NONSPREADING MAPPINGS ON MODULAR VECTOR SPACES

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We introduce the notion of a nonspreading mapping in the setting of modular vector spaces, having as starting source the elegant article by Kohsaka and Takahashi [Arch. Math., 2008, 91, 166-177]. We establish some properties of this class of mappings and suggest a way to reckon their fixed points. More accurately, to estimate the solutions of fixed point equations involving this kind of operators, we use a suitable iterative process introduced by Sintunavarat and Pitea [J. Nonlinear Sci. Appl., 2016, 2553-2562].

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1. Introduction

The beginning of modular analysis was given by some practical examples of generalized function and sequence spaces provided by Orlicz and Birnbaum in the early 1930's. A deep analysis regarding modular function spaces and their suitability for fixed point theory was realized by Kozlowski (1988) in [10] and by Khamsi and Kozlowski in [8]. Still, the formal definition of modular vector spaces (not necessarily function-type spaces), as it is known and used today, was settled by Orlicz and Musielack in [12] and [13]. From that moment on, the modular setting became an interesting and nontrivial alternative to classical Banach spaces. Recent papers, using this particular framework as underlying setting are related with various modular nonexpansivenes conditions, please see: Abdou and Khamsi [1], Bejenaru and Postolache [2], Kassab and Turcanu [7].

Iteration based procedures provide important instruments in nonlinear analysis. They can produce approximate solutions for certain classes of problems, which can be thought of in terms of fixed point theory, whenever analytical methods fail. For instance, they can be useful for approximating the zeros of complex polynomials, for studying general variational inequalities, solving classes of split problems, finding solutions to optimization problems or designing algorithms for processing signals and images: please, see Usurelu *et al.* [19, 20], Yao *et al.* [21, 22].

The necessity of elaborated iteration procedures came with the study of generalized contractive conditions and the major limitation of the Picard sequence under the aspect of reaching the fixed point. Important results in this direction were obtained by Mann (1953) [11], Ishikawa (1974) [5], Noor (2000) [14], Sahu *et al.* (2020) [15] and many others, in the context of fixed point theory or variational inequalities. For instance, Suzuki (2008)

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[17] proved a convergence result for a mapping satisfying condition C using a Krasnoselskij iterative process; Karahan and Ozdemir (2013) [6] introduced the S^* iteration for numerical reckoning fixed points of contractive or nonexpansive mappings; Thakur *et al.* (2016) [18] used a newly defined iteration process for approximating a fixed point of nonexpansive mappings; Sahu *et al.* (2020) [15] utilise their new iteration technique for nonlinear operators as concerns convex programming and feasibility problems. And the list above may continue. Lately, classical methods of numerical analysis have been combined with some of these new iterative procedures, resulting interesting and valuable new approximation methods.

However, there is an odd thing about iterative schemes. Despite the significant interest a certain iterative process could produce, no one could say for sure that its study is being completed. Each newly defined iterative scheme is almost immediately absorbed and reused in different setting. For instance, the iteration process S_n defined by Sintunavarat and Pitea (2016) [16] was initially used for approximating the fixed points of mappings satisfying Berinde (2004) contractive condition; in [4] convergence, stability and data dependence were analyzed in connection with operators with condition (D), while in [3], the same procedure was used to solve split feasibility problems.

In 2008, Kohsaka and Takahashi ([9]) introduced a new class of operators on Banach spaces, namely the nonspreading mappings. This way, they generalized the class of firmly nonexpansive type mappings. An interesting fact about the newly introduced operators concerns their appearance on Hilbert spaces. Starting from this particular expression, we adapt the definition to convex modular vector spaces. Further on, we evaluate the solutions of fixed point equations involving this kind of operators based on the S_n iterative process.

2. **Preliminaries**

We initiate our approach by revealing the main features of the analytical setting, as well as some instrumental definitions and lemmas.

