AN ELEMENTARY PROOF OF THE WEIGHTED GEOMETRIC MEAN BEING A BERNSTEIN FUNCTION

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In the paper, the authors supply an elementary proof for the assertion that the weighted geometric mean is a Bernstein function.

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1. Introduction

An infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies

$$(-1)^n f^{(n)}(t) \ge 0$$

for $x \in I$ and $n \ge 0$. See [10, Definition 1.3]. An infinitely differentiable function $f : I \subseteq (-\infty, \infty) \to [0, \infty)$ is called a Bernstein function on I if its derivative $f'(t)$ is completely monotonic on I. See [10, Definition 3.1]. The Bernstein functions on $(0, \infty)$ can be characterized by the assertion that a function $f : (0, \infty) \to \mathbb{R}$ is a Bernstein function if and only if it admits the representation

$$f(x) = a + bx + \int_0^\infty \left(1 - e^{-xt}\right) \mathrm{d}\,\mu(t), \tag{1.1}$$

where $a, b \ge 0$ and μ is a Radon measure on $(0, \infty)$ satisfying

$$\int_0^\infty \min\{1, t\} \,\mathrm{d}\,\mu(t) < \infty. \tag{1.2}$$

See [10, Theorem 3.2]. The triplet (a, b, μ) determines f uniquely and vice versa. The representing measure μ and the characteristic triplet (a, b, μ) from the expression (1.1) are often called the Lévy measure and the Lévy triplet of the Bernstein function f. The formula (1.1) is called the Lévy-Khintchine representation of f. If the Lévy measure μ satisfying (1.1) and (1.2) has a completely monotonic density $m(t)$ with respect to the Lebesgue measure, that is, the integral representation

$$f(x) = a + bx + \int_0^\infty \left(1 - e^{-xt}\right) m(t) \,\mathrm{d}\,t \tag{1.3}$$

holds for $a, b \ge 0$ and $m(t)$ is a completely monotonic function on $(0, \infty)$, then f is said to be a complete Bernstein function on $(0, \infty)$. See [10, Definition 6.1]. For $f(t)$ being

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a nonconstant infinitely differentiable function on $(0,\infty)$, for $f(\infty)=\lim_{x\to\infty} f(x)$, and for some $r\in\mathbb{R}$, if the function $t^r[f(t)-f(\infty)]$ is completely monotonic on $(0,\infty)$ but $t^{r+\varepsilon}[f(t)-f(\infty)]$ is not for any positive number $\varepsilon>0$, then we say that the number r is the completely monotonic degree of $f(t)$ with respect to $t\in(0,\infty)$; if for all $r\in\mathbb{R}$ each $t^r[f(t)-f(\infty)]$ is completely monotonic on $(0,\infty)$, then we say that the completely monotonic degree of $f(t)$ with respect to $t\in(0,\infty)$ is ∞. See [2, Definition 1.1], or [4, Definition 1.1], or [8, Definiton 1.3]. If the density $m(t)$ of the representing measure $\mu(t)$ is of the completely monotonic degree r, then f is said to be a complete Bernstein function of degree r, or say, r is said to be the degree of the complete Bernstein function f on $(0,\infty)$. See [1, Definition 1.6] and [8, Definiton 1.4].

Recall that the quantity $G(x,y;\lambda)=x^\lambda y^{1-\lambda}$ for $x,y>0$ and $\lambda\in(0,1)$ is called the weighted geometric mean. For $\lambda\in(0,1)$ and $x,y\in\mathbb{R}$, define

$$G_{x,y;\lambda}(t)=G(x+t,y+t;\lambda) \tag{1.4}$$

on $(-\min\{x,y\},\infty)$. From the facts that x^λ is a complete Bernstein function for $\lambda\in(0,1)$ on $(0,\infty)$, see [10, Remark 7.8], and that, if f_1 and f_2 are complete Bernstein functions on $(0,\infty)$, then $f_1^\lambda f_2^{1-\lambda}$ for $\lambda\in(0,1)$ is also a complete Bernstein function on $(0,\infty)$, see [10, Proposition 7.10], it follows that $G_{x,y;\lambda}(t)$ for $\lambda\in(0,1)$ is a complete Bernstein function on $(0,\infty)$. In the proof of [3, Theorem 1], it was essentially recovered that the weighted geometric mean $G_{x,y;\lambda}(t)$ is a Bernstein function of $t>-\min\{x,y\}$. In [9], see also [11, Chapter 2], the statement that the geometric mean $G_{x,y;1/2}(t)$ is a complete Bernstein function was rediscovered by several approaches. Recently, among other things, it was found in [1, Theorem 2.5] that $G_{x,y;\lambda}(t)$ for $\lambda\in(0,1)$ is a complete Bernstein function of degree 0 on $(0,\infty)$. For more information, please refer to [5, 7] and closely related references therein.

