STRONG CONVERGENCE OF A MULTI-STEP ITERATIVE PROCESS FOR RELATIVELY QUASI-NONEXPANSIVE MULTIVALUED MAPPINGS AND EQUILIBRIUM PROBLEM IN BANACH SPACES

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In this paper, we introduce a multi-step iterative process which converges strongly to a common element of a set of common fixed points of a finite family of relatively quasi-nonexpansive multivalued mappings and the solution set of an equilibrium problem in Banach spaces. Our results extend some important recent results.

Keywords: Relatively quasi-nonexpansive, multivalued mapping, equilibrium problem, common fixed point.

MSC2010: 47H10, 47H09

1. Introduction

Let E be a real Banach space with norm $\|.\|$ and let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}$$

for all $x \in E$, where E^* denotes the dual space of E and $\langle ., . \rangle$ denotes the generalized duality pairing between E and E^*. A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided that

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in E$. It is well known that if E^* is strictly convex then J is single valued, and if E is uniformly smooth then J is uniformly continuous on bounded subsets of E. Moreover, if E is a reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is single valued, one-to-one, surjective, and it is the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$. We note that in a Hilbert space H, J is the identity operator.

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Let E be a smooth Banach space and J be the normalized duality mapping from E to E^*. Alber [1] considered the following function $\phi : E \times E \to [0, \infty)$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \qquad x, y \in E.$$

It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(x, y) \leq (\|y\| + \|x\|)^2 \qquad \forall x, y \in E. \tag{1.1}$$

Observe that in a Hilbert space H, $\phi(x, y) = \|x-y\|^2, x, y \in H$. Let E be a reflexive, strictly convex and smooth Banach space and let C be a nonempty closed and convex subset of E. The generalized projection mapping, introduced by Alber [1], is a mapping $\Pi_C : E \to C$, that assigns to an arbitrary point $x \in E$ the minimum point of the function $\phi(y, x)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \inf_{y \in C} \phi(y, x).$$

Lemma 1.1. *(see [1]) Let C be a nonempty closed and convex subset of a real reflexive and strictly convex Banach space E and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = \inf_{y \in C} \phi(y, x)$.*

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point $p \in C$ is said to be an asymptotic [2] fixed point of T, if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F(T)}$. A mapping T is said to be relatively nonexpansive [3, 4], if $\widetilde{F(T)} = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. T is said to be relatively quasi-nonexpansive ([5, 6]) if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The class of relatively quasi-nonexpansive mappings is bigger than the class of relatively nonexpansive mappings which requires the strong restriction: $\widetilde{F(T)} = F(T)$.

Let f be a bifunction from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$f(x, y) \geq 0, \qquad \forall y \in C.$$

We shall denote the set of solutions of this equilibrium problem by $EP(f)$. The equilibrium problems include fixed point problems, optimization problems and variational inequality problems as special cases. Some methods have been proposed to solve the equilbrium problem, see for example, [7-10].

Recently, many authors studied the problem of finding a common element of the set of fixed points of nonexpansive or relatively nonexpansive single valued mappings and the set of solutions of an equilibrium problem in the framework of Hilbert or Banach spaces, respectively: see, for instance, [11-19] and the references therein.

A subset $C \subset E$ is called proximinal if for each $x \in E$, there exists an element $y \in C$ such that

$$\| x - y \| = dist(x, C) = inf\{\| x - z \|: z \in C\}.$$

We denote by $N(C), CB(C)$ and $P(C)$ the collection of all nonempty subsets, nonempty closed bounded subsets and nonempty proximinal bounded subsets of C, respectively. The Hausdorff metric H on $CB(C)$ is defined by

$$H(A,B) := \max\{\sup_{x\in A} dist(x,B), \sup_{y\in B} dist(y,A)\},$$

for all $A, B \in CB(C)$.
Let $T : E \to N(E)$ be a multivalued mapping. An element $x \in E$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points of T will be denoted by $F(T)$.

Definition 1.2. A multivalued mapping $T : E \to CB(E)$ is called

(i) nonexpansive if

$$H(Tx,Ty) \le \|x-y\|, \quad x,y \in E.$$

(ii) quasi-nonexpansive if

$$F(T) \neq \emptyset \qquad and \qquad H(Tx,Tp) \le \|x-p\|, \quad x \in E, \quad p \in F(T).$$

In recent years, approximation of fixed points of nonexpansive multivalued mappings by iteration has been studied by many authors, see [20-24]. The theory of multivalued mappings has applications in control theory, convex optimization, differential equations and economics.

In this paper we intend to modify the concept of relatively nonexpansivness to incorporate the multivalued case as well. This will be done in the following definition.