Definition 2.1 ([12],[13])**.** Let X be a real vector space. A function $\rho\colon X \to [0,\infty]$ is called a modular if it satisfies:

(i) $\rho(x) = 0$ if and only if $x = 0$;

(ii) $\rho(\alpha x) = \rho(x)$, for $|\alpha| = 1,\ \forall x \in X$;

(iii) $\rho(\alpha x + (1-\alpha)y) \leq \rho(x) + \rho(y)$, $\forall \alpha \in [0,1]$, for all $x, y \in X$.

By replacing condition (iii) with

$$\rho(\alpha x + (1-\alpha)y) \leq \alpha\rho(x) + (1-\alpha)\rho(y),$$

for all $\alpha \in [0,1]$ and for all $x, y \in X$, we find the so-called convex modular.

Definition 2.2 ([12])**.** Let ρ be a convex modular function defined on a vector space X.

The vector subspace X_ρ is called a modular space, where

$$X_\rho = \left\{ x \in X : \lim_{\alpha \to 0} \rho(\alpha x) = 0 \right\}.$$

Definition 2.3 ([1])**.** Let ρ be a convex modular on a vector space X.

(1) A sequence $\{x_n\} \subset X_\rho$ is called ρ-convergent to some $x \in X_\rho$ if and only if $\lim\limits_{n\to\infty} \rho(x_n - x) = 0$.

(2) A sequence $\{x_n\} \subset X_\rho$ is called ρ-Cauchy if $\lim\limits_{m,n\to\infty} \rho(x_m - x_n) = 0$.

(3) We say that X_ρ is ρ-complete if any ρ-Cauchy sequence in X_ρ is ρ-convergent.

(4) A set $C \subset X_\rho$ is called ρ-closed if for any sequence $\{x_n\} \subset C$ which ρ - converges to some point x, one has $x \in C$.

(5) A set $C \subset X_\rho$ is called ρ-bounded if $\delta_\rho(C) = \sup_{x,y\in C} \rho(x-y) < \infty$.

(6) A set $K \subset X_\rho$ is called ρ-compact if any sequence $\{x_n\} \subset K$ has a subsequence which ρ-converges to a point in K.

(7) The modular ρ is said to satisfy the Fatou property if $\rho(x-y) \leq \liminf_{n\to\infty} \rho(x-y_n)$ whenever $\{y_n\}$ ρ-converges to y, for any $x, y, y_n \in X_\rho$.

Definition 2.4 ([8])**.** The uniform convexity type properties of the modular ρ are defined for every $r > 0$ and every $\varepsilon > 0$ as follows:

(1) Define

$$D_1(r,\varepsilon) = \{(x,y) : x, y \in X_\rho, \rho(x) \leq r, \rho(y) \leq r, \rho(x-y) \geq \varepsilon r\}.$$

If $D_1(r,\varepsilon) \neq \emptyset$, let

$$\delta_1(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{x+y}{2}\right) : (x,y) \in D_1(r,\varepsilon)\right\}.$$

If $D_1(r,\varepsilon) = \emptyset$, set $\delta_1(r,\varepsilon) = 1$.

We say that ρ satisfies (UUC1) if for every $s \geq 0$ and $\varepsilon > 0$, there exists $\eta_1(s,\varepsilon) > 0$, depending on s and ε, such that

$$\delta_1(r,\varepsilon) > \eta_1(s,\varepsilon) > 0.$$

for $r > s$.

(2) Define

$$D_2(r,\varepsilon) = \left\{(x,y) : x, y \in X_\rho, \rho(x) \leq r, \rho(y) \leq r, \rho\left(\frac{x-y}{2}\right) \geq \varepsilon r\right\}.$$

If $D_2(r,\varepsilon) \neq \emptyset$, let

$$\delta_2(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{x+y}{2}\right) : (x,y) \in D_2(r,\varepsilon)\right\}.$$

If $D_2(r,\varepsilon) = \emptyset$, set $\delta_2(r,\varepsilon) = 1$.