In this paper, we will provide an elementary proof for the assertion that for $\lambda\in(0,1)$ and $x,y\in\mathbb{R}$ the weighted geometric mean $G_{x,y;\lambda}(t)$ is a Bernstein function on $(-\min\{x,y\},\infty)$.

Theorem 1.1. *For $\lambda\in(0,1)$ and $x,y\in\mathbb{R}$ with $x\ne y$, the weighted geometric mean $G_{x,y;\lambda}(t)$ defined by* (1.4) *is a Bernstein function of $t>-\min\{x,y\}$.*

2. A lemma

Our elementary proof for Theorem 1.1 bases on the following lemma.

Lemma 2.1. *For $t>0$ and $\alpha\in(-1,1)$, let*

$$h_\alpha(t)=\left(1+\frac{1}{t}\right)^\alpha. \tag{2.1}$$

Then the derivatives of $h_\alpha(t)$ can be computed by

$$h_\alpha^{(i)}(t)=\frac{(-1)^i}{t^i(1+t)^i}\left(1+\frac{1}{t}\right)^\alpha\sum_{k=0}^{i-1}a_{\alpha,i,k}t^k, \tag{2.2}$$

where $i\in\mathbb{N}$ and

$$a_{\alpha,i,k}=k!\binom{i}{k}\binom{i-1}{k}\prod_{\ell=0}^{i-k-1}(\alpha+\ell). \tag{2.3}$$

Consequently,

(1) *if $\alpha\in(0,1)$, the function $h_\alpha(t)$ is completely monotonic on $(0,\infty)$;*
(2) *if $\alpha\in(-1,0)$, the function $h_\alpha(t)$ is a Bernstein function on $(0,\infty)$;*

(3) *the derivatives of the function*

$$H_\alpha(t)=\frac{h_\alpha(t)}{\alpha}-\frac{h_{\alpha-1}(t)}{\alpha-1} \tag{2.4}$$

may be calculated by

$$H_\alpha^{(i)}(t)=\frac{(-1)^i}{t^i(1+t)^{i+1}}\biggl(1+\frac1t\biggr)^\alpha\sum_{k=0}^{i-1}b_{\alpha,i,k}t^k, \tag{2.5}$$

where $i\in\mathbb{N}$ *and*

$$b_{\alpha,i,k}=k!\binom{i+1}{k}\binom{i-1}{k}\prod_{\ell=1}^{i-k-1}(\alpha+\ell); \tag{2.6}$$

(4) *the function* $H_\alpha(t)$ *is completely monotonic for all* $\alpha\in(0,1)$ *on* $(0,\infty)$.

Proof. It is easy to see that

$$h_\alpha'(t)=-\alpha\biggl(1+\frac1t\biggr)^{\alpha-1}\frac1{t^2}=-\alpha\biggl(1+\frac1t\biggr)^\alpha\frac1{t(1+t)}.$$

This means that the formulas (2.2) and (2.3) are valid for $i=1$.

Assume that the formulas (2.2) and (2.3) are valid for some $i>1$. By this inductive hypothesis, a simple calculation gives

