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ON THE ESTIMATES OF WARPING FUNCTIONS ON ISOMETRIC IMMERSIONS

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Using the results of [11]*, we get some estimates of warping functions for isometric immersions by changing the target manifolds by some types of Riemannian manifolds: constant space forms and Hermitian symmetric spaces. We deal with equality cases and obtain applications. Finally, we present some open problems.*

Keywords: warping function, warped product, isometric immersion, eigenvalue
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1. Introduction

Let (B, g_B) and (F, g_F) be Riemannian manifolds. Given a warped product manifold $M = B \times_f F$ with a warping function f (See [11]), we can consider an isometric immersion $\psi : M \mapsto (\overline{M}, \overline{g})$, where $(\overline{M}, \overline{g})$ is a Riemannian manifold.

In 2018, B. Y. Chen [5] proposed two Fundamental Questions on the isometric immersion $\psi : M \mapsto (\overline{M}, \overline{g})$, where $(\overline{M}, \overline{g})$ is a Kähler manifold and gave some recent results on these problems.

In 2014, as a generalization of Chen's works ([3],[4]), the author [11] obtained two inequalities, which give the upper bound and the lower bound of the function $\frac{\triangle f}{f}$. Replacing the Riemannian manifold $(\overline{M}, \overline{g})$ with several types of Riemannian manifolds (i.e., real space forms, complex space forms, quaternionic space forms, Sasakian space forms, Kenmotsu space forms, Hermitian symmetric spaces: complex two-plane Grassmannians, complex hyperbolic two-plane Grassmannians, complex quadrics), we will obtain the upper bounds and the lower bounds of the functions $\frac{\triangle f}{f}$. And by using these results, we will get some equality cases of these relations and obtain their applications.

We also know that warped product manifolds take an important position in differential geometry and in physics, in particular in general relativity. And Nash's result [9] implies that each warped product manifold can be isometrically embedded in a Euclidean space.

The paper is organized as follows. In section 2 we recall some notions, which will be used in the following sections. In section 3 we estimate the upper bounds and the lower bounds of the functions $\frac{\triangle f}{f}$ for constant space forms $(\overline{M}, \overline{g})$ and have some equality cases and their applications. In section 4 we do the works for Hermitian symmetric spaces $(\overline{M}, \overline{g})$. In section 5 we present some open problems.

2. Preliminaries

In this section we recall some notions, which will be used in the following sections.

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Let $(\overline{M}, \overline{g})$ be an n-dimensional Riemannian manifold and let M be an m-dimensional submanifold of $(\overline{M}, \overline{g})$. We denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections of $\overline{M}$ and M, respectively.

Then we get the *Gauss formula* and the *Weingarten formula*

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1}$$
$$\overline{\nabla}_X N = -A_N X + D_X N, \tag{2}$$

respectively, for tangent vector fields $X, Y \in \Gamma(TM)$ and a normal vector field $N \in \Gamma(TM^\perp)$, where h, A, D denote the *second fundamental form*, the *shape operator*, and the *normal connection* of M in $\overline{M}$, respectively.

Then we know

$$\overline{g}(A_N X, Y) = \overline{g}(h(X, Y), N). \tag{3}$$

Fix a local orthonormal frame $\{v_1, \cdots, v_n\}$ of $T\overline{M}$ with $v_i \in \Gamma(TM)$, $1 \leq i \leq m$ and $v_\alpha \in \Gamma(TM^\perp)$, $m+1 \leq \alpha \leq n$. We define the mean curvature vector field H, the squared mean curvature H^2, the squared norm $||h||^2$ of the second fundamental form h as follows:

$$H = \frac{1}{m}\text{trace}h = \frac{1}{m}\sum_{i=1}^{m} h(v_i, v_i), \tag{4}$$
$$H^2 = \overline{g}(H, H), \tag{5}$$
$$||h||^2 = \sum_{i,j=1}^{m} \overline{g}(h(v_i, v_j), h(v_i, v_j)). \tag{6}$$

We call the submanifold $M \subset (\overline{M}, \overline{g})$ *totally geodesic* if the second fundamental form h vanishes identically. Denote by R, $\overline{R}$ the Riemannian curvature tensors of M, $\overline{M}$, respectively.

Let

$$K(X \wedge Y) := \frac{g(R(X, Y)Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2},$$
$$\overline{K}(X \wedge Y) := \frac{\overline{g}(\overline{R}(X, Y)Y, X)}{\overline{g}(X, X)\overline{g}(Y, Y) - \overline{g}(X, Y)^2}$$

for $X, Y \in \Gamma(TM)$, where g denotes the induced metric on M of $(\overline{M}, \overline{g})$. i.e., given a plane $V \subset T_pM$, $p \in M$, spanned by the vectors $X, Y \in T_pM$, $K(V) = K(X \wedge Y)$ and $\overline{K}(V) = \overline{K}(X \wedge Y)$ denote the sectional curvatures of a plane V in M and in $\overline{M}$, respectively.

Let

$$(\inf \overline{K})(p) := \inf\{\overline{K}(V) \mid V \subset T_pM, \dim V = 2\}, \tag{7}$$
$$(\sup \overline{K})(p) := \sup\{\overline{K}(V) \mid V \subset T_pM, \dim V = 2\}. \tag{8}$$

Let $\overline{R}(X, Y, Z, W) := \overline{g}(\overline{R}(X, Y)Z, W)$ for $X, Y, Z, W \in \Gamma(T\overline{M})$.