We say that ρ satisfies (UUC2) if for every $s \geq 0$ and $\varepsilon > 0$, there exists $\eta_2(s,\varepsilon) > 0$, depending on s and ε, such that

$$\delta_2(r,\varepsilon) > \eta_2(s,\varepsilon) > 0,$$

for $r > s$. It is important to point out that (UUC1) property also implies (UUC2).

Lemma 2.1 ([7])**.** *Let ρ be a convex modular which is* (UUC1) *and let $\{t_n\} \in (0,1)$ be a sequence bounded away from 0 to 1. If there exists $r > 0$ such that*

$$\limsup_{n\to\infty} \rho(x_n) \leq r,$$

$$\limsup_{n\to\infty} \rho(y_n) \leq r,$$

$$\lim_{n\to\infty} \rho(t_n x_n + (1-t_n)y_n) = r,$$

where $\{x_n\}$ and $\{y_n\}$ are sequences in X_ρ, then

$$\lim_{n\to\infty} \rho(x_n - y_n) = 0.$$

Definition 2.5 ([1])**.** Let $\{x_n\}$ be a sequence in X_ρ and C a nonempty subset of X_ρ. The function

$$\tau\colon C \to [0,\infty]\,, \tau(x) = \limsup_{n\to\infty} \rho(x - x_n)$$

is called a ρ-type function.

Furthermore, a sequence $\{c_n\} \subset C$ is called a minimizing sequence of τ if

$$\lim_{n\to\infty} \tau(c_n) = \inf_{x\in C} \tau(x).$$

Lemma 2.2 ([1])**.** *Assume that the modular space X_ρ is ρ-complete and ρ satisfies Fatou property. Let C be a nonempty convex and ρ-closed subset of X_ρ. Consider the ρ-type function $\tau\colon C \to [0,\infty]$ given by a sequence $\{x_n\}$ in X_ρ. Assume that $\tau_0 = \inf_{x\in C} \tau(x) < \infty$. If ρ is* (UUC1)*, then all minimizing sequences of τ are ρ-convergent to the same ρ-limit.*

Definition 2.6. Let X_ρ be a modular space. It is said that the modular ρ satisfies the Δ_2-condition if there exists a constant $K \geq 0$ such that

$$\rho(2x) \leq K\rho(x),$$

for any $x \in X_\rho$.

The smallest such constant K will be denoted by $\omega(2)$. In addition, one can also consider $\mu = \dfrac{\omega(2)}{2}$, known as the modular factor (see [2]).

Remark 2.1 ([2])**.** The modular factor μ satisfies the properties:

(P1) $\mu \geq 1$;

(P2) $\rho(x + y) \leq \mu\,[\rho(x) + \rho(y)]$, for all $x, y \in X_\rho$.

3. **Main results**

Throughout this part we will assume that ρ is a convex modular, satisfying the Δ_2-condition. Moreover, μ stands always for the modular factor.

Definition 3.1. Let C be a nonempty subset of a modular space X_ρ. A mapping $T\colon C \to X_\rho$ with

$$(1 + \mu^2)\mu^2\rho^2(Tx - Ty) \leq \rho^2(Tx - y) + \rho^2(x - Ty),$$

for all $x, y \in X_\rho$ is called a modular nonspreading mapping.

Remark 3.1. The definition of nonspreading mappings on a Hilbert space H (see [9]) is recovered by taking $\rho(x) = ||x||$. Indeed, in this particular case one has $\mu = 1$, so the inequality above becomes:

$$2||Tx - Ty||^2 \leq ||Tx - y||^2 + ||x - Ty||^2,$$

for all $x, y \in H$.

In the following, we will present an example of nonspreading modular mapping.

Example 3.1. In $\mathbb{R}$, we consider the convex modular

$$\rho(x) = |x|\,\sqrt{|x|},$$

with the modular factor $\mu = \sqrt{2}$.