Definition 1.3. Let C be a closed convex subset of a smooth Banach space E, and $T : C \to N(C)$ be a multivalued mapping. We set

$$\Phi(Tx,Tp) = max\{\sup_{q\in Tp}\inf_{y\in Tx} \phi(y,q), \sup_{y\in Tx}\inf_{q\in Tp} \phi(y,q)\}.$$

We call T relatively quasi-nonexpansive multivalued mapping if $F(T) \neq \emptyset$ and

$$\Phi(Tx,Tp) \le \phi(x,p), \qquad \forall p \in F(T), \quad \forall x \in C.$$

Remark : In a Hilbert space, $\Phi(Tx,Ty) = H(Tx,Ty)^2$, and hence relatively quasi-nonexpansivness is equivalent to quasi-nonexpansivness.

In this paper, a multi-step iterative process by hybrid method is constructed. Strong convergence of the iterative process to a common element of the set of common fixed points of a finite family of relatively quasi-nonexpansive multivalued mappings and the solution set of an equilibrium problem in a uniformly convex real Banach space which is also uniformly smooth is proved. Our results extend some important recent results.

2. **Preliminaries**

Lemma 2.1. *([25, 26]) If E is a reflexive, strictly convex and smooth Banach space, then for $x,y \in E$, $\phi(x,y) = 0$ if and only if $x = y$.*

Lemma 2.2. *([27]) Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If $\phi(x_n,y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.3. *([1]) Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \qquad \forall y \in C.$$

Lemma 2.4. *([1]) Let E be a reflexive, strictly convex and smooth Banach space, Let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \qquad \forall y \in C.$$

Lemma 2.5. *Let C be a nonempty closed convex subset of a uniformly convex and smooth Banach space E. Suppose $T : C \to P(C)$ is a multivalued mapping such that P_T is a relatively quasi-nonexpansive multivalued mapping where*

$$P_T(x) = \{y \in Tx : \|x - y\| = dist(x, Tx)\}.$$

If $F(T) \neq \emptyset$, then $F(T)$ is closed and convex.

Proof. Let $\{p_n\}$ be a sequence in $F(T)$, such that $p_n \to p$ as $n \to \infty$. Then we have $P_T(p_n) = \{p_n\}$. Since P_T is relatively quasi-nonexpansive, we have

$$\sup_{z \in P_T(p)} \phi(p_n, z) = \Phi(P_T(p_n), P_T(p)) \leq \phi(p_n, p).$$

Hence for all $z \in P_T(p)$,

$$0 \leq \phi(p, z) = \lim_{n \to \infty} \phi(p_n, z) \leq \lim_{n \to \infty} \phi(p_n, p) \leq \phi(p, p) = 0.$$

This implies that $p = z \in P_T(p) \subset T(p)$. Therefore $F(T)$ is closed. Now, we show that $F(T)$ is convex. Let $p_1, p_2 \in F(T)$, then $P_T(p_1) = \{p_1\}$ and $P_T(p_2) = \{p_2\}$. Take $t \in (0, 1)$, and put $p = tp_1 + (1 - t)p_2$. Let $w \in P_T(p)$, then we have

$$\begin{aligned}
\phi(p, w) &= \|p\|^2 - 2\langle p, Jw \rangle + \|w\|^2 \\
&= \|p\|^2 - 2\langle tp_1 + (1-t)p_2, Jw \rangle + \|w\|^2 \\
&= \|p\|^2 - 2t\langle p_1, Jw \rangle - 2(1-t)\langle p_2, Jw \rangle + \|w\|^2 \\
&= \|p\|^2 + t\phi(p_1, w) + (1-t)\phi(p_2, w) - t\|p_1\|^2 - (1-t)\|p_2\|^2 \\
&= \|p\|^2 + t \inf_{p_1 \in P_T(p_1)} \phi(p_1, w) + (1-t) \inf_{p_2 \in P_T(p_2)} \phi(p_2, w) - t\|p_1\|^2 - (1-t)\|p_2\|^2 \\
&\leq \|p\|^2 + t\Phi(P_T(p_1), P_T(p)) + (1-t)\Phi(P_T(p_2), P_T(p)) - t\|p_1\|^2 - (1-t)\|p_2\|^2 \\
&\leq \|p\|^2 + t\phi(p_1, p) + (1-t)\phi(p_2, p) - t\|p_1\|^2 - (1-t)\|p_2\|^2 \\
&= \|p\|^2 - 2\langle tp_1 + (1-t)p_2, Jp \rangle + \|p\|^2 \\
&= \|p\|^2 - 2\langle p, Jp \rangle + \|p\|^2 = \phi(p, p) = 0.
\end{aligned}$$

This implies, using Lemma 2.1, that $p = w \in P_T(p) \subset T(p)$, i.e., $p \in F(T)$. Hence $F(T)$ is convex. □

Similarly we can prove the following lemma.