Given a $C^\infty-$function $f \in C^\infty(M)$, we define the *Laplacian* $\triangle f$ of f by

$$\triangle f := \sum_{i=1}^{m}((\nabla_{v_i} v_i)f - v_i^2 f).$$

Let (B, g_B) and (F, g_F) be Riemannian manifolds.

Throughout this paper, we will denote by $(M, g) := (B \times_f F, g_B + f^2 g_F)$ the warped product manifold of Riemannian manifolds (B, g_B) and (F, g_F) with the warping function $f : B \mapsto \mathbb{R}^+$ (See [11]).

By Theorem 3.1, Theorem 3.4, and their proofs of [11], we have

Lemma 2.1. *Let* $(M,g)=(B\times_f F, g_B+f^2g_F)$ *be a warped product manifold and let* $(\overline{M},\overline{g})$ *be a Riemannian manifold. Let* $\psi:(M,g)\mapsto(\overline{M},\overline{g})$ *be an isometric immersion. Then we get*

$$\frac{m_1m^2}{2(m-1)}H^2-\frac{m_1}{2}||h||^2+m_1\inf\overline{K}\le\frac{\triangle f}{f}\le\frac{m^2}{4m_2}H^2+m_1\sup\overline{K}, \tag{9}$$

where $m_1=\dim B$, $m_2=\dim F$ *and* $m=m_1+m_2$.

3. **Constant space forms**

In this section, we will estimate the functions $\frac{\triangle f}{f}$ for isometric immersions $\psi:(M,g)=(B\times_f F,g_B+f^2g_F)\mapsto(\overline{M},\overline{g})$ with constant space forms $(\overline{M},\overline{g})$. We also deal with equality cases and obtain their applications.

Using Lemma 2.1, we obtain

Theorem 3.1. *Let* $(M,g)=(B\times_f F,g_B+f^2g_F)$ *be a warped product manifold and* $(\overline{M},\overline{g})=(\overline{M}(c),\overline{g})$ *a real space form of constant sectional curvature c. Let* $\psi:(M,g)\mapsto(\overline{M},\overline{g})$ *be an isometric immersion. Then we have*

$$\frac{m_1m^2}{2(m-1)}H^2-\frac{m_1}{2}||h||^2+m_1c\le\frac{\triangle f}{f}\le\frac{m^2}{4m_2}H^2+m_1c, \tag{10}$$

where $m_1=\dim B$, $m_2=\dim F$ *and* $m=m_1+m_2$.

Proof. We know that the Riemannian curvature tensor $\overline{R}$ [8] of $(\overline{M},\overline{g})$ is given by

$$\overline{R}(X,Y)Z=c(\overline{g}(Y,Z)X-\overline{g}(X,Z)Y) \tag{11}$$

for $X,Y,Z\in\Gamma(T\overline{M})$. Since $\inf\overline{K}=\sup\overline{K}=c$, by Lemma 2.1, we get the result. □

Then we easily obtain

Corollary 3.1. *Let* $(M,g)=(B\times_f F,g_B+f^2g_F)$ *be a warped product manifold and* $(\overline{M},\overline{g})=(\overline{M}(c),\overline{g})$ *a real space form of constant sectional curvature c. Let* $\psi:(M,g)\mapsto(\overline{M},\overline{g})$ *be an isometric immersion. Assume that the manifold* (M,g) *is a totally geodesic submanifold of* $(\overline{M},\overline{g})$*. Then we get*

$$m_1c\le\frac{\triangle f}{f}\le m_1c\ .$$

Remark 3.1. *Let* $(M,g)=(B\times_f F,g_B+f^2g_F)$ *be a warped product manifold and* $(\overline{M},\overline{g})=(\overline{M}(c),\overline{g})$ *a real space form of constant sectional curvature c. Let* $\psi:(M,g)\mapsto(\overline{M},\overline{g})$ *be an isometric immersion. Assume that the manifold* (M,g) *is a totally geodesic submanifold of* $(\overline{M},\overline{g})$*. Then the warping function f is an eigen-function with eigenvalue* m_1c*.*

In particular, if $c=0$ *(i.e.,* $(\overline{M},\overline{g})$ *is a Euclidean space* $\mathbb{E}^n$*.), then the warping function f is a harmonic function.*

Lemma 3.1. *Let* $(M,g)=(B\times_f F,g_B+f^2g_F)$ *be a warped product manifold and* $(\overline{M},\overline{g})=(\overline{M}(c),\overline{g})$ *a real space form of constant sectional curvature c. Let* $\psi:(M,g)\mapsto(\overline{M},\overline{g})$ *be an isometric immersion.*

There does not exist a totally geodesic submanifold (M,g) *of* $(\overline{M},\overline{g})$ *such that either the warping function f is not an eigen-function or the eigenvalue of f is not equal to* m_1c*.*

Theorem 3.2. *Let* $(M,g)=(B\times_f F,g_B+f^2g_F)$ *be a warped product manifold and* $(\overline{M},\overline{g})=(\overline{M}(c),\overline{g},J)$ *a complex space form of constant holomorphic sectional curvature c. Let* ψ :

$(M,g) \mapsto (\overline{M},\overline{g})$ be an isometric immersion. Then we have

$$\frac{m_1 m^2}{2(m-1)}H^2 - \frac{m_1}{2}||h||^2 + m_1\frac{c}{4} \leq \frac{\triangle f}{f} \leq \frac{m^2}{4m_2}H^2 + m_1 c, \quad c \geq 0, \tag{12}$$

$$\frac{m_1 m^2}{2(m-1)}H^2 - \frac{m_1}{2}||h||^2 + m_1 c \leq \frac{\triangle f}{f} \leq \frac{m^2}{4m_2}H^2 + m_1\frac{c}{4}, \quad c < 0, \tag{13}$$

where $m_1 = \dim B$, $m_2 = \dim F$ and $m = m_1 + m_2$.