To prove that a mapping $T\colon \mathbb{R} \to \mathbb{R}$ is a nonspreading modular mapping, we need to prove

$$6\,|Tx - Ty|^3 \leq |Tx - y|^3 + |Ty - x|^3\,, \tag{3.1}$$

for all $x, y \in \mathbb{R}$.

We take now the mapping

$$T\colon \mathbb{R} \to \mathbb{R}, \quad Tx = \frac{x}{2}$$

and we check the inequality (3.1), meaning

$$6\left[\frac{1}{2}\left|x-y\right|\right]^3 \le \left|\frac{1}{2}x - y\right|^3 + \left|\frac{1}{2}y - x\right|^3,$$

or, in other words,

$$6\left|x-y\right|^3 \le \left|x-2y\right|^3 + \left|y-2x\right|^3, \tag{3.2}$$

for all $x, y \in \mathbb{R}$.

In the following, we will assume that $x \ge y$ and we mention that the case $y > x$ can be proved absolutely similar.

Let $a = x - y \ge 0$. This implies that $x = a + y$ and the condition (3.2) becomes

$$6a^3 \le \left|a-y\right|^3 + \left|2a+y\right|^3.$$

Case I: Let $y \in [-2a, a]$. In this case, we have

$$6a^3 \le (a-y)^3 + (2a+y)^3,$$

which is equivalent to

$$3a(a^2 + 3ay + 3y^2) \ge 0. \tag{3.3}$$

But inequality (3.3) is true for all $a \ge 0$.

Case II: Let $y > a$. Because $a \ge 0$, we obtain $y > 0$. We have

$$6a^3 \le (y-a)^3 + (2a+y)^3,$$

whish is equivalent to

$$a^3 + 3y^2a + 15a^2y + 2y^3 \ge 0. \tag{3.4}$$

But inequality (3.4) is true for all $a, y \ge 0$.

Case III: Let $y < -2a$. It is obvious that $y < 0$, because $a \ge 0$. In this case, we have

$$6a^3 \le (a-y)^3 + (-2a-y)^3,$$

which is equivalent to

$$15a^2(y+2a) - 17a^3 + \frac{3}{2}y^2(y+2a) + \frac{1}{2}y^3 \le 0. \tag{3.5}$$

But inequality (3.5) is true for all $a > 0$ and $y < 0$, given that $y + 2a < 0$.

Based on relationships (3.3), (3.4) and (3.5), which turned out to be true, we conclude that $Tx = \frac{1}{2}x$, for $x \in \mathbb{R}$ is a nonspreading modular mapping in relation to the modular $\rho(x) = |x|\sqrt{|x|}$.

We consider that this example is extremely significant for the type of operators that will be worked on in the following results regarding the convergence of the chosen iterative process and comes as a natural complement to other examples previously presented in the literature.

Next, we introduce some characteristic properties of the newly introduced class of operators.

Lemma 3.1. *Let C be a nonempty subset of a modular space X_ρ and let $T\colon C \to C$ be a modular nonspreading mapping with $F(T) \neq \emptyset$. Then T is a modular quasi-nonexpansive mapping (i.e. $\rho(Tx - p) \le \rho(x - p)$, $\forall x \in C$, $\forall p \in F(T)$).*

Proof. Let $p \in F(T)$. As T is a nonspreading mapping, we have

$$(1+\mu^2)\mu^2\rho^2(Tx-p) \leq \rho^2(Tx-p)+\rho^2(x-p),$$

so

$$\left[(1+\mu^2)\mu^2-1\right]\rho^2(Tx-p) \leq \rho^2(x-p).$$

Using (P1) from Remark 2.1 we get that $\mu \geq 1$, therefore $(1+\mu^2)\mu^2-1 \geq 1$.