Lemma 2.6. *Let C be a nonempty closed convex subset of a uniformly convex and smooth Banach space E. Suppose $T : C \to N(C)$ is a relatively quasi- nonexpansive multivalued mapping. If $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for all $p \in F(T)$, then $F(T)$ is closed and convex.*

Now we present an example of a multivalued mapping such that P_T is relatively quasi-nonexpansive, but T is not relatively quasi-nonexpansive.
Example : Let $I = [0,1]$, $E = L^p(I)$, $1 < p < \infty$ and $C = \{f \in E : f(x) \geq 0, \forall x \in I\}$. Let $T : C \to CB(C)$ be defined by

$$T(f) = \{g \in C : f(x) \leq g(x) \leq 3\, f(x)\}.$$

Then we have

$$P_T(f) = \{g \in T(f), \|g - f\|_p = dist(T(f), f)\} = \{f\}$$

and hence

$$\Phi(P_T(f_1), P_T(f_2)) \leq \phi(f_1, f_2), \qquad \forall f_1, f_2 \in C.$$

Therefore P_T is relatively quasi-nonexpansive. Now putting $f_1(x) = 0$ and $f_2(x) = 1$ we have $T(f_1) = 0$ and $T(f_2) = \{g \in C : 1 \leq g(x) \leq 3\}$, hence $\Phi(T0, T1) = \sup_{g \in T1} \phi(0, g) = \phi(0, 3)$. On the other hand $\phi(0, 1) = \|1\|_p^2 = 1$ and $\phi(0, 3) = \|3\|_p^2 = 9$, which shows that

$$\Phi(T0, T1) > \phi(0, 1).$$

Hence T is not relatively quasi-nonexpansive.

Definition 2.7. A multivalued mapping T is called closed if $x_n \to w$ and $\lim_{n\to\infty} dist(x_n, Tx_n) = 0$, then $w \in T(w)$.

Lemma 2.8. *([28]) Let E be a uniformly convex Banach space and let $B_r(0) = \{x \in E : \| x \| \leq r\}$, for $r > 0$. Then there exists a continuous, strictly increasing convex function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\varphi(\|x - y\|)$$

for all $x, y \in B_r(0)$.

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$,
(A2) f is monotone, i.e. $f(x, y) + f(y, x) \leq 0$ for any $x, y \in C$,
(A3) f is upper-hemicontinuous, i.e. for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} f(tz + (1-t)x, y) \leq f(x, y),$$

(A4) $f(x, .)$ is convex and lower semicontinuous for each $x \in C$.

The following lemma was proved in [7].

Lemma 2.9. *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \qquad \forall y \in C.$$

The following lemma was given in [12].

Lemma 2.10. *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $r > 0$ and $x \in E$. Define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\}.$$

Then, the following hold:

(i) *T_r is single valued,*

(ii) *T_r is a firmly nonexpansive-type mapping, i.e., for any $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle,$$

(iii) *$F(T_r) = EP(f)$,*

(iv) *$EP(f)$ is closed and convex.*

Lemma 2.11. *([12]) Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction of $C\times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$, and let $r > 0$. Then for all $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

3. Main Result

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $T_i : C \to N(C)$, $i = 1, 2, ..., m$, be a finite family of closed relatively quasi-nonexpansive multivalued mappings such that $\mathcal{F} = \bigcap_{i=1}^m F(T_i) \bigcap EP(f) \neq \emptyset$ and $T_i(p) = \{p\}$ for all $p \in \mathcal{F}$. For $x_0 \in C$ and $C_0 = C$, let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} y_{n,1} = J^{-1}((1 - a_{n,1})Jx_n + a_{n,1}Jz_{n,1}), \\ y_{n,2} = J^{-1}((1 - a_{n,2})Jx_n + a_{n,2}Jz_{n,2}), \\ \ldots \\ y_{n,m} = J^{-1}((1 - a_{n,m})Jx_n + a_{n,m}Jz_{n,m}), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_{n,m}\rangle \geq 0, \qquad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \prod_{C_{n+1}} x_0, \qquad \forall n \geq 0 \end{cases}$$

where $z_{n,1} \in T_1 x_n$ and $z_{n,i} \in T_i y_{n,i-1}$ for $i = 2, ..., m$ and J is the duality mapping on E. Assume that $\sum_{i=1}^m a_{n,i} = 1$, $\{a_{n,i}\} \in [a, b] \subset (0, 1)$ and $\{r_n\} \subset [c, \infty)$ for some $c > 0$. Suppose that each T_i is uniformly continuous with respect to the Hausdorff metric for $i = 2, ..., m$. Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_0$, where $\Pi_{\mathcal{F}}$ is the projection of E onto $\mathcal{F}$.