Proof. The Riemannian curvature tensor $\overline{R}$ [8] of $(\overline{M},\overline{g})$ is given by

$$\begin{aligned} &\overline{R}(X,Y)Z \\ &= \frac{c}{4}(\overline{g}(Y,Z)X - \overline{g}(X,Z)Y + \overline{g}(JY,Z)JX - \overline{g}(JX,Z)JY - 2\overline{g}(JX,Y)JZ) \end{aligned} \tag{14}$$

for $X, Y, Z \in \Gamma(T\overline{M})$. Given orthonormal vectors $X, Y \in T_p\overline{M}$, $p \in \overline{M}$, we get

$$\overline{K}(X \wedge Y) = \overline{R}(X,Y,Y,X) = \frac{c}{4}(1 + 3\overline{g}(JX,Y)^2) \tag{15}$$

so that since $0 \leq |\overline{g}(JX,Y)| \leq 1$, we easily obtain

$$\frac{c}{4} \leq \overline{K}(X \wedge Y) \leq c, \quad c \geq 0,$$
$$c \leq \overline{K}(X \wedge Y) \leq \frac{c}{4}, \quad c < 0.$$

From Lemma 2.1, the result follows. □

Corollary 3.2. *Let $(M,g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M},\overline{g}) = (\overline{M}(c),\overline{g},J)$ a complex space form of constant holomorphic sectional curvature c. Let $\psi : (M,g) \mapsto (\overline{M},\overline{g})$ be an isometric immersion. Assume that the manifold (M,g) is a totally geodesic totally real submanifold of $(\overline{M},\overline{g})$ (i.e., $J(TM) \subset TM^{\perp}$).*

Then we have

$$m_1\frac{c}{4} \leq \frac{\triangle f}{f} \leq m_1\frac{c}{4}.$$

Proof. By Lemma 2.1 and (15), we obtain the result. □

Remark 3.2. *Let $(M,g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M},\overline{g}) = (\overline{M}(c),\overline{g},J)$ a complex space form of constant holomorphic sectional curvature c. Let $\psi : (M,g) \mapsto (\overline{M},\overline{g})$ be an isometric immersion. Assume that the manifold (M,g) is a totally geodesic totally real submanifold of $(\overline{M},\overline{g})$.*

Then the warping function f is an eigen-function with eigenvalue $\frac{m_1 c}{4}$.

Lemma 3.2. *Let $(M,g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M},\overline{g}) = (\overline{M}(c),\overline{g},J)$ a complex space form of constant holomorphic sectional curvature c. Let $\psi : (M,g) \mapsto (\overline{M},\overline{g})$ be an isometric immersion.*

There does not exist a totally geodesic totally real submanifold (M,g) of $(\overline{M},\overline{g})$ such that either the warping function f is not an eigen-function or the eigenvalue of f is not equal to $\frac{m_1 c}{4}$.

Corollary 3.3. *Let $(M,g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M},\overline{g}) = (\overline{M}(c),\overline{g},J)$ a complex space form of constant holomorphic sectional curvature c. Let $\psi : (M,g) \mapsto (\overline{M},\overline{g})$ be an isometric immersion. Assume that the manifold (M,g) is a 2-dimensional totally geodesic complex submanifold of $(\overline{M},\overline{g})$ (i.e., $J(TM) = TM$).*

Then we have

$$m_1 c \leq \frac{\triangle f}{f} \leq m_1 c.$$

Remark 3.3. *Let $(M,g)=(B\times_f F, g_B+f^2g_F)$ be a warped product manifold and $(\overline{M},\overline{g})=(\overline{M}(c),\overline{g},J)$ a complex space form of constant holomorphic sectional curvature c. Let $\psi:(M,g)\mapsto(\overline{M},\overline{g})$ be an isometric immersion. Assume that the manifold (M,g) is a 2-dimensional totally geodesic complex submanifold of $(\overline{M},\overline{g})$.*

Then the warping function f is an eigen-function with eigenvalue m_1c.

Lemma 3.3. *Let $(M,g)=(B\times_f F, g_B+f^2g_F)$ be a warped product manifold and $(\overline{M},\overline{g})=(\overline{M}(c),\overline{g},J)$ a complex space form of constant holomorphic sectional curvature c. Let $\psi:(M,g)\mapsto(\overline{M},\overline{g})$ be an isometric immersion.*

There does not exist a 2-dimensional totally geodesic complex submanifold (M,g) of $(\overline{M},\overline{g})$ such that either the warping function f is not an eigen-function or the eigenvalue of f is not equal to m_1c.

Theorem 3.3. *Let $(M,g)=(B\times_f F, g_B+f^2g_F)$ be a warped product manifold and $(\overline{M},\overline{g})=(\overline{M}(c),E,\overline{g})$ a quaternionic space form of constant quaternionic sectional curvature c. Let $\psi:(M,g)\mapsto(\overline{M},\overline{g})$ be an isometric immersion. Then we obtain*

$$\frac{m_1m^2}{2(m-1)}H^2-\frac{m_1}{2}||h||^2+m_1\frac{c}{4}\leq\frac{\triangle f}{f}\leq\frac{m^2}{4m_2}H^2+m_1c,\quad c\geq 0, \tag{16}$$

$$\frac{m_1m^2}{2(m-1)}H^2-\frac{m_1}{2}||h||^2+m_1c\leq\frac{\triangle f}{f}\leq\frac{m^2}{4m_2}H^2+m_1\frac{c}{4},\quad c<0, \tag{17}$$

where $m_1=\dim B$, $m_2=\dim F$ and $m=m_1+m_2$.