$$\begin{aligned}
h_\alpha^{(i+1)}(t)&=\bigl[h_\alpha^{(i)}(t)\bigr]'=\Biggl[\frac{(-1)^i}{t^i(1+t)^i}\biggl(1+\frac1t\biggr)^\alpha\sum_{k=0}^{i-1}a_{\alpha,i,k}t^k\Biggr]'\\
&=(-1)^{i+1}\sum_{k=0}^{i-1}\frac{a_{\alpha,i,k}}{[t^{i+\alpha-k}(1+t)^{i-\alpha}]^2}\\
&\quad\times\bigl[(i+\alpha-k)t^{i+\alpha-k-1}(1+t)^{i-\alpha}+(i-\alpha)t^{i+\alpha-k}(1+t)^{i-\alpha-1}\bigr]\\
&=\frac{(-1)^{i+1}}{t^{i+1}(1+t)^{i+1}}\biggl(1+\frac1t\biggr)^\alpha\sum_{k=0}^{i-1}a_{\alpha,i,k}[i+\alpha-k+(2i-k)t]t^k\\
&=\frac{(-1)^{i+1}}{t^{i+1}(1+t)^{i+1}}\biggl(1+\frac1t\biggr)^\alpha\Biggl\{(i+\alpha)a_{\alpha,i,0}+(i+1)a_{\alpha,i,i-1}t^i\\
&\quad+\sum_{k=1}^{i-1}[(i+\alpha-k)a_{\alpha,i,k}+(2i-k+1)a_{\alpha,i,k-1}]t^k\Biggr\}\\
&=\frac{(-1)^{i+1}}{t^{i+1}(1+t)^{i+1}}\biggl(1+\frac1t\biggr)^\alpha\sum_{k=0}^{i}a_{\alpha,i+1,k}t^k.
\end{aligned}$$

This shows that the formulas (2.2) and (2.3) are valid for all $i\ge1$.

The rest of the proof is straightforward. The proof of Lemma 2.1 is completed. □

3. An elementary proof of Theorem 1.1

Now we are in a position to provide an elementary proof of Theorem 1.1.

When $x>y>0$, a direct differentiation yields

$$\begin{aligned}
G_{x,y;\lambda}'(t)&=\lambda(1-\lambda)\Biggl[\frac1\lambda\biggl(\frac{x+t}{y+t}\biggr)^\lambda+\frac1{1-\lambda}\biggl(\frac{y+t}{x+t}\biggr)^{1-\lambda}\Biggr]\\
&=\lambda(1-\lambda)\Biggl[\frac1\lambda\biggl(1+\frac{x-y}{y+t}\biggr)^\lambda-\frac1{\lambda-1}\biggl(1+\frac{x-y}{y+t}\biggr)^{\lambda-1}\Biggr]
\end{aligned}$$

$$= \lambda(1-\lambda)\left[\frac{1}{\lambda}h_\lambda\left(\frac{y+t}{x-y}\right) - \frac{1}{\lambda-1}h_{\lambda-1}\left(\frac{y+t}{x-y}\right)\right] = \lambda(1-\lambda)H_\lambda\left(\frac{y+t}{x-y}\right).$$

By the complete monotonicity of the function H_α obtained in Lemma 2.1, it is immediate to see that the derivative $G'_{x,y;\lambda}(t)$ is completely monotonic, and so the geometric mean $G_{x,y;\lambda}(t)$ is a Bernstein function for $x > y > 0$ and $\lambda \in (0,1)$. Considering the symmetry property $G_{x,y;\lambda}(t) = G_{y,x;1-\lambda}(t)$ reveals that, no matter $y > x > 0$ or $x > y > 0$, the geometric mean $G_{x,y;\lambda}(t)$ is a Bernstein function of $t > -\min\{x, y\}$.

Remark 1. In [10, Corollary 7.9 (i)], it was given that, if f_1 is a complete Bernstein function on $(0,\infty)$ and f_2 is a Stieltjes function, then the composition $f_1 \circ f_2$ is a Stieltjes function. For $\alpha \in (0,1)$, since x^α is a complete Bernstein function on $(0,\infty)$, see [10, Remark 7.8], and $1+\frac{1}{x}$ is a Stieltjes function, see [10, Theorem 2.2 (ii) and Remark 2.4], the function $h_\alpha(t)$ is a Stieltjes function, and so a completely monotonic function on $(0,\infty)$.

In [10, Corollary 7.6 (iii)], it was stated that, if f_1, f_2 are complete Bernstein functions on $(0,\infty)$, then the composition $f_1 \circ f_2$ is also a complete Bernstein function on $(0,\infty)$. Therefore, for $\alpha \in (-1,0)$, since both of the functions $f_1(x) = x^{-\alpha}$ and $f_2(x) = \frac{x}{1+x}$ are clearly complete Bernstein functions on $(0,\infty)$, the function $h_\alpha(t) = \left(\frac{t}{1+t}\right)^{-\alpha}$ is a complete Bernstein function on $(0,\infty)$.

In [8, Lemma 2.1], the above two conclusions were directly recovered by using the Cauchy integral formula in the theory of complex functions.

Remark 2. This paper is abstracted from the preprint [6] and the thesis [11].

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