Proof. At first, we show that C_n is closed and convex for each $n \geq 0$. From the definition, it is obvious that C_n is closed. Moreover, since $\phi(z, u_n) \leq \phi(z, x_n)$ is equivalent to

$$2\langle z, Jx_n - Ju_n\rangle - \|x_n\|^2 + \|u_n\|^2 \leq 0,$$

it follows that C_n is convex for each $n \geq 0$. Next, we show by induction that $\mathcal{F} = \bigcap_{i=1}^m F(T_i) \bigcap EP(f) \subset C_n$ for all $n \geq 0$. From $C_0 = C$, we have $\mathcal{F} \subset C_0$. We suppose that $\mathcal{F} \subset C_n$ for some $n \geq 0$. Let $u \in \mathcal{F}$. Since for each $1 \leq i \leq m$, T_i is relatively quasi- nonexpansive, we have

$$\begin{aligned}\phi(u, y_{n,1}) &= \phi(u, J^{-1}((1-a_{n,1})Jx_n + a_{n,1}Jz_{n,1}))\\ &= \|u\|^2 - 2\langle u, (1-a_{n,1})Jx_n + a_{n,1}Jz_{n,1}\rangle + \|(1-a_{n,1})Jx_n + a_{n,1}Jz_{n,1}\|^2\\ &\leq \|u\|^2 - 2(1-a_{n,1})\langle u, Jx_n\rangle - 2a_{n,1}\langle u, Jz_{n,1}\rangle + (1-a_{n,1})\|x_n\|^2 + a_{n,1}\|z_{n,1}\|^2\\ &= (1-a_{n,1})\phi(u, x_n) + a_{n,1}\phi(u, z_{n,1})\\ &\leq (1-a_{n,1})\phi(u, x_n) + a_{n,1}\Phi(T_1u, T_1x_n)\\ &\leq (1-a_{n,1})\phi(u, x_n) + a_{n,1}\phi(u, x_n) = \phi(u, x_n),\end{aligned}$$

and

$$\begin{aligned}\phi(u, y_{n,2}) &= \phi(u, J^{-1}((1-a_{n,2})Jx_n + a_{n,2}Jz_{n,2}))\\ &= \|u\|^2 - 2\langle u, (1-a_{n,2})Jx_n + a_{n,2}Jz_{n,2}\rangle + \|(1-a_{n,2})Jx_n + a_{n,2}Jz_{n,2}\|^2\\ &\leq \|u\|^2 - 2(1-a_{n,2})\langle u, Jx_n\rangle - 2a_{n,2}\langle u, Jz_{n,2}\rangle + (1-a_{n,2})\|x_n\|^2 + a_{n,2}\|z_{n,2}\|^2\\ &= (1-a_{n,2})\phi(u, x_n) + a_{n,2}\phi(u, z_{n,2})\\ &\leq (1-a_{n,2})\phi(u, x_n) + a_{n,2}\Phi(T_2u, T_2y_{n,1})\\ &\leq (1-a_{n,2})\phi(u, x_n) + a_{n,2}\phi(u, y_{n,1}) = \phi(u, x_n).\end{aligned}$$

By continuing this process we obtain

$$\begin{aligned}\phi(u, u_n) &= \phi(u, T_{r_n}y_{n,m}) \leq \phi(u, y_{n,m}) = \phi(u, J^{-1}((1-a_{n,m})Jx_n + a_{n,2m}Jz_{n,m}))\\ &= \|u\|^2 - 2\langle u, (1-a_{n,m})Jx_n + a_{n,m}Jz_{n,m}\rangle + \|(1-a_{n,m})Jx_n + a_{n,m}Jz_{n,m}\|^2\\ &\leq \|u\|^2 - 2(1-a_{n,m})\langle u, Jx_n\rangle - 2a_{n,m}\langle u, Jz_{n,m}\rangle + (1-a_{n,m})\|x_n\|^2 + a_{n,m}\|z_{n,m}\|^2\\ &= (1-a_{n,m})\phi(u, x_n) + a_{n,m}\phi(u, z_{n,2})\\ &\leq (1-a_{n,m})\phi(u, x_n) + a_{n,m}\Phi(T_mu, T_my_{n,m-1})\\ &\leq (1-a_{n,m})\phi(u, x_n) + a_{n,m}\phi(u, y_{n,m-1})\\ &\leq (1-a_{n,m})\phi(u, x_n) + a_{n,m}\phi(u, x_n) = \phi(u, x_n), \qquad (3.1)\end{aligned}$$

hence, we have $u \in C_{n+1}$. This implies that

$$\mathcal{F} = \bigcap_{i=1}^{m} F(T_i) \bigcap EP(f) \subset C_n, \qquad \forall n \geq 0.$$