Proof. We know that the Riemannian curvature tensor $\overline{R}$ [6] of $(\overline{M},\overline{g})$ is given by

$$\begin{aligned}\overline{R}(X,Y)Z=&\frac{c}{4}(\overline{g}(Y,Z)X-\overline{g}(X,Z)Y\\&+\sum_{\alpha=1}^{3}(\overline{g}(J_\alpha Y,Z)J_\alpha X-\overline{g}(J_\alpha X,Z)J_\alpha Y-2\overline{g}(J_\alpha X,Y)J_\alpha Z))\end{aligned} \tag{18}$$

for $X,Y,Z\in\Gamma(T\overline{M})$. Given orthonormal vectors $X,Y\in T_p\overline{M}$, $p\in\overline{M}$, we have

$$\overline{K}(X\wedge Y)=\overline{R}(X,Y,Y,X)=\frac{c}{4}(1+3\sum_{\alpha=1}^{3}\overline{g}(J_\alpha X,Y)^2). \tag{19}$$

Since $\{J_1X,J_2X,J_3X\}$ is orthonormal, we get $0\leq\sum_{\alpha=1}^{3}\overline{g}(J_\alpha X,Y)^2\leq|Y|^2=1$ so that

$$\begin{aligned}&\frac{c}{4}\leq\overline{K}(X\wedge Y)\leq c,\quad c\geq 0,\\&c\leq\overline{K}(X\wedge Y)\leq\frac{c}{4},\quad c<0.\end{aligned}$$

From Lemma 2.1, we obtain the result. □

Corollary 3.4. *Let $(M,g)=(B\times_f F, g_B+f^2g_F)$ be a warped product manifold and $(\overline{M},\overline{g})=(\overline{M}(c),E,\overline{g})$ a quaternionic space form of constant quaternionic sectional curvature c. Let $\psi:(M,g)\mapsto(\overline{M},\overline{g})$ be an isometric immersion. Assume that the manifold (M,g) is a totally geodesic totally real submanifold of $(\overline{M},\overline{g})$ (i.e., $J_\alpha(TM)\subset TM^\perp$, $\forall\alpha\in\{1,2,3\}$).*

Then we have

$$m_1\frac{c}{4}\leq\frac{\triangle f}{f}\leq m_1\frac{c}{4}.$$

Proof. By Lemma 2.1 and (19), we obtain the result. □

Lemma 3.4. *Let $(M,g)=(B\times_f F, g_B+f^2g_F)$ be a warped product manifold and $(\overline{M},\overline{g})=(\overline{M}(c),E,\overline{g})$ a quaternionic space form of constant quaternionic sectional curvature c. Let $\psi:(M,g)\mapsto(\overline{M},\overline{g})$ be an isometric immersion.*

There does not exist a totally geodesic totally real submanifold (M, g) *of* $(\overline{M}, \overline{g})$ *such that either the warping function* f *is not an eigen-function or the eigenvalue of* f *is not equal to* $\frac{m_1 c}{4}$.

Corollary 3.5. *Let* $(M, g) = (B \times_f F, g_B + f^2 g_F)$ *be a warped product manifold and* $(\overline{M}, \overline{g}) = (\overline{M}(c), E, \overline{g})$ *a quaternionic space form of constant quaternionic sectional curvature* c. *Let* $\psi : (M, g) \mapsto (\overline{M}, \overline{g})$ *be an isometric immersion. Assume that the manifold* (M, g) *is a 4-dimensional totally geodesic quaternionic submanifold of* $(\overline{M}, \overline{g})$ *(i.e.,* $J_\alpha(TM) = TM$, $\forall \alpha \in \{1, 2, 3\}$*).*

Then we have

$$m_1 c \leq \frac{\triangle f}{f} \leq m_1 c.$$

Lemma 3.5. *Let* $(M, g) = (B \times_f F, g_B + f^2 g_F)$ *be a warped product manifold and* $(\overline{M}, \overline{g}) = (\overline{M}(c), E, \overline{g})$ *a quaternionic space form of constant quaternionic sectional curvature* c. *Let* $\psi : (M, g) \mapsto (\overline{M}, \overline{g})$ *be an isometric immersion.*

There does not exist a 4-dimensional totally geodesic quaternionic submanifold (M, g) *of* $(\overline{M}, \overline{g})$ *such that either the warping function* f *is not an eigen-function or the eigenvalue of* f *is not equal to* $m_1 c$.

Theorem 3.4. *Let* $(M, g) = (B \times_f F, g_B + f^2 g_F)$ *be a warped product manifold and* $(\overline{M}, \overline{g}) = (\overline{M}(c), \phi, \xi, \eta, \overline{g})$ *a Sasakian space form of constant* ϕ*-sectional curvature* c. *Let* $\psi : (M, g) \mapsto (\overline{M}, \overline{g})$ *be an isometric immersion. Then we obtain*

$$\frac{m_1 m^2}{2(m-1)} H^2 - \frac{m_1}{2} ||h||^2 + m_1 \leq \frac{\triangle f}{f} \leq \frac{m^2}{4m_2} H^2 + m_1 c, \quad c \geq 1, \tag{20}$$

$$\frac{m_1 m^2}{2(m-1)} H^2 - \frac{m_1}{2} ||h||^2 + m_1 c \leq \frac{\triangle f}{f} \leq \frac{m^2}{4m_2} H^2 + m_1, \quad c < 1, \tag{21}$$

where $m_1 = \dim B$, $m_2 = \dim F$ *and* $m = m_1 + m_2$.