In conclusion,

$$\rho^2(Tx-p) \leq \left[(1+\mu^2)\mu^2-1\right]\rho^2(Tx-p) \leq \rho^2(x-p),$$

so T is a quasi-nonexpansive mapping. □

Lemma 3.2. *Let C be a nonempty ρ-bounded subset of a modular space X_ρ and $T\colon C \to C$ a modular nonspreading mapping. If $\{x_n\}$ is a sequence in C such that $\lim_{n\to\infty}\rho(Tx_n-x_n)=0$, and τ is the ρ-type function of $\{x_n\}$, then:*

(i) *$\tau(Tx) \leq \tau(x)$, for all $x \in X_\rho$;*

(ii) *T leaves the minimizing sequences invariant (i.e. if $\{c_n\}$ is a minimizing sequence for τ, then so is $\{Tc_n\}$).*

Proof. (i) First, we will apply the Definition 3.1. Based on this, we get

$$(1+\mu^2)\mu^2\rho^2(Tx_n-Tx) \leq \rho^2(Tx_n-x)+\rho^2(x_n-Tx),$$

that is

$$\begin{aligned}(1+\mu^2)\left[\mu^2\rho^2(Tx_n-Tx)-\rho^2(x_n-Tx)\right] &\leq \rho^2(Tx_n-x)-\mu^2\rho^2(x_n-Tx)\\ &= \left[\rho^2(Tx_n-x)-\mu^2\rho^2(x_n-x)\right]\\ &\quad +\mu^2\left[\rho^2(x_n-x)-\rho^2(x_n-Tx)\right].\end{aligned}$$

It follows

$$\begin{aligned}\mu^2\left[\rho^2(x_n-x)-\rho^2(x_n-Tx)\right] &\geq (1+\mu^2)\left[\mu^2\rho^2(Tx_n-Tx)-\rho^2(x_n-Tx)\right]\\ &\quad +\left[\mu^2\rho^2(x_n-x)-\rho^2(Tx_n-x)\right].\end{aligned} \tag{3.6}$$

Next we will prove that

$$\limsup_{n\to\infty}\left[\mu^2\rho^2(Tx_n-Tx)-\rho^2(x_n-Tx)\right] \geq 0.$$

Indeed, using property (P2), we find that

$$\rho(x_n-Tx)=\rho\left((x_n-Tx_n)+(Tx_n-Tx)\right) \leq \mu\rho(x_n-Tx_n)+\mu\rho(Tx_n-Tx),$$

so

$$\mu\rho(Tx_n-Tx)-\rho(x_n-Tx) \geq -\mu\rho(x_n-Tx_n).$$

By multiplying this last inequality with $\mu\rho(Tx_n-Tx)+\rho(x_n-Tx)$, which is obviously positive, we obtain

$$\mu^2\rho^2(Tx_n-Tx)-\rho^2(x_n-Tx) \geq -\mu\rho(x_n-Tx_n)\cdot\left[\mu\rho(Tx_n-Tx)+\rho(x_n-Tx)\right].$$

Because C is ρ-bounded it follows that $\mu\rho(Tx_n-Tx)+\rho(x_n-Tx)$ is also bounded. Moreover, $\lim_{n\to\infty}\rho(Tx_n-x_n)=0$ and, by applying lim sup, it follows that

$$\limsup_{n\to\infty}\left[\mu^2\rho^2(Tx_n-Tx)-\rho^2(x_n-Tx)\right] \geq 0.$$

Similarly, one obtains

$$\limsup_{n\to\infty}\left[\mu^2\rho^2(x_n-x)-\rho^2(Tx_n-x)\right] \geq 0.$$

Taking lim sup in (3.6), we find

$$\mu^2 \cdot \limsup_{n\to\infty} \left[\rho^2(x_n - x) - \rho^2(x_n - Tx)\right] \geq 0.$$

From this

$$\limsup_{n\to\infty} \rho^2(x_n - x) \geq \limsup_{n\to\infty} \rho^2(x_n - Tx),$$

so

$$\limsup_{n\to\infty} \rho(x_n - x) \geq \limsup_{n\to\infty} \rho(x_n - Tx).$$

which closes the proof.