From $x_n = \Pi_{C_n}x_0$, we have

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \qquad \forall z \in C_n. \tag{3.2}$$

Since $\mathcal{F} \subset C_n$ for all $n \geq 0$, we obtain that

$$\langle x_n - u, Jx_0 - Jx_n\rangle \geq 0 \qquad \forall u \in \mathcal{F}.$$

From Lemma 2.4 we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n}x_0, x_0) \leq \phi(u, x_0) - \phi(u, \Pi_{C_n}x_0) \leq \phi(u, x_0)$$

for all $u \in \mathcal{F} \subset C_n$. Then the sequence $\phi(x_n, x_0)$ is bounded. Therefore $\{x_n\}$ is bounded. We show that $\{z_{n,i}\}$ is bounded for $i = 1, 2, ..., m$. Indeed, for $u \in \mathcal{F}$ we have

$$(\|z_{n,i}\| - \|u\|)^2 \le \phi(z_{n,i}, u) \le \phi(x_n, u) \le (\|x_n\| + \|u\|)^2.$$

Since $\{x_n\}$ is bounded, we conclude that $\{z_{n,i}\}$ is bounded for $i = 1, 2, ..., m$. From $x_n = \Pi_{C_n} x_0$ and $x_{n+1} \in C_{n+1} \subset C_n$ we have

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0), \qquad \forall n \ge 0.$$

Thus, $\{\phi(x_n, x_0)\}$ is nondecreasing. So the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of C_n for any positive integer $m \ge n$ we have

$$x_m = \Pi_{C_m} x_0 \in C_m \subset C_n.$$

It follows that

$$\begin{aligned}\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\le \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0)\end{aligned}$$

Letting $m, n \to \infty$ we have

$$\lim_{n\to\infty} \phi(x_m, x_n) = 0. \tag{3.3}$$

It follows from Lemma 2.2 that $x_m - x_n \to 0$ as $m, n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. Since C is a closed and convex subset of the Banach space E, we can assume that $x_n \to p$ as $n \to \infty$. Next we show $p \in \bigcap_{i=1}^m F(T_i)$. By taking $m = n+1$ in (3.4) we get

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.4}$$

It follows from Lemma 2.2 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.5}$$

From $x_{n+1} = \Pi_{C_{n+1}} x \in C_{n+1}$, we have

$$\phi(x_{n+1}, u_n) \le \phi(x_{n+1}, x_n), \qquad n \ge 0$$

It follows from (3.5) that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

By Lemma 2.1 we have

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{3.6}$$

Combining (3.6) with (3.7) one observes that

$$\lim_{n\to\infty} \|x_n - u_n\| \le \lim_{n\to\infty} (\|x_{n+1} - x_n\| + \|x_{n+1} - u_n\|) = 0. \tag{3.7}$$

It follows from $x_n \to p$ that $u_n \to p$ as $n \to \infty$. Since J is uniformly norm-to-norm continuous on bounded sets and $\lim_{n\to\infty} \|x_n - u_n\| = 0$, we have

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.8}$$

Let

$$r = sup_{n\ge0}\{\|x_n\|, \|z_{n,i}\| : i = 1, 2, ..., m\}.$$

Since E is a uniformly smooth Banach space, we know that E^* is a uniformly convex Banach space. Therefore from Lemma 2.8 there exists a continuous strictly increasing, and convex function g with $g(0) = 0$ such that for $i = 1, 2, ..., m$,

$$\begin{aligned}
\phi(u, y_{n,1}) &= \phi(u, J^{-1}((1 - a_{n,1})Jx_n + a_{n,1}Jz_{n,1})) \\
&= \|u\|^2 - 2\langle u, (1 - a_{n,1})Jx_n + a_{n,1}Jz_{n,1}\rangle + \|(1 - a_{n,1})Jx_n + a_{n,1}Jz_{n,1}\|^2 \\
&\leq \|u\|^2 - 2(1 - a_{n,1})\langle u, Jx_n\rangle - 2a_{n,1}\langle u, Jz_{n,1}\rangle \\
&\quad + (1 - a_{n,1})\|x_n\|^2 + a_{n,1}\|z_{n,1}\|^2 - a_{n,1}(1 - a_{n,1})g(\|Jx_n - Jz_{n,1}\|) \\
&= (1 - a_{n,1})\phi(u, x_n) + a_{n,1}\phi(u, z_{n,1}) - a_{n,1}(1 - a_{n,1})g(\|Jx_n - Jz_{n,1}\|) \\
&\leq (1 - a_{n,1})\phi(u, x_n) + a_{n,1}\Phi(T_1u, T_1x_n) - a_{n,1}(1 - a_{n,1})g(\|Jx_n - Jz_{n,1}\|) \\
&\leq (1 - a_{n,1})\phi(u, x_n) + a_{n,1}\phi(u, x_n) - a_{n,1}(1 - a_{n,1})g(\|Jx_n - Jz_{n,1}\|) \\
&= \phi(u, x_n) - a_{n,1}(1 - a_{n,1})g(\|Jx_n - Jz_{n,1}\|)
\end{aligned}$$