Proof. We see that the Riemannian curvature tensor $\overline{R}$ [10] of $(\overline{M}, \overline{g})$ is given by

$$\overline{R}(X, Y)Z = \frac{c+3}{4}(\overline{g}(Y, Z)X - \overline{g}(X, Z)Y) \tag{22}$$
$$+ \frac{c-1}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)\overline{g}(X, Z)\xi - \eta(X)\overline{g}(Y, Z)\xi$$
$$+ \overline{g}(\phi Y, Z)\phi X - \overline{g}(\phi X, Z)\phi Y - 2\overline{g}(\phi X, Y)\phi Z)$$

for $X, Y, Z \in \Gamma(T\overline{M})$. Given orthonormal vectors $X, Y \in T_p\overline{M}$, $p \in \overline{M}$, we have

$$\overline{K}(X \wedge Y) = \overline{R}(X, Y, Y, X) = \frac{c+3}{4} + \frac{c-1}{4}(-\eta(Y)^2 - \eta(X)^2 + 3\overline{g}(\phi X, Y)^2). \tag{23}$$

If $\xi \in \mathrm{Span}(X, Y)$, then $-\eta(Y)^2 - \eta(X)^2 + 3\overline{g}(\phi X, Y)^2 = -1$. If $Y = \phi X$ and $\eta(X) = 0$, then $-\eta(Y)^2 - \eta(X)^2 + 3\overline{g}(\phi X, Y)^2 = 3$. Hence we get $-1 \leq -\eta(Y)^2 - \eta(X)^2 + 3\overline{g}(\phi X, Y)^2 \leq 3$ so that

$$1 \leq \overline{K}(X \wedge Y) \leq c, \quad c \geq 1,$$
$$c \leq \overline{K}(X \wedge Y) \leq 1, \quad c < 1.$$

From Lemma 2.1, the result follows. □

Corollary 3.6. *Let* $(M, g) = (B \times_f F, g_B + f^2 g_F)$ *be a warped product manifold and* $(\overline{M}, \overline{g}) = (\overline{M}(c), \phi, \xi, \eta, \overline{g})$ *a Sasakian space form of constant* ϕ*-sectional curvature* c. *Let* $\psi : (M, g) \mapsto (\overline{M}, \overline{g})$ *be an isometric immersion. Assume that the manifold* (M, g) *is a totally geodesic* ϕ*-totally real submanifold of* $(\overline{M}, \overline{g})$ *with* $\xi \in \Gamma(TM^\perp)$ *(i.e.,* $\phi(TM) \subset TM^\perp$*).*

Then we get

$$m_1\frac{c+3}{4} \le \frac{\triangle f}{f} \le m_1\frac{c+3}{4}.$$

Proof. By Lemma 2.1 and (23), we obtain the result. □

Lemma 3.6. *Let $(M,g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M},\overline{g}) = (\overline{M}(c),\phi,\xi,\eta,\overline{g})$ a Sasakian space form of constant ϕ-sectional curvature c. Let $\psi : (M,g) \mapsto (\overline{M},\overline{g})$ be an isometric immersion.*

There does not exist a totally geodesic ϕ-totally real submanifold (M,g) of $(\overline{M},\overline{g})$ such that either the warping function f is not an eigen-function or the eigenvalue of f is not equal to $\frac{m_1(c+3)}{4}$.

Corollary 3.7. *Let $(M,g) = (B\times_f F, g_B+f^2 g_F)$ be a warped product manifold and $(\overline{M},\overline{g}) = (\overline{M}(c),\phi,\xi,\eta,\overline{g})$ a Sasakian space form of constant ϕ-sectional curvature c. Let $\psi : (M,g) \mapsto (\overline{M},\overline{g})$ be an isometric immersion. Assume that the manifold (M,g) is a 2-dimensional totally geodesic submanifold of $(\overline{M},\overline{g})$ with $\xi \in \Gamma(TM)$.*

Then we get

$$m_1 \cdot 1 \le \frac{\triangle f}{f} \le m_1 \cdot 1.$$

Lemma 3.7. *Let $(M,g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M},\overline{g}) = (\overline{M}(c),\phi,\xi,\eta,\overline{g})$ a Sasakian space form of constant ϕ-sectional curvature c. Let $\psi : (M,g) \mapsto (\overline{M},\overline{g})$ be an isometric immersion.*

There does not exist a 2-dimensional totally geodesic submanifold (M,g) of $(\overline{M},\overline{g})$ with $\xi \in \Gamma(TM)$ such that either the warping function f is not an eigen-function or the eigenvalue of f is not equal to m_1.

Corollary 3.8. *Let $(M,g) = (B\times_f F, g_B+f^2 g_F)$ be a warped product manifold and $(\overline{M},\overline{g}) = (\overline{M}(c),\phi,\xi,\eta,\overline{g})$ a Sasakian space form of constant ϕ-sectional curvature c. Let $\psi : (M,g) \mapsto (\overline{M},\overline{g})$ be an isometric immersion. Assume that the manifold (M,g) is a 2-dimensional totally geodesic ϕ-invariant submanifold of $(\overline{M},\overline{g})$ with $\xi \in \Gamma(TM^\perp)$ (i.e., $\phi(TM) = TM$).*

Then we have

$$m_1 c \le \frac{\triangle f}{f} \le m_1 c.$$

Lemma 3.8. *Let $(M,g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M},\overline{g}) = (\overline{M}(c),\phi,\xi,\eta,\overline{g})$ a Sasakian space form of constant ϕ-sectional curvature c. Let $\psi : (M,g) \mapsto (\overline{M},\overline{g})$ be an isometric immersion.*

There does not exist a 2-dimensional totally geodesic ϕ-invariant submanifold (M,g) of $(\overline{M},\overline{g})$ with $\xi \in \Gamma(TM^\perp)$ such that either the warping function f is not an eigen-function or the eigenvalue of f is not equal to $m_1 c$.

