets as an extension of fuzzy sets [24] in which the values of the membership degree are intervals of numbers instead of the numbers. Akram et al. [1, 2, 3, 4, 5] defined interval-valued fuzzy graphs, regular

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bipolar fuzzy graphs, certain types of interval-valued fuzzy graphs, intuitionistic fuzzy hypergraphs with applications and regularity in vague intersection graphs and vague line graphs. Kaufmann's initial definition of a fuzzy graph [6] was based on Zadeh's fuzzy relations [24]. Later Rosenfeld [16] introduced the fuzzy analogue of several basic graph-theoretic concepts. Mordeson and Nair [8] defined the concept of complement of fuzzy graph and studied some operations on fuzzy graphs. Wu [23] discussed fuzzy digraphs. Shahzamanian et al. [17] introduced the notion of roughness in Cayley graphs. Namboothiri et al. [9] discussed Cayley fuzzy graphs. Pal and Rashmanlou [10] studied irregular interval-valued fuzzy graphs. Also, they defined antipodal interval-valued fuzzy graphs [11], balanced interval-valued fuzzy graphs [12], some properties of highly irregular interval-valued fuzzy graphs [13] and a study on bipolar fuzzy graphs [14]. Rashmanlou and Yong Bae Jun investigated complete interval valued fuzzy graphs [15]. Samanta and Pal defined fuzzy tolerance graphs [18], fuzzy threshold graphs [19], fuzzy planar graphs [20], fuzzy k-competition graphs and p-competition fuzzy graphs [22] and irregular bipolar fuzzy graphs [21]. In this paper, we introduce a class of Cayley interval-valued fuzzy graphs and then study its various graph theoretic properties in terms of algebraic properties. Moreover, we introduce the concept of α-connectedness, semi α-connectedness and quasi α-connectedness in interval-valued fuzzy graphs and then study these concepts in terms of algebraic properties.

2. **Preliminaries**

A digraph is a $D^* = (V, E)$, where V is a finite set and $E \subseteq V \times V$. Let G be a finite group and let S be a minimal generating set of G. A Cayley graph (G, S) has elements of G as its vertices; the edge set is given by $\{(g, gs) : g \in G, s \in S\}$. Two vertices g_1 and g_2 are adjacent if $g_2 = g_1 \cdot s$, where $s \in S$. Note that a generating set S is minimal if S generates G but no proper subset of S does. Let $(V, *)$ be a group and A be any subset of V. Then the Cayley graph induced by $(V, *, A)$ is the graph $G = (V, R)$ where $R = \{(x, y) : x^{-1}y \in A\}$. A fuzzy subset μ on a set X is a map $\mu : X \to [0, 1]$. A fuzzy binary relation on X is a fuzzy subset μ on $X \times X$. By a fuzzy relation we mean a fuzzy binary relation given by $\mu : X \times X \to [0, 1]$.

Definition 2.1. [9] *Let $(V, *)$ be a group and let μ be a fuzzy subset of V. Then the fuzzy relation R on $V \times V$ defined by $R(x, y) = \mu(x^{-1} * y)$ $\forall x, y \in V$ induces a fuzzy graph $G = (V, R)$, called the Cayley fuzzy graph induced by the $(V, *, \mu)$.*
The interval-valued fuzzy set A in V is defined by $A = \{(x, [\mu_{A^-}(x), \mu_{A^+}(x)]) : x \in V\}$, where $\mu_{A^-}(x)$ and $\mu_{A^+}(x)$ are fuzzy subsets of V such that $\mu_{A^-}(x) \leq \mu_{A^+}(x)$ for all $x \in V$. For any two interval-valued sets $A = \{(x, [\mu_{A^-}(x), \mu_{A^+}(x)]) : x \in V\}$ and $B = \{(x, [\mu_{B^-}(x), \mu_{B^+}(x)]) : x \in V\}$ we define:

$$A \cup B = \{(x, \max(\mu_{A^-}(x), \mu_{B^-}(x)), \max(\mu_{A^+}(x), \mu_{B^+}(x))) : x \in V\},$$

$$A \cap B = \{(x, \min(\mu_{A^-}(x), \mu_{B^-}(x)), \min(\mu_{A^+}(x), \mu_{B^+}(x))) : \ x \in V\}.$$

For the sake of simplicity, we shall use the symbol $A = [\mu_{A^-}, \mu_{A^+}]$ *for the interval-valued fuzzy set* $A = \{(x, [\mu_{A^-}(x), \mu_{A^+}(x)]); \ x \in V\}$. *If* $G^* = (V, E)$ *is a graph, then by an interval-valued fuzzy relation* R *on a set* E *we mean an interval-valued fuzzy set such that* $\mu_{B^-}(xy) \leq \min(\mu_{A^-}(x), \mu_{A^-}(y))$, $\mu_{B^+}(xy) \leq \min(\mu_{A^+}(x), \mu_{A^+}(y))$ *for all* $xy \in E$. *By an interval-valued fuzzy graph of a graph* $G^* = (V, E)$ [1], *we mean a pair* $G = (A, R)$, *where* $A = [\mu_{A^-}, \mu_{A^+}]$ *is an interval-valued fuzzy set on* V *and* $R = [\mu_{B^-}, \mu_{B^+}]$ *is an interval-valued fuzzy relation on* E.

Definition 2.2. *Let* X *be interval-valued fuzzy set. For any subset* A *and for* $\alpha \in [0, 1]$,
(i) $\{x | \mu_{A^-}(x) \geq \alpha, \mu_{A^+}(x) \leq \alpha\}$ *is called* α*-cut of* A *and it is denoted by* A_α.
(ii) $\{x | \mu_{A^-}(x) > \alpha, \mu_{A^+}(x) < \alpha\}$ *is called strong* α*-cut of* A *and it is denoted by* A_α^+.
(iii) Support of A *is the set* $\{x \in X | \mu_{A^-}(x) \geq 0, \mu_{A^+}(x) > 0\}$. *It is denoted by* **supp**(A). *It can denoted as* **supp**$(A) = A_0^+$, *too.*

3. Cayley interval-valued fuzzy graphs

The concept of a Cayley fuzzy graph has become a standard part of the toolkit used to investigate and describe groups. It has become particularly important in the study of infinite finitely generated groups, where the Cayley fuzzy graph and related concepts provide a way to treat the group as a geometric object. Also, Cayley fuzzy graphs are good models for interconnection networks, and they are useful in semigroup theory for establishing which elements are $\mathcal{L}$ and R related. In this section, we introduce Cayley interval-valued fuzzy graphs and prove that all vertex transitive interval-valued fuzzy graphs are regular. Also, we define the concepts of α-connectedness, weakly α-connectedness, semi α-connectedness, locally α-connectedness, quasi α-connectedness, and strength of connectivity in interval-valued fuzzy graphs.

Definition 3.1. *By an interval-valued fuzzy digraph of a graph* $G^* = (V, E)$ *we mean a pair* $G = (A, R)$, *where* $A = [\mu_{A^-}, \mu_{A^+}]$ *is an interval-valued fuzzy set on* V *and* $R = [\mu_{B^-}, \mu_{B^+}]$ *is an interval-valued fuzzy relation on* E. *An interval valued fuzzy digraph* G *is said to be: (i) connected if for all* $x, y \in V$, *there is a directed path from* x *to* y, *(ii) weakly connected if* $(V, R \vee R^{-1})$ *is connected, (iii) semi-connected if for all* $x, y \in V$ *there is a directed path from* x *to* y *or there is a directed path from* y *to* x *in* G, *(iv) locally connected if for any* $x, y \in V$ *there is a directed path from* x *to* y *whenever there is directed path from* y *to* x *in* G, *(v) quasi-connected if for every pair* $x, y \in V$, *there is some* $z \in V$ *such that there is a directed path from* z *to* x *and there is a directed path from* z *to* y.

Definition 3.2. *Let* G *be an interval-valued fuzzy digraph. The in-degree of a vertex* u *in* G *is defined by* $ind(u) = (ind_\mu^-(u), ind_\mu^+(u))$, *where* $ind_\mu^-(u) =$

$\sum_{u\neq v}\mu_{B^-}(vu)$ and $ind^+_\mu(u)=\sum_{u\neq v}\mu_{B^+}(vu)$. Similarly, the out-degree of a vertex u in G is defined by $outd(u)=(outd^-_\mu(u),$ $outd^+_\mu(u))$, where $outd^-_\mu(u)=\sum_{u\neq v}\mu_{B^-}(uv)$ and $outd^+_\mu(u)=\sum_{u\neq v}\mu_{B^+}(uv)$. An interval-valued fuzzy digraph in which each vertex has the same out-degree r is called an out-regular digraph with index of out-regularity r. In-regular digraphs are defined similarly.

Example 3.1. *Consider a graph $G^*=(V,E)$ such that $V=\{v_1,v_2,v_3,v_4\}$ and $E=\{v_1v_2,v_2v_3,v_3v_4,v_4v_1\}$. Let A be an interval-valued fuzzy set of V and let R be an interval-valued fuzzy relation on E which is defined by*

$$A=\langle(\frac{v_1}{0.2},\frac{v_2}{0.2},\frac{v_3}{0.3},\frac{v_4}{0.1}),(\frac{v_1}{0.3},\frac{v_2}{0.5},\frac{v_3}{0.4},\frac{v_4}{0.5})\rangle,$$
$$R=\langle(\frac{v_1v_2}{0.2},\frac{v_2v_3}{0.2},\frac{v_3v_4}{0.1},\frac{v_4v_1}{0.1}),(\frac{v_1v_2}{0.3},\frac{v_2v_3}{0.3},\frac{v_3v_4}{0.3},\frac{v_4v_1}{0.2})\rangle.$$

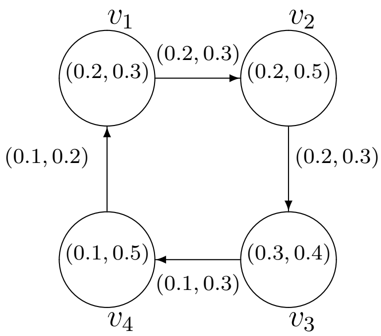

Figure 1. Interval-valued fuzzy digraph G

By routine computations, it is easy to see that $G=(A,R)$ is an interval-valued fuzzy digraph. Also, we have $ind(v_1)=(0.1,0.2), outd(v_1)=(0.2,0.3)$, $ind(v_3)=(0.2,0.3), outd(v_3)=(0.1,0.3)$. It is easy to show that G is both connected and semi-connected but it is not quasi-connected because if we consider two vertices v_1 and v_3, then there is no vertex which has a directed path to both v_1 and v_3. Clearly, G is locally connected.

Definition 3.3. *Let $(G,*)$ be a group and let S be a non-empty finite subset of G. Then the Cayley interval-valued fuzzy graph $G=(V,R)$ is an interval valued fuzzy graph with the vertex set $V=G$ and the interval valued fuzzy relation $R(x,y)$ on V which is defined by*

$$R(x,y)=\big(\mu_{A^-}(x^{-1}y),\mu_{A^+}(x^{-1}y)\big),\ x,y\in G \text{ and } x^{-1}y\in S$$

which $A=(\mu_{A^-},\mu_{A^+})$ be an interval valued fuzzy subset of V.

Example 3.2. *Consider the group $\mathbb{Z}_4$ and take $V=\mathbb{Z}_4$. Define $\mu_{A^-}:V\to[0,1]$ and $\mu_{A^+}:V\to[0,1]$ by $\mu_{A^-}(0)=1$, $\mu_{A^-}(1)=0.5$, $\mu_{A^-}(2)=0.4$, $\mu_{A^-}(3)=0.3$, $\mu_{A^+}(0)=1$, $\mu_{A^+}(1)=0.6$, $\mu_{A^+}(2)=0.5$, $\mu_{A^+}(3)=0.4$. Then the Cayley interval-valued fuzzy graphs $G=(V,R)$ induced by $(\mathbb{Z}_4,+,A)$ is given by Table 1, Table 2 and Figure 2.*

Table1

a	*0*	*0*	*0*	*0*	*1*	*1*	*1*	*1*
b	*0*	*1*	*2*	*3*	*0*	*1*	*2*	*3*
(-a)+b	*0*	*1*	*2*	*3*	*3*	*0*	*1*	*2*
R(a,b)	*(1,1)*	*(0.5,0.6)*	*(0.4,0.5)*	*(0.3,0.4)*	*(0.3,0.4)*	*(1,1)*	*(0.5,0.6)*	*(0.4,0.5)*

Table2

a	*2*	*2*	*2*	*2*	*3*	*3*	*3*	*3*
b	*0*	*1*	*2*	*3*	*0*	*1*	*2*	*3*
(-a)+b	*2*	*3*	*0*	*1*	*1*	*2*	*3*	*0*
R(a,b)	*(0.4,0.5)*	*(0.3,0.4)*	*(1,1)*	*(0.5,0.6)*	*(0.5,0.6)*	*(0.4,0.5)*	*(0.3,0.4)*	*(1,1)*

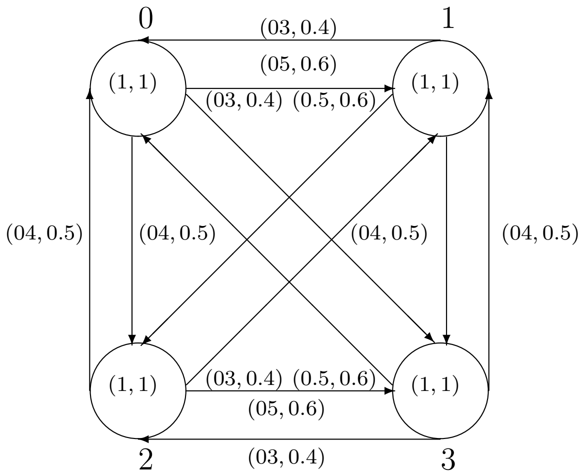

Figure 2. Cayley interval-valued fuzzy graph

Note1: Observe that the Cayley interval-valued fuzzy graphs are actually interval-valued fuzzy digraphs. Furthermore, the relation R in the Definition 3.3 describes the strength of each directed edge.

Note2: Symmetries in graphs and networks are closely related to the fields of group theory (more specifically, permutation group theory) and graph theory. Networks modelled on vertex-transitive graphs have been shown to be very good in their balance of cost (measured by the degree of each vertex in the network) against performance (how easy they are to disconnect, and the efficiency of algorithms run on them). Vertex-transitive graphs also provide a beautiful context in which to study many of the general problems of graph theory; beautiful not only because of the symmetric pictures, but because of the interactions with group theory, and permutation group theory in particular. We define an interval valued fuzzy graph G to be vertex-transitive, if for $x, y \in V$, there is an automorphism f on G such that $f(x) = y$. Let G denote an interval-valued fuzzy graph $G = (V, R)$ induced by the triple $(V, *, A)$. First, we will show that G is vertex transitive.

Theorem 3.1. *The Cayley interval-valued fuzzy graph G is vertex transitive.*

Proof. Let $u, v \in V$. Define $f : V \to V$ by $f(x) = vu^{-1}x$ for all $x \in V$. Clearly, f is a bijective map. For each $x, y \in V$, $R(f(x), f(y)) = (R_{\mu^-}(f(x), f(y)), R_{\mu^+}(f(x), f(y)))$. Now

$$\begin{aligned} R_{\mu^-}(f(x), f(y)) &= R_{\mu^-}(vu^{-1}x, vu^{-1}y) = \mu_{A^-}\big((vu^{-1}x)^{-1})(vu^{-1}y)\big) \\ &= \mu_{A^-}(x^{-1}y) = R_{\mu^-}(x, y) \\ R_{\mu^+}(f(x), f(y)) &= R_{\mu^+}(vu^{-1}x, vu^{-1}y) = \mu_{A^+}\big((vu^{-1}x)^{-1})(vu^{-1}y)\big) \\ &= \mu_{A^+}(x^{-1}y) = R_{\mu^+}(x, y). \end{aligned}$$

Therefore, $R(f(x), f(y)) = R(x, y)$. Hence f is an automorphism on G. Also, $f(u) = v$. Hence G is vertex transitive. □

One of the most widely studied classes of fuzzy graphs is regular fuzzy graphs. They show up in many contexts. Fore example, r-regular fuzzy graphs with connectivity and edge-connectivity equal to r play a key role in designing reliable communication networks. Now we show that every vertex transitive interval-valued fuzzy graph is regular.

Theorem 3.2. *Every vertex transitive interval-valued fuzzy graph is regular.*

Proof. Let $G = (V, R)$ be a vertex transitive interval-valued fuzzy graph and $u, v \in V$. Then there is an automorphism f on G such that $f(u) = v$. Since

$$\begin{aligned} ind(u) &= \sum_{x\in V} R(x, u) = \sum_{x\in V} \big(R_{\mu^-}(x, u), R_{\mu^+}(x, u)\big) \\ &= \sum_{x\in V} \big(R_{\mu^-}(f(x), f(u)), R_{\mu^+}(f(x), f(u))\big) \\ &= \sum_{x\in V} \big(R_{\mu^-}(f(x), v), R_{\mu^+}(f(x), v)\big) \\ &= \sum_{x\in V} \big(R_{\mu^-}(y, v), R_{\mu^+}(y, v)\big) = ind(v), \end{aligned}$$

and

$$\begin{aligned} outd(u) &= \sum_{x\in V} R(u, x) = \sum_{x\in V} \big(R_{\mu^-}(u, x), R_{\mu^+}(u, x)\big) \\ &= \sum_{x\in V} \big(R_{\mu^-}(f(u), f(x)), R_{\mu^+}(f(u), f(x))\big) \\ &= \sum_{x\in V} \big(R_{\mu^-}(v, f(x)), R_{\mu^+}(v, f(x))\big) \\ &= \sum_{x\in V} \big(R_{\mu^-}(v, y), R_{\mu^+}(v, y)\big) = outd(v). \end{aligned}$$

Then G is regular. □

Theorem 3.3. *Cayley interval-valued fuzzy graphs are regular, too.*

Proof. It follows from Theorem 3.1 and Theorem 3.2. □

Theorem 3.4. *Let $G = (V, R)$ be an interval-valued fuzzy graph. Then the interval-valued fuzzy relation R is symmetric if and only if $(\mu_{A^-}(x), \mu_{A^+}(x)) = (\mu_{A^-}(x^{-1}), \mu_{A^+}(x^{-1}))$ for all $x \in V$.*

Proof. Suppose that R is symmetric. Then for any $x \in V$,

$$\begin{aligned}\big(\mu_{A^-}(x), \mu_{A^+}(x)\big) &= \big(\mu_{A^-}(x^{-1}x^2), \mu_{A^+}(x^{-1}x^2)\big) = R(x, x^2)\\ &= R(x^2, x)(\text{Since } R \text{ is symmetric})\\ &= \big(\mu_{A^-}((x^2)^{-1}x), \mu_{A^+}((x^2)^{-1}x)\big) = \big(\mu_{A^-}(x^{-2}x), \mu_{A^+}(x^{-2}x)\big)\\ &= \big(\mu_{A^-}(x^{-1}), \mu_{A^+}(x^{-1})\big).\end{aligned}$$

Conversely, suppose that $\big(\mu_{A^-}(x), \mu_{A^+}(x)\big) = \big(\mu_{A^-}(x^{-1}), \mu_{A^+}(x^{-1})\big)$ for all $x \in V$. Then for all $x, y \in V$,

$$R(x, y) = \big(\mu_{A^-}(x^{-1}y), \mu_{A^+}(x^{-1}y)\big) = \big(\mu_{A^-}(y^{-1}x), \mu_{A^+}(y^{-1}x)\big) = R(y, x).$$

Hence R is symmetric. □

Definition 3.4. [7] *Let $(S, *)$ be a semigroup and $A = (\mu_{A^-}, \mu_{A^+})$ be an interval-valued fuzzy subset of S. Then A is said to be an interval-valued fuzzy subsemigroup of S if for all $x, y \in S$, $\mu_{A^-}(xy) \geq \mu_{A^-}(x) \wedge \mu_{A^-}(y)$ and $\mu_{A^+}(xy) \geq \mu_{A^+}(x) \wedge \mu_{A^+}(y)$.*

Remark 3.1. *Let $G = (V, R)$ be an interval valued fuzzy graph. Then G is connected (weakly connected, semi-connected, locally connected or quasi-connected) if and only if the induce fuzzy graph (V, R_0^+) is connected (weakly connected, semi-connected, locally connected or quasi-connected).*

Definition 3.5. *Let $(S, *)$ be a semigroup and $A = (\mu_{A^-}, \mu_{A^+})$ be an interval valued fuzzy subset of S. Then the subsemigroup generated by A is defined the meet of all interval valued fuzzy subsemigroup of S which contains A. It is denoted by $\langle A \rangle$.*

Proposition 3.1. *Let $(S, *)$ be a semigroup and $A = (\mu_{A^-}, \mu_{A^+})$ be an interval-valued fuzzy subset of S. Then the interval-valued fuzzy subset $\langle A \rangle$ is precisely given by; for any $x \in S$*

$$\langle \mu_{A^-} \rangle(x) = \vee\{\mu_{A^-}(x_1) \wedge \mu_{A^-}(x_2) \wedge \cdots \wedge \mu_{A^-}(x_n) : \ x = x_1x_2 \cdots x_n$$
$$\textit{with } \mu_{A^-}(x_i) > 0 \textit{ for } i = 1, 2, \cdots n\},$$
$$\langle \mu_{A^+} \rangle(x) = \vee\{\mu_{A^+}(x_1) \wedge \mu_{A^+}(x_2) \wedge \cdots \wedge \mu_{A^+}(x_n) : \ x = x_1x_2 \cdots x_n$$
$$\textit{with } \mu_{A^+}(x_i) > 0 \textit{ for } i = 1, 2, \cdots n\}.$$

Proof. Let $x \in S$ and $A' = (\mu'_{A^-}, \mu'_{A^+})$ be an interval-valued fuzzy subset of S defined by

$$\mu'_{A^-}(x) = \vee\{\mu_{A^-}(x_1) \wedge \mu_{A^-}(x_2) \wedge \cdots \wedge \mu_{A^-}(x_n) : \ x = x_1x_2 \cdots x_n$$

with $\mu_{A^-}(x_i) > 0$ for $i = 1, 2, \cdots n\}$,

$$\mu'_{A^+}(x) = \vee\{\mu_{A^+}(x_1) \wedge \mu_{A^+}(x_2) \wedge \cdots \wedge \mu_{A^+}(x_n) : \ x = x_1x_2\cdots x_n$$

with $\mu_{A^+}(x_i) > 0$ for $i = 1, 2, \cdots n\}$.

Let $x, y \in S$. If $\mu_{A^-}(x) = 0$ or $\mu_{A^-}(y) = 0$ then $\mu_{A^-}(x) \wedge \mu_{A^-}(y) = 0$ and $\mu_{A^+}(x) = 0$ or $\mu_{A^+}(y) = 0$, then $\mu_{A^+}(x) \wedge \mu_{A^+}(y) = 0$. Therefore, $\mu'_{A^-}(xy) \geq \mu_{A^-}(x) \wedge \mu_{A^-}(y)$ and $\mu'_{A^+}(xy) \geq \mu_{A^+}(x) \wedge \mu_{A^+}(y)$. Moreover, if $\mu_{A^-}(x) \neq 0$ and $\mu_{A^+}(x) \neq 0$, then by definition of $\mu'_{A^-}(x)$ and $\mu'_{A^+}(x)$, we have

$$\mu'_{A^-}(xy) \geq \mu_{A^-}(x) \wedge \mu_{A^-}(y) \text{ and } \mu'_{A^+}(xy) \geq \mu_{A^+}(x) \wedge \mu_{A^+}(y).$$

Hence, (μ'_{A^-}, μ'_{A^+}) is an interval-valued fuzzy subsemigroup of S containing (μ_{A^-}, μ_{A^+}). Now let L be any interval-valued fuzzy subsemigroup of S containing (μ_{A^-}, μ_{A^+}). Then for any $x \in S$ with $x = x_1x_2\cdots x_n$, $\mu_{A^-}(x_i) > 0$, $\mu_{A^+}(x_i) > 0$, for $i = 1, 2, \cdots n$, we have

$$\mu_{L^-}(x_i) \geq \mu_{L^-}(x_1)\wedge\mu_{L^-}(x_2)\wedge\cdots\wedge\mu_{L^-}(x_n) \geq \mu_{A^-}(x_1)\wedge\mu_{A^-}(x_2)\wedge\cdots\wedge\mu_{A^-}(x_n)$$

and $\mu_{L^+}(x_i) \geq \mu_{L^+}(x_1) \wedge \mu_{L^+}(x_2) \wedge \cdots \wedge \mu_{L^+}(x_n) \geq \mu_{A^+}(x_1) \wedge \mu_{A^+}(x_2) \wedge \cdots \wedge \mu_{A^+}(x_n)$. Then

$$\mu_{L^-}(x) \geq \vee\{\mu_{A^-}(x_1) \wedge \mu_{A^-}(x_2) \wedge \cdots \wedge \mu_{A^-}(x_n) : \ x = x_1x_2\cdots x_n$$

with $\mu_{A^-}(x_i) > 0$ for $i = 1, 2, \cdots n\}$,

$$\mu_{L^+}(x) \geq \vee\{\mu_{A^+}(x_1) \wedge \mu_{A^+}(x_2) \wedge \cdots \wedge \mu_{A^+}(x_n) : \ x = x_1x_2\cdots x_n$$

with $\mu_{A^+}(x_i) > 0$ for $i = 1, 2, \cdots n\}$.

Hence $\mu_{L^-}(x) \geq \mu'_{A^-}(x)$, $\mu_{L^+}(x) \geq \mu'_{A^+}(x)$, for all $x \in S$. Thus $\mu'_{A^-}(x) \leq \mu_{L^-}(x)$, $\mu'_{A^+}(x) \leq \mu_{L^+}(x)$, for all $x \in S$. Therefore, $A' = (\mu'_{A^-}, \mu'_{A^+})$ is the meeting of all interval-valued fuzzy subsemigroups containing (μ_{A^-}, μ_{A^+}). □

Theorem 3.5. *Let $(S, *)$ be a semigroup and $A = (\mu_{A^-}, \mu_{A^+})$ be an interval-valued fuzzy subset of S. Then for any $\alpha \in [0, 1]$, $(\langle\mu^-_\alpha\rangle, \langle\mu^+_\alpha\rangle) = (\langle\mu^-\rangle_\alpha, \langle\mu^+\rangle_\alpha)$, where $(\langle\mu^-_\alpha\rangle, \langle\mu^+_\alpha\rangle)$ denotes the subsemigroup generated by $(\mu^-_\alpha, \mu^+_\alpha)$ and $\langle(\mu^-, \mu^+)\rangle$ denotes the interval-valued fuzzy subsemigroup generated by (μ^-, μ^+).*

Proof. By Proposition 3.1, we have

$$\begin{aligned} x \in (\langle\mu^-_\alpha\rangle, \langle\mu^+_\alpha\rangle) &\iff \exists\, x_1, x_2, \cdots, x_n \in (\mu^-_\alpha, \mu^+_\alpha), \text{st. } x = x_1x_2\cdots x_n \\ &\iff \exists\, x_1, x_2, \cdots, x_n \in S, \text{ st. } \mu^-(x_i) \geq \alpha, \\ &\qquad \mu^+(x_i) \leq \alpha, \forall\, i = 1, 2, \cdots, n, \ x = x_1x_2\cdots x_n \\ &\iff \langle\mu^-\rangle(x) \geq \alpha, \ \langle\mu^+\rangle(x) \leq \alpha \iff x \in \langle\mu^-\rangle_\alpha, \ x \in \langle\mu^+\rangle_\alpha. \end{aligned}$$

Therefore, $(\langle\mu^-_\alpha\rangle, \langle\mu^+_\alpha\rangle) = (\langle\mu^-\rangle_\alpha, \langle\mu^+\rangle_\alpha)$. □

Remark 3.2. *Let $(S, *)$ be a semigroup and $A = (\mu_{A^-}, \mu_{A^+})$ be an interval-valued fuzzy subset of S. Then by Theorem 3.5, we have $\langle\mathbf{supp}(A)\rangle = \mathbf{supp}\langle A\rangle$.*

Definition 3.6. *Let $(S, *)$ be a group and A be an interval-valued fuzzy subset of S. Then we define A^{-1} as interval-valued fuzzy subset of S given by $A^{-1}(x) = A(x^{-1})$ for all $x \in S$.*

Theorem 3.6. [9] *Let A be any subset of V' and $G' = (V', R')$ be the Cayley graph induced by $(V', *, A)$. Then G' is*
(i) connected $\Longleftrightarrow \langle A\rangle \supseteq V - v_1$,
(ii) weakly connected $\Longleftrightarrow \langle A \cup A^{-1}\rangle \supseteq V - v_1$, *where* $A^{-1} = \{x^{-1} : x \in A\}$,
(iii) semi-connected $\Longleftrightarrow \langle A\rangle \cup \langle A^{-1}\rangle \supseteq V - v_1$,
(iv) locally connected $\Longleftrightarrow \langle A\rangle = \langle A^{-1}\rangle$,
(v) quasi-connected $\Longleftrightarrow$ *it is connected.*

Theorem 3.7. *Let G denote the Cayley interval-valued fuzzy graph $G = (V, R)$ induced by $(V, *, \mu^-, \mu^+)$. Then we have the following results.*
(i) G is weakly connected $\Longleftrightarrow \mathbf{supp}(\langle A \cup A^{-1}\rangle) \supseteq V - v_1$,
(ii) G is semi-connected $\Longleftrightarrow \mathbf{supp}(\langle A\rangle \cup \langle A^{-1}\rangle) \supseteq V - v_1$,
(iii) G is locally connected $\Longleftrightarrow \mathbf{supp}(\langle A\rangle) = \mathbf{supp}(\langle A^{-1}\rangle)$,
(iv) G is quasi-connected $\Longleftrightarrow$ *it is connected.*

Proof. (i) G is weakly connected $\Longleftrightarrow \langle A_0^+ \cup (A_0^+)^{-1}\rangle \supseteq V - v_1$

$$\begin{aligned}
&\Longleftrightarrow \langle \mathbf{supp}(A) \cup \mathbf{supp}(A)^{-1}\rangle \supseteq V - v_1\\
&\Longleftrightarrow \mathbf{supp}\langle A \cup (A)^{-1}\rangle \supseteq V - v_1\\
&\Longleftrightarrow \mathbf{supp}\langle A \cup A^{-1}\rangle \supseteq V - v_1.
\end{aligned}$$

$$\begin{aligned}
(ii)\ \text{G is semi-connected} &\Longleftrightarrow (V, R_0^+)\text{is semi-connected}\\
&\Longleftrightarrow \langle A_0^+\rangle \cup \langle (A_0^+)^{-1}\rangle \supseteq V - v_1\\
&\Longleftrightarrow \langle \mathbf{supp}(A)\rangle \cup \langle \mathbf{supp}(A)^{-1}\rangle \supseteq V - v_1\\
&\Longleftrightarrow \mathbf{supp}\big(\langle A\rangle \cup \langle (A)^{-1}\rangle\big) \supseteq V - v_1\\
&\Longleftrightarrow \mathbf{supp}\big(\langle A\rangle \cup \langle A^{-1}\rangle\big) \supseteq V - v_1.\\
(iii)\ \text{G is locally connected} &\Longleftrightarrow (V, R_0^+)\text{is locally connected}\\
&\Longleftrightarrow \langle A_0^+\rangle = \langle (A_0^+)^{-1}\rangle\\
&\Longleftrightarrow \langle \mathbf{supp}(A)\rangle = \langle \mathbf{supp}(A)^{-1}\rangle\\
&\Longleftrightarrow \mathbf{supp}\langle A\rangle = \mathbf{supp}\langle A^{-1}\rangle.\\
(iv)\ G\ \text{is quasi-connected} &\Longleftrightarrow (V, R_0^+)\text{is quasi-connected}\\
&\Longleftrightarrow (V, R_0^+)\ \text{is connected} \Longleftrightarrow G\ \text{is connected.}
\end{aligned}$$

□

Definition 3.7. *The μ^- strength of a path $P = v_1, v_2, \cdots, v_n$ is defined as* $\min\{\mu_{B^-}(v_iv_j)|$
$i, j \in \{1, 2, \cdots, n\}\}$ *and is denoted by S_{μ^-}. The μ^+ strength of a path $P = v_1, v_2, \cdots, v_n$ is defined as* $\max\{\mu_{B^+}(v_iv_j)|\ i, j \in \{1, 2, \cdots, n\}\}$ *and is denoted by S_{μ^+}. The strength of path P is denote by $P = \{S_{\mu^-}, S_{\mu^+}\}$.*

Definition 3.8. *Let $G = (V, \mu^-, \mu^+)$ be an interval-valued fuzzy graph. Then G is said to be*
(i) α-connected if for every pair of vertices $x, y \in G$, there is a path P from x to y such that strength$(P) \geq \alpha$,

(ii) weakly α-connected if an interval-valued fuzzy graph $(V, R \vee R^{-1})$ is α-connected,
(iii) semi α-connected if for every $x, y \in V$, there is a path of strength greater than or equal to α from x to y or from y to x in G,
(iv) locally α-connected if for every pair of vertices x and y, there is a path P of strength greater than or equal to α from x to y and a path P' of strength greater than or equal to α from y to x,
(v) quasi α-connected if for every pair $x, y \in V$, there is $z \in V$ and a directed path from z to x of strength greater than or equal to α and a directed path from z to y of strength greater than or equal to α.

Example 3.3. *Consider an interval-valued fuzzy graph G as shown in the Figure 3.*

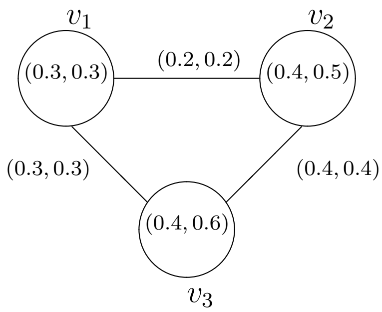

Figure 3. Interval-valued fuzzy graph G

By routine computation it is easy to show that G is α-connected, semi α-connected and quasi α-connected for $\alpha = (0.2, 0.2)$.

Remark 3.3. *Let $G = (V, R)$ be an interval-valued fuzzy graph. Then G is α-connected (weakly α-connected, semi α-connected, locally α-connected or quasi α-connected) if and only if the induce fuzzy graph (V, R_α) is connected (weakly connected, semi connected, locally connected or quasi connected).*

Theorem 3.8. *Let $G = (V, R)$ be a Cayley interval-valued fuzzy graph induced by $(V, *, \mu^-, \mu^+)$. Then for any $\alpha \in [0, 1]$,*
(i) G is α-connected $\Longleftrightarrow \langle A\rangle_\alpha \supseteq V - v_1$,
(ii) G is semi α-connected $\Longleftrightarrow (\langle A\rangle_\alpha \cup \langle A^{-1}\rangle_\alpha) \supseteq V - v_1$,
(iii) G is locally α-connected $\Longleftrightarrow \langle A\rangle_\alpha = \langle A_\alpha^{-1}\rangle$.

Proof. (i) By Remark 3.3 and Theorems 3.5 and 3.6,

$$\begin{aligned} G \text{ is } \alpha\text{-connected} &\Longleftrightarrow (V, R_\alpha) \text{ is connected} \Longleftrightarrow \langle A_\alpha\rangle \supseteq V - v_1 \\ &\Longleftrightarrow \langle A\rangle_\alpha \supseteq V - v_1. \end{aligned}$$

(ii) By Remark 3.3 and Theorems 3.5 and 3.7

$$\begin{aligned} G \text{ is semi } \alpha\text{-connected} &\Longleftrightarrow (V, R_\alpha) \text{ is semi } \alpha\text{-connected} \\ &\Longleftrightarrow (\langle A_\alpha\rangle \cup \langle A_\alpha^{-1}\rangle) \supseteq V - v_1 \\ &\Longleftrightarrow (\langle A\rangle_\alpha \cup \langle A^{-1}\rangle_\alpha) \supseteq V - v_1. \end{aligned}$$

(iii) By Remark 3.3 and Theorems 3.5 and 3.7

$$\begin{aligned} G \text{ is locally } \alpha\text{-connected} &\iff (V, R_\alpha) \text{ is locally } \alpha\text{-connected} \\ &\iff \langle A_\alpha \rangle = \langle A_\alpha^{-1} \rangle \iff \langle A \rangle_\alpha = \langle A^{-1} \rangle_\alpha. \end{aligned}$$

□

4. Conclusion

Graph theory is an extremely useful tool in solving the combinatorial problems in different areas including geometry, algebra, topology, optimization, and computer science. The interval-valued fuzzy sets consitute a generalization of Zadeh's fuzzy set theory. In this paper, we introduced a class of Cayley interval-valued fuzzy graphs and then studied its various graph theoretic properties in terms of algebraic properties. Moreover, we defined the concepts of α-connectedness, weakly α-connectedness, semi α-connectedness, locally α-connectedness and quasi α-connectedness in interval-valued fuzzy graphs. In our future work, we will focus on vague planar graphs and define the other relevant terms such as vague multigraphs, strong edges, vague faces, and strong vague faces. We will use the term degree of planarity to measure the nature of planarity of a vague graph.

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