(ii) If $\{c_n\}$ is a minimizing sequence for τ, we find that

$$\lim_{n\to\infty} \tau(c_n) = \inf_{x\in C} \tau(x).$$

Using the conclusion in (i) we have

$$\inf_{x\in C} \tau(x) \leq \lim_{n\to\infty} \tau(Tc_n) \leq \lim_{n\to\infty} \tau(c_n) = \inf_{x\in C} \tau(x) \tag{3.7}$$

From (3.7) it is clear that

$$\lim_{n\to\infty} \tau(Tc_n) = \inf_{x\in C} \tau(x),$$

so $\{Tc_n\}$ is a minimizing sequence for τ. □

Proposition 3.1. *Let C be a nonempty, convex and ρ-closed subset of a ρ-complete modular space X_ρ. Assume that ρ is* (UUC1) *and satisfies Fatou property. Consider the ρ-type function $\tau\colon C \to [0, \infty]$ of a sequence $\{x_n\} \subset X_\rho$ and suppose $\tau_0 = \inf_{x\in C} \tau(x) < \infty$. Let $\{c_n\}$ and $\{d_n\}$ be two minimizing sequence for τ. Then,*

(i) *any convex combination of $\{c_n\}$ and $\{d_n\}$ is a minimizing sequence for τ as well;*

(ii) $\lim_{n\to\infty} \rho(c_n - d_n) = 0$.

Proof. The proof does not differ at all from the proof of Proposition 1 from [7].

(i) We consider

$$e_n = \lambda c_n + (1 - \lambda)d_n,$$

for $\lambda \in (0, 1)$ and $n \geq 1$.

For any $x \in C$, we have

$$\rho(e_n - x) \leq \lambda\rho(c_n - x) + (1 - \lambda)\rho(d_n - x),\ n \geq 1,$$

therefore

$$\limsup_{m\to\infty} \rho(e_n - x_m) \leq \lambda\limsup_{m\to\infty} \rho(c_n - x_m) + (1 - \lambda)\limsup_{m\to\infty} \rho(d_n - x_m),\ n \geq 1,$$

meaning that

$$\tau(e_n) \leq \lambda\tau(c_n) + (1 - \lambda)\tau(d_n).$$

Passing to the limit and keeping in mind that $\{c_n\}$ and $\{d_n\}$ are minimizing sequences, we obtain

$$\tau_0 = \inf_{x\in C} \tau(x) \leq \lim_{n\to\infty} \tau(e_n) \leq \lambda\tau_0 + (1 - \lambda)\tau_0 = \tau_0,$$

where we get the conclusion.

(ii) Notice that, for $e_n = \dfrac{1}{2}(c_n + d_n)$, $n \geq 1$, we have $c_n - d_n = 2(e_n - d_n)$, $n \geq 1$.

From (i), $\{e_n\}$ is a minimizing sequence and, according to Lemma 2.2, all minimizing sequences ρ-converge to the same point, which we denote by z. Hence,

$$\rho(e_n - d_n) = \rho\left(\frac{c_n - d_n}{2}\right) \leq \frac{1}{2}\left(\rho(c_n - z) + \rho(d_n - z)\right),\ n \geq 1.$$

Using (i), we deduce that $\lim_{n\to\infty} \rho(e_n - d_n) = 0$. From Δ_2 - condition, we will also have

$$\rho(c_n - d_n) \leq \omega(2)\rho(e_n - d_n).$$

Taking $n \to \infty$, we obtain the conclusion in this case. □

In 2016, Sintunavarat and Pitea ([16]) introduced the S_n iteration procedure defined as follows: for an arbitrary $x_1 \in C$, a sequence $\{x_n\}$ is obtained by the rule:

$$\begin{cases} y_n = (1-\beta_n)\, x_n + \beta_n T x_n \\ z_n = (1-\gamma_n)\, x_n + \gamma_n y_n \\ x_{n+1} = (1-\alpha_n)\, T z_n + \alpha_n T y_n, \end{cases} \tag{3.8}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences $(0,1)$.