Theorem 3.5. *Let $(M,g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M},\overline{g}) = (\overline{M}(c),\phi,\xi,\eta,\overline{g})$ a Kenmotsu space form of constant ϕ-sectional curvature c. Let $\psi : (M,g) \mapsto (\overline{M},\overline{g})$ be an isometric immersion. Then we obtain*

$$\frac{m_1 m^2}{2(m-1)}H^2 - \frac{m_1}{2}||h||^2 - m_1 \le \frac{\triangle f}{f} \le \frac{m^2}{4m_2}H^2 + m_1 c, \quad c \ge -1, \tag{24}$$

$$\frac{m_1 m^2}{2(m-1)}H^2 - \frac{m_1}{2}||h||^2 + m_1 c \le \frac{\triangle f}{f} \le \frac{m^2}{4m_2}H^2 - m_1, \quad c < -1, \tag{25}$$

where $m_1 = \dim B$, $m_2 = \dim F$ and $m = m_1 + m_2$.

Proof. We know that the Riemannian curvature tensor $\overline{R}$ [7] of $(\overline{M}, \overline{g})$ is given by

$$\overline{R}(X,Y)Z = \frac{c-3}{4}(\overline{g}(Y,Z)X - \overline{g}(X,Z)Y) \tag{26}$$
$$+\frac{c+1}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)\overline{g}(X,Z)\xi - \eta(X)\overline{g}(Y,Z)\xi$$
$$+\overline{g}(\phi Y, Z)\phi X - \overline{g}(\phi X, Z)\phi Y - 2\overline{g}(\phi X, Y)\phi Z)$$

for $X, Y, Z \in \Gamma(T\overline{M})$. Given orthonormal vectors $X, Y \in T_p\overline{M}$, $p \in \overline{M}$, we have

$$\overline{K}(X \wedge Y) = \overline{R}(X,Y,Y,X) = \frac{c-3}{4} + \frac{c+1}{4}(-\eta(Y)^2 - \eta(X)^2 + 3\overline{g}(\phi X, Y)^2) \tag{27}$$

so that since $-1 \leq -\eta(Y)^2 - \eta(X)^2 + 3\overline{g}(\phi X, Y)^2 \leq 3$, we get

$$-1 \leq \overline{K}(X \wedge Y) \leq c, \quad c \geq -1,$$
$$c \leq \overline{K}(X \wedge Y) \leq -1, \quad c < -1.$$

From Lemma 2.1, we obtain the result. □

Corollary 3.9. *Let $(M, g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M}, \overline{g}) = (\overline{M}(c), \phi, \xi, \eta, \overline{g})$ a Kenmotsu space form of constant ϕ-sectional curvature c. Let $\psi : (M, g) \mapsto (\overline{M}, \overline{g})$ be an isometric immersion. Assume that the manifold (M, g) is a 2-dimensional totally geodesic submanifold of $(\overline{M}, \overline{g})$ with $\xi \in \Gamma(TM)$.*

Then we get

$$m_1 \cdot (-1) \leq \frac{\triangle f}{f} \leq m_1 \cdot (-1).$$

Lemma 3.9. *Let $(M, g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M}, \overline{g}) = (\overline{M}(c), \phi, \xi, \eta, \overline{g})$ a Kenmotsu space form of constant ϕ-sectional curvature c. Let $\psi : (M, g) \mapsto (\overline{M}, \overline{g})$ be an isometric immersion.*

There does not exist a 2-dimensional totally geodesic submanifold (M, g) of $(\overline{M}, \overline{g})$ with $\xi \in \Gamma(TM)$ such that either the warping function f is not an eigen-function or the eigenvalue of f is not equal to $-m_1$.

Corollary 3.10. *Let $(M, g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M}, \overline{g}) = (\overline{M}(c), \phi, \xi, \eta, \overline{g})$ a Kenmotsu space form of constant ϕ-sectional curvature c. Let $\psi : (M, g) \mapsto (\overline{M}, \overline{g})$ be an isometric immersion. Assume that the manifold (M, g) is a 2-dimensional totally geodesic ϕ-invariant submanifold of $(\overline{M}, \overline{g})$ with $\xi \in \Gamma(TM^\perp)$ (i.e., $\phi(TM) = TM$).*

Then we get

$$m_1 c \leq \frac{\triangle f}{f} \leq m_1 c.$$

Lemma 3.10. *Let $(M, g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M}, \overline{g}) = (\overline{M}(c), \phi, \xi, \eta, \overline{g})$ a Kenmotsu space form of constant ϕ-sectional curvature c. Let $\psi : (M, g) \mapsto (\overline{M}, \overline{g})$ be an isometric immersion.*

There does not exist a 2-dimensional totally geodesic ϕ-invariant submanifold (M, g) of $(\overline{M}, \overline{g})$ with $\xi \in \Gamma(TM^\perp)$ such that either the warping function f is not an eigen-function or the eigenvalue of f is not equal to $m_1 c$.