and

$$\begin{aligned}
\phi(u, y_{n,2}) &= \phi(u, J^{-1}((1 - a_{n,2})Jx_n + a_{n,2}Jz_{n,2})) \\
&= \|u\|^2 - 2\langle u, (1 - a_{n,2})Jx_n + a_{n,2}Jz_{n,2}\rangle + \|(1 - a_{n,2})Jx_n + a_{n,2}Jz_{n,2}\|^2 \\
&\leq \|u\|^2 - 2(1 - a_{n,2})\langle u, Jx_n\rangle - 2a_{n,2}\langle u, Jz_{n,2}\rangle \\
&\quad + (1 - a_{n,2})\|x_n\|^2 + a_{n,2}\|z_{n,2}\|^2 - a_{n,2}(1 - a_{n,2})g(\|Jx_n - Jz_{n,2}\|) \\
&= (1 - a_{n,2})\phi(u, x_n) + a_{n,2}\phi(u, z_{n,2}) - a_{n,2}(1 - a_{n,2})g(\|Jx_n - Jz_{n,2}\|) \\
&\leq (1 - a_{n,2})\phi(u, x_n) + a_{n,2}\Phi(T_2u, T_2y_{n,1}) - a_{n,2}(1 - a_{n,2})g(\|Jx_n - Jz_{n,2}\|) \\
&\leq (1 - a_{n,2})\phi(u, x_n) + a_{n,2}\phi(u, y_{n,1}) - a_{n,2}(1 - a_{n,2})g(\|Jx_n - Jz_{n,2}\|) \\
&\leq \phi(u, x_n) - a_{n,2}(1 - a_{n,2})g(\|Jx_n - Jz_{n,2}\|) - a_{n,2}a_{n,1}(1 - a_{n,1})g(\|Jx_n - Jz_{n,1}\|).
\end{aligned}$$

By continuing this process we obtain

$$\begin{aligned}
\phi(u, u_n) &= \phi(u, T_{r_n}y_{n,m}) \leq \phi(u, y_{n,m}) = \phi(u, J^{-1}((1 - a_{n,m})Jx_n + a_{n,m}Jz_{n,m})) \\
&= \|u\|^2 - 2\langle u, (1 - a_{n,m})Jx_n + a_{n,m}Jz_{n,m}\rangle + \|(1 - a_{n,m})Jx_n + a_{n,m}Jz_{n,m}\|^2 \\
&\leq \|u\|^2 - 2(1 - a_{n,m})\langle u, Jx_n\rangle - 2a_{n,m}\langle u, Jz_{n,m}\rangle + (1 - a_{n,m})\|x_n\|^2 + a_{n,m}\|z_{n,m}\|^2 \\
&\quad - a_{n,m}(1 - a_{n,m})g(\|Jx_n - Jz_{n,m}\|) \\
&= (1 - a_{n,m})\phi(u, x_n) + a_{n,m}\phi(u, z_{n,m}) - a_{n,m}(1 - a_{n,m})g(\|Jx_n - Jz_{n,m}\|) \\
&\leq 1 - a_{n,m})\phi(u, x_n) + a_{n,m}\Phi(T_mu, T_my_{n,m-1}) - a_{n,m}(1 - a_{n,m})g(\|Jx_n - Jz_{n,m}\|) \\
&\leq (1 - a_{n,m})\phi(u, x_n) + a_{n,m}\phi(u, y_{n,m-1}) - a_{n,m}(1 - a_{n,m})g(\|Jx_n - Jz_{n,m}\|) \\
&\leq (1 - a_{n,m})\phi(u, x_n) + a_{n,m}\phi(u, x_n) - a_{n,m}(1 - a_{n,m})g(\|Jx_n - Jz_{n,m}\|) \\
&\quad -a_{n,m}a_{n,m-1}(1-a_{n,m-1})g(\|Jx_n-Jz_{n,m-1}\|)-...-a_{n,m}a_{n,m-1}...a_{n,1}(1-a_{n,1})g(\|Jx_n-Jz_{n,1}\|) \\
&\leq \phi(u, x_n) - a_{n,m}(1 - a_{n,m})g(\|Jx_n - Jz_{n,m}\|) \\
&\quad - a_{n,m}a_{n,m-1}(1 - a_{n,m-1})g(\|Jx_n - Jz_{n,m-1}\|) - ... \\
&\quad - a_{n,m}a_{n,m-1}...a_{n,1}(1 - a_{n,1})g(\|Jx_n - Jz_{n,1}\|). \quad (3.9)
\end{aligned}$$