Lemma 3.3. *Let C be a nonempty ρ-bounded and convex subset of X_ρ and let $T\colon C \to C$ be a modular nonspreading mapping with $F(T) \neq \emptyset$. For an arbitrary chosen $x_1 \in C$, let the sequence $\{x_n\}$ be generated by the iterative process* (3.8).

Then, $\lim_{n\to\infty} \rho(x_n - p)$ exists for any $p \in F(T)$.

Proof. Let $p \in F(T)$. From Lemma 3.1 we have

$$\rho(Tx - p) \leq \rho(x - p),$$

for all $x \in C$.

Now using this inequality and the convexity of ρ, we find that

$$\begin{aligned} \rho(y_n - p) &= \rho((1-\beta_n)x_n + \beta_n T x_n - p) \\ &\leq (1-\beta_n)\rho(x_n - p) + \beta_n \rho(Tx_n - p) \\ &\leq (1-\beta_n)\rho(x_n - p) + \beta_n \rho(x_n - p) \\ &= \rho(x_n - p). \end{aligned} \tag{3.9}$$

Using (3.9), we get

$$\begin{aligned} \rho(z_n - p) &= \rho((1-\gamma_n)x_n + \gamma_n y_n - p) \\ &\leq (1-\gamma_n)\rho(x_n - p) + \gamma_n \rho(y_n - p) \\ &\leq (1-\gamma_n)\rho(x_n - p) + \gamma_n \rho(x_n - p) \\ &= \rho(x_n - p). \end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we have

$$\begin{aligned} \rho(x_{n+1} - p) &= \rho((1-\alpha_n)Tz_n + \alpha_n T y_n - p) \\ &\leq (1-\alpha_n)\rho(Tz_n - p) + \alpha_n \rho(Ty_n - p) \\ &\leq (1-\alpha_n)\rho(z_n - p) + \alpha_n \rho(y_n - p) \\ &\leq (1-\alpha_n)\rho(x_n - p) + \alpha_n \rho(x_n - p) \\ &= \rho(x_n - p). \end{aligned} \tag{3.11}$$

This involves that the sequence $\{\rho(x_n - p)\}_{n\geq k}$ is bounded and nonincreasing for any $p \in F(T)$, so $\lim_{n\to\infty} \rho(x_n - p)$ exists for any $p \in F(T)$. □

Theorem 3.1. *Let X_ρ be a ρ-complete modular space and C be a nonempty convex ρ-closed and ρ-bounded subset of X_ρ. Suppose ρ is* (UUC1) *and satisfies Fatou property. Let $T\colon C \to C$ be a modular nonspreading mapping and let the sequence $\{x_n\}$ be generated by the iterative process* (3.8) *with $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ and $\{\beta_n\}$ bounded away from* 0 *and* 1.

Then $F(T) \neq \emptyset$ if and only if $\lim_{n\to\infty} \rho(x_n - Tx_n) = 0$.

Proof. First of all, suppose $F(T) \neq \emptyset$ and take $p \in F(T)$. According to Lemma 3.3, the limit $\lim\limits_{n\to\infty} \rho(x_n - p)$ exists and we will denote its value with r.

Using Lemma 3.1, we obtain

$$\limsup_{n\to\infty} \rho(Tx_n - p) \leq \lim_{n\to\infty} \rho(x_n - p) = r.$$