Corollary 3.11. *Let $(M, g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M}, \overline{g}) = (\overline{M}(c), \phi, \xi, \eta, \overline{g})$ a Kenmotsu space form of constant ϕ-sectional curvature c. Let $\psi : (M, g) \mapsto (\overline{M}, \overline{g})$ be an isometric immersion. Assume that the manifold (M, g) is a totally geodesic ϕ-totally real submanifold of $(\overline{M}, \overline{g})$ with $\xi \in \Gamma(TM^\perp)$ (i.e., $\phi(TM) \subset TM^\perp$).*

Then we have

$$m_1 \frac{c-3}{4} \leq \frac{\triangle f}{f} \leq m_1 \frac{c-3}{4}.$$

Lemma 3.11. *Let $(M,g)=(B\times_f F, g_B+f^2g_F)$ be a warped product manifold and $(\overline{M},\overline{g})=(\overline{M}(c),\phi,\xi,\eta,\overline{g})$ a Kenmotsu space form of constant ϕ-sectional curvature c. Let $\psi:(M,g)\mapsto(\overline{M},\overline{g})$ be an isometric immersion.*

There does not exist a totally geodesic ϕ-totally real submanifold (M,g) of $(\overline{M},\overline{g})$ with $\xi\in\Gamma(TM^{\perp})$ such that either the warping function f is not an eigen-function or the eigenvalue of f is not equal to $\frac{m_1(c-3)}{4}$.

4. **Hermitian symmetric spaces**

Theorem 4.1. *Let $(M,g)=(B\times_f F, g_B+f^2g_F)$ be a warped product manifold and $(\overline{M},\overline{g})=G_2(\mathbb{C}^{m+2})=SU_{m+2}/S(U_mU_2)$ the complex two-plane Grassmannian. Let $\psi:(M,g)\mapsto(\overline{M},\overline{g})$ be an isometric immersion. Then we have*

$$\frac{m_1m^2}{2(m-1)}H^2-\frac{m_1}{2}||h||^2-m_1\leq\frac{\triangle f}{f}\leq\frac{m^2}{4m_2}H^2+8m_1, \tag{28}$$

where $m_1=\dim B$, $m_2=\dim F$ and $m=m_1+m_2$.

Proof. The Riemannian curvature tensor $\overline{R}$ [12] of $(\overline{M},\overline{g})$ is given by

$$\begin{aligned}\overline{R}(X,Y)Z&=\overline{g}(Y,Z)X-\overline{g}(X,Z)Y\\&+\overline{g}(JY,Z)JX-\overline{g}(JX,Z)JY-2\overline{g}(JX,Y)JZ\\&+\sum_{\alpha=1}^{3}(\overline{g}(J_\alpha Y,Z)J_\alpha X-\overline{g}(J_\alpha X,Z)J_\alpha Y-2\overline{g}(J_\alpha X,Y)J_\alpha Z)\\&+\sum_{\alpha=1}^{3}(\overline{g}(J_\alpha JY,Z)J_\alpha JX-\overline{g}(J_\alpha JX,Z)J_\alpha JY)\end{aligned} \tag{29}$$

for $X,Y,Z\in\Gamma(T\overline{M})$. Given orthonormal vectors $X,Y\in T_p\overline{M}$, $p\in\overline{M}$, we get

$$\begin{aligned}\overline{K}(X\wedge Y)&=\overline{R}(X,Y,Y,X)=1+3\overline{g}(JX,Y)^2\\&+\sum_{\alpha=1}^{3}(3\overline{g}(J_\alpha X,Y)^2+\overline{g}(J_\alpha JY,Y)\overline{g}(J_\alpha JX,X)-\overline{g}(J_\alpha JX,Y)^2).\end{aligned} \tag{30}$$

With simple computations, we obtain

$\overline{g}(JX,Y)^2\leq|JX|^2|Y|^2=1,$

$\sum_{\alpha=1}^{3}\overline{g}(J_\alpha X,Y)^2\leq|Y|^2=1$ (since $\{J_1X,J_2X,J_3X\}$ is orthonormal),

$$\begin{aligned}|\sum_{\alpha=1}^{3}\overline{g}(J_\alpha JY,Y)\overline{g}(J_\alpha JX,X)|&\leq\sqrt{\sum_{\alpha=1}^{3}\overline{g}(J_\alpha JY,Y)^2}\cdot\sqrt{\sum_{\alpha=1}^{3}\overline{g}(J_\alpha JX,X)^2}\\&\leq\sqrt{|Y|^2}\sqrt{|X|^2}=1\end{aligned}$$

(by Cauchy-Schwarz inequality and since $\{J_1JY,J_2JY,J_3JY\}$ and $\{J_1JX,J_2JX,J_3JX\}$ are orthonormal)

$\Rightarrow\ -1\leq\sum_{\alpha=1}^{3}\overline{g}(J_\alpha JY,Y)\overline{g}(J_\alpha JX,X)\leq1,$

$\sum_{\alpha=1}^{3}\overline{g}(J_\alpha JX,Y)^2\leq|Y|^2=1$ (since $\{J_1JX,J_2JX,J_3JX\}$ is orthonormal).

By using the above relations, we obtain

$$\overline{K}(X\wedge Y)\leq1+3\cdot1+3\cdot1+1=8. \tag{31}$$

On the other hand, by the above relations, we have

$$\overline{K}(X \wedge Y) \geq 1 + \sum_{\alpha=1}^{3} (\overline{g}(J_\alpha JY, Y)\overline{g}(J_\alpha JX, X) - \overline{g}(J_\alpha JX, Y)^2) \geq 1 - 1 - 1 = -1. \quad (32)$$

From Lemma 2.1, by using (31) and (32), the result follows. □

Remark 4.1. *Choose orthonormal vectors $X, Y \in T_p\overline{M}$, $p \in \overline{M}$ such that $Y = JX$ and X is a singular vector, i.e., conveniently, $JX = J_1X$ (See [1]). From (30), we get*

$$\overline{K}(X \wedge Y) = 1 + 3 + 3 + 1 + 0 = 8.$$

So, the upper bound of the function $\overline{K}(X \wedge Y)$ is rigid.