It follows that

$$a_{n,m}a_{n,m-1}...a_{n,1}(1 - a_{n,1})g(\|Jx_n - Jz_{n,1}\|) \leq \phi(u, x_n) - \phi(u, u_n) \qquad n \geq 0. \quad (3.10)$$

On the other hand

$$\begin{aligned}\phi(u,x_n)-\phi(u,u_n) &= \|x_n\|^2-\|u_n\|^2-2\langle u, Jx_n-Ju_n\rangle\\ &\le |\, \|x_n\|^2-\|u_n\|^2\,| + 2|\langle u, Jx_n-Ju_n\rangle\,|\\ &\le |\, \|x_n\|-\|u_n\|\,|(\|x_n\|+\|u_n\|)+2\|u\|\|Jx_n-Ju_n\|\\ &\le \|x_n-u_n\|(\|x_n\|+\|u_n\|)+2\|u\|\|Jx_n-Ju_n\|.\end{aligned}$$

It follows from (3.8) and (3.9) that

$$\lim_{n\to\infty}(\phi(u,x_n)-\phi(u,u_n))=0. \tag{3.11}$$

By our assumption we have

$$a^m(1-b)g(\|Jx_n-Jz_{n,1}\|)\le a_{n,m}a_{n,m-1}...a_{n,1}(1-a_{n,1})g(\|Jx_n-Jz_{n,1}\|),$$

which implies, by (3.10), that

$$\lim_{n\to\infty} g(\|Jx_n-Jz_{n,1}\|)=0.$$

Therefore from the property of g, we have

$$\lim_{n\to\infty}\|Jx_n-Jz_{n,1}\|=0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded subsets, we have

$$\lim_{n\to\infty}\|x_n-z_{n,1}\|=0.$$

By a similar way, for $i=2,...m$ we obtain that

$$\lim_{n\to\infty}\|x_n-z_{n,i}\|=0.$$

Therefore we have

$$\lim_{n\to\infty} dist(x_n,T_1x_n)\le\lim_{n\to\infty}\|x_n-z_{n,1}\|=0,$$

and also

$$\lim_{n\to\infty} dist(x_n,T_iy_{n,i-1})\le\lim_{n\to\infty}\|x_n-z_{n,i}\|=0, \quad i=2,...,m.$$

For $k=1,2,...,m$ we have

$$\lim_{n\to\infty}\|Jy_{n,k}-Jx_n\|=\lim_{n\to\infty}a_{n,k}\|Jz_{n,k}-Jx_n\|=0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|x_n-y_{n,k}\|=0.$$

Since T_i is uniformly continuous for $k=2,3,...,m$ we have

$$\begin{aligned}dist(x_n,T_kx_n) &\le dist(x_n,T_ky_{n,k-1})+H(Ty_{n.k-1},T_kx_n)\\ &\le \|x_n-z_{n,k}\|+H(Ty_{n.k-1},T_kx_n)\to 0 \qquad n\to\infty\end{aligned} \tag{3.12}$$

Now by the closedness of T_i we obtain that $p\in\bigcap_{i=1}^m F(T_i)$. We shall show that $p\in EP(f)$. From (3.2) we have

$$\phi(u,y_n)\le\phi(u,x_n). \tag{3.13}$$

From $u_n = T_{r_n} y_{n,m}$ and Lemma 2.10 we have that

$$\phi(u_n, y_{n,m}) = \phi(T_{r_n} y_{n,m}, y_{n,m})$$
$$\leq \phi(u, y_{n,m}) - \phi(u, T_{r_n} y_{n,m})$$
$$\leq \phi(u, x_n) - \phi(u, T_{r_n} y_{n,m})$$
$$= \phi(u, x_n) - \phi(u, u_n)$$

So, we have from (3.11) that

$$\lim_{n\to\infty} \phi(u_n, y_{n,m}) = 0.$$

Since E is uniformly convex and smooth and $\{u_n\}$ is bounded, we have from Lemma 2.2 that

$$lim_{n\to\infty} \|u_n - y_{n,m}\| = 0. \tag{3.14}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Ju_n - Jy_{n,m}\| = 0. \tag{3.15}$$

From the assumption $r_n \geq c$ we have

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_{n,m}\|}{r_n} = 0.$$

By $u_n = T_{r_n} y_{n,m}$ we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_{n,m}\rangle \geq 0 \qquad \forall y \in C.$$