On the other hand, using the relation (3.11) and the Lemma 3.1, together with the convexity of ρ, we obtain

$$\begin{aligned} \rho(x_{n+1} - p) &\leq (1-\alpha_n)\rho(z_n - p) + \alpha_n \rho(y_n - p) \\ &= (1-\alpha_n)\rho((1-\gamma_n)x_n + \gamma_n y_n - p) + \alpha_n \rho(y_n - p) \\ &\leq (1-\alpha_n)(1-\gamma_n)\rho(x_n - p) + [(1-\alpha_n)\gamma_n + \alpha_n]\, \rho(y_n - p) \\ &= \rho(x_n - p) + [1 - (1-\gamma_n)(1-\alpha_n)] \left(\rho(y_n - p) - \rho(x_n - p)\right), \end{aligned}$$

which implies

$$\frac{\rho(x_{n+1} - p) - \rho(x_n - p)}{1 - (1-\gamma_n)(1-\alpha_n)} \leq \rho(y_n - p) - \rho(x_n - p).$$

Therefore

$$\rho(x_{n+1} - p) - \rho(x_n - p) \leq \frac{\rho(x_{n+1} - p) - \rho(x_n - p)}{1 - (1-\gamma_n)(1-\alpha_n)} \leq \rho(y_n - p) - \rho(x_n - p),$$

so

$$\rho(x_{n+1} - p) \leq \rho(y_n - p).$$

It is worth noting that, according to inequality (3.9), $\rho(y_n - p) \leq \rho(x_n - p)$, which implies that

$$r = \lim_{n\to\infty} \rho(y_n - p).$$

It follows

$$\lim_{n\to\infty} \rho\left(\beta_n(Tx_n - p) + (1-\beta_n)(x_n - p)\right) = \lim_{n\to\infty} \rho(y_n - p) = r$$

and, since the conditions of Lemma 2.1 are now checked, we see that $\lim\limits_{n\to\infty} \rho(Tx_n - x_n) = 0$.

Conversely, let τ denote the ρ-type function of $\{x_n\}$ and let $\{c_n\}$ be a minimizing sequence for τ converging to a point $z \in C$, which implies that $\lim\limits_{n\to\infty} \rho(c_n - z) = 0$ (Lemma 2.2 ensures this convergence).

Using Lemma 3.2 (ii), $\{Tc_n\}$ is a minimizing sequence as well and by Proposition 3.1 it is easily observed that $\lim\limits_{n\to\infty} \rho(c_n - Tc_n) = 0$.

Using now Lemma 3.2 (i), we have

$$0 \leq \limsup_{n\to\infty} \rho(c_n - Tz) \leq \limsup_{n\to\infty} \rho(c_n - z) = 0,$$

which involves that $\lim\limits_{n\to\infty} \rho(c_n - Tz) = 0$.

By the uniqueness of the limit, we have $Tz = z$. □

Theorem 3.2. *Let C be a nonempty ρ-compact and convex subset of a complete modular space X_ρ and let ρ, T and $\{x_n\}$ be as in Theorem* 3.1*. Then, the sequence $\{x_n\}$ ρ-converges to a fixed point of T.*

Proof. The ρ-compactness of C implies the existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which ρ-converges to a point $z \in C$.

From Lemma 3.2, we obtain

$$0 \leq \limsup_{n\to\infty} \rho(x_{n_k} - Tz) \leq \limsup_{n\to\infty} \rho(x_{n_k} - z) = 0,$$

hence $\lim\limits_{n\to\infty} \rho(x_{n_k} - Tz) = 0$. By the uniqueness of the limit, we have $Tz = z$.

From Lemma 3.3, it follows that the limit $\lim_{n\to\infty} \rho(x_n - z)$ exists and then the sequence $\{x_n\}$ ρ-converges to z, where $z \in F(T)$. □

4. **Conclusions**

Quadratic nonexpansiveness conditions of nonspreading or hybrid type are a recent direction in fixed point theory. Initiated in the setting of Banach spaces, directly related with the duality map, they reach a more approachable expression in the particular setting of a Hilbert space. This paper adapted the expression of nospreadingness in Hilbert setting to a modular framework. The analysis specifically looked at issues related to necessary and sufficient conditions for the existence of fixed points and was performed via the S_n iteration procedure. A convergence criterion was also established under modular compactness assumption.

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