Corollary 4.1. *Let $(M, g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M}, \overline{g}) = G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_m U_2)$ the complex two-plane Grassmannian. Let $\psi : (M, g) \mapsto (\overline{M}, \overline{g})$ be an isometric immersion. Assume that the manifold (M, g) is a 2-dimensional totally geodesic J-invariant submanifold of $(\overline{M}, \overline{g})$ with a singular vector field $X \in \Gamma(TM)$ (i.e., $J(TM) = TM$).*

Then we get

$$m_1 \cdot 8 \leq \frac{\triangle f}{f} \leq m_1 \cdot 8.$$

Proof. By Lemma 2.1 and (30), we obtain the result. □

Lemma 4.1. *Let $(M, g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M}, \overline{g}) = G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_m U_2)$ the complex two-plane Grassmannian. Let $\psi : (M, g) \mapsto (\overline{M}, \overline{g})$ be an isometric immersion.*

There does not exist a 2-dimensional totally geodesic J-invariant submanifold (M, g) of $(\overline{M}, \overline{g})$ with a singular vector field $X \in \Gamma(TM)$ such that either the warping function f is not an eigen-function or the eigenvalue of f is not equal to $8m_1$.

Theorem 4.2. *Let $(M, g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M}, \overline{g}) = SU_{2,m}/S(U_2 \cdot U_m)$ the complex hyperbolic two-plane Grassmannian. Let $\psi : (M, g) \mapsto (\overline{M}, \overline{g})$ be an isometric immersion. Then we obtain*

$$\frac{m_1 m^2}{2(m-1)} H^2 - \frac{m_1}{2} ||h||^2 - 4m_1 \leq \frac{\triangle f}{f} \leq \frac{m^2}{4m_2} H^2 + \frac{1}{2} m_1, \quad (33)$$

where $m_1 = \dim B$, $m_2 = \dim F$ and $m = m_1 + m_2$.

Proof. We know that the Riemannian curvature tensor $\overline{R}$ [12] of $(\overline{M}, \overline{g})$ is given by

$$\begin{aligned} \overline{R}(X, Y)Z = &-\frac{1}{2}(\overline{g}(Y, Z)X - \overline{g}(X, Z)Y \\ &+ \overline{g}(JY, Z)JX - \overline{g}(JX, Z)JY - 2\overline{g}(JX, Y)JZ \\ &+ \sum_{\alpha=1}^{3} (\overline{g}(J_\alpha Y, Z)J_\alpha X - \overline{g}(J_\alpha X, Z)J_\alpha Y - 2\overline{g}(J_\alpha X, Y)J_\alpha Z) \\ &+ \sum_{\alpha=1}^{3} (\overline{g}(J_\alpha JY, Z)J_\alpha JX - \overline{g}(J_\alpha JX, Z)J_\alpha JY)) \end{aligned} \quad (34)$$

for $X, Y, Z \in \Gamma(T\overline{M})$.

Hence, in a similar way with Theorem 4.1, we easily get the result. □

Remark 4.2. *We choose orthonormal vectors $X, Y \in T_p\overline{M}$, $p \in \overline{M}$, such that $Y = JX$ and X is a singular vector. i.e., conveniently, $JX = J_1X$ (See [2]). In a similar way with Remark 4.1, we obtain*

$$\overline{K}(X \wedge Y) = -4.$$

So, the lower bound of the function $\overline{K}(X \wedge Y)$ is rigid.

Corollary 4.2. *Let $(M, g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M}, \overline{g}) = SU_{2,m}/S(U_2 \cdot U_m)$ the complex hyperbolic two-plane Grassmannian. Let $\psi : (M, g) \mapsto (\overline{M}, \overline{g})$ be an isometric immersion. Assume that the manifold (M, g) is a 2-dimensional totally geodesic J-invariant submanifold of $(\overline{M}, \overline{g})$ with a singular vector field $X \in \Gamma(TM)$.*

Then we get

$$m_1 \cdot (-4) \leq \frac{\triangle f}{f} \leq m_1 \cdot (-4).$$

Lemma 4.2. *Let $(M, g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M}, \overline{g}) = SU_{2,m}/S(U_2 \cdot U_m)$ the complex hyperbolic two-plane Grassmannian. Let $\psi : (M, g) \mapsto (\overline{M}, \overline{g})$ be an isometric immersion.*

There does not exist a 2-dimensional totally geodesic J-invariant submanifold (M, g) of $(\overline{M}, \overline{g})$ with a singular vector field $X \in \Gamma(TM)$ such that either the warping function f is not an eigen-function or the eigenvalue of f is not equal to $-4m_1$.

Theorem 4.3. *Let $(M, g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M}, \overline{g}) = Q^m = SO_{m+2}/SO_m SO_2$ the complex quadric. Let $\psi : (M, g) \mapsto (\overline{M}, \overline{g})$ be an isometric immersion. Then we get*

$$\frac{m_1 m^2}{2(m-1)} H^2 - \frac{m_1}{2} ||h||^2 - 2.3m_1 \leq \frac{\triangle f}{f} \leq \frac{m^2}{4m_2} H^2 + 5m_1, \tag{35}$$

where $m_1 = \dim B$, $m_2 = \dim F$ and $m = m_1 + m_2$.

Proof. We see that the Riemannian curvature tensor $\overline{R}$ [13] of $(\overline{M}, \overline{g})$ is given by

$$\begin{aligned} &\overline{R}