From (A2), we have

$$\|y - u_n\| \frac{\|Ju_n - Jy_{n,m}\|}{r_n} \geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_{n,m}\rangle \geq -f(u_n, y) \geq f(y, u_n).$$

By taking the limit as $n \to \infty$, in the above inequality and from (A4) we have

$$0 \geq f(y, p),\ \forall\, y \in C.$$

For all $t \in (0, 1)$ and $y \in C$, define $y_t = ty + (1-t)p$. Since $y, p \in C$, and C is convex we have $y_t \in C$ and hence $f(y_t, p) \leq 0$. So, from (A1) we have

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1 - t)f(y_t, p) \leq tf(y_t, y),$$

which gives $f(y_t, y) \geq 0$. From (A3) we have $0 \leq f(p, y), \forall y \in C$ which implies that $p \in EP(f)$, and therefore $p \in \mathcal{F}$. Finally we prove $p = \Pi_{\mathcal{F}} x_0$. By taking limit in (3.3) we have

$$\langle p - u, Jx_0 - Jp\rangle \geq 0, \qquad \forall u \in \mathcal{F}.$$

Hence by Lemma 2.3 we have $p = \Pi_{\mathcal{F}} x_0$. □

As a result for single valued mappings we obtain the following corollary.

Theorem 3.2. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1) − (A4). Let $T_i : C \to C$, $i = 1, 2, ..., m$ be a finite family of closed relatively quasi-nonexpansive mappings such that $\mathcal{F} = \bigcap_{i=1}^{m} F(T_i) \bigcap EP(f) \neq$*

$\emptyset$. For $x_0 \in C$ and $C_0 = C$, let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{cases} y_{n,1} = J^{-1}((1-a_{n,1})Jx_n + a_{n,1}JT_1x_n), \\ y_{n,2} = J^{-1}((1-a_{n,2})Jx_n + a_{n,2}JT_2y_{n,1}), \\ ... \\ y_{n,m} = J^{-1}((1-a_{n,m})Jx_n + a_{n,m}JT_my_{n,m-1}), \\ u_n \in C \, such \, that \, f(u_n, y) + \frac{1}{r_n}\langle y-u_n, Ju_n - Jy_{n,m}\rangle \geq 0, \qquad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \prod_{C_{n+1}} x_0, \qquad \forall n \geq 0. \end{cases}$$

Assume that $\sum_{i=1}^m a_{n,i} = 1$, $\{a_{n,i}\} \in [a,b] \subset (0,1)$ and $\{r_n\} \subset [c,\infty)$ for some $c > 0$. Suppose that T_i is uniformly continuous for $i = 2,3,...,m$. Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}}x_0$.

Theorem 3.3. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $T_i : C \to P(C)$, $i = 1,2,...,m$, be a finite family of multivalued mappings such that P_{T_i} is closed and relatively quasi-nonexpansive. Assume that $\mathcal{F} = \bigcap_{i=1}^m F(T_i) \bigcap EP(f) \neq \emptyset$. For $x_0 \in C$ and $C_0 = C$, let $\{x_n\}$ be a sequences generated by the following algorithm:*

$$\begin{cases} y_{n,1} = J^{-1}((1-a_{n,1})Jx_n + a_{n,1}Jz_{n,1}), \\ y_{n,2} = J^{-1}((1-a_{n,2})Jx_n + a_{n,2}Jz_{n,2}), \\ ... \\ y_{n,m} = J^{-1}((1-a_{n,m})Jx_n + a_{n,m}Jz_{n,m}), \\ u_n \in C \, such \, that \, f(u_n, y) + \frac{1}{r_n}\langle y-u_n, Ju_n - Jy_{n,m}\rangle \geq 0, \qquad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \prod_{C_{n+1}} x_0, \qquad \forall n \geq 0 \end{cases}$$

where $z_{n,1} \in P_{T_1}x_n$ and $z_{n,i} \in P_{T_i}y_{n,i-1}$ for $i = 2,...,m$ and J is the duality mapping on E. Assume that $\sum_{i=1}^m a_{n,i} = 1$, $\{a_{n,i}\} \in [a,b] \subset (0,1)$ and $\{r_n\} \subset [c,\infty)$ for some $c > 0$. Suppose that P_{T_i} is uniformly continuous with respect to the Hausdorff metric for $i = 2,3,...,m$. Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}}x_0$.

Proof. Let $p \in \mathcal{F}$, then $P_{T_i}(p) = \{p\}$, (i=1,2,...,m). Also we have $F(T_i) = F(P_{T_i})$. Now by substituting P_{T_i} instead of T_i and similar argument as in the proof of Theorem 3.1 we obtain the result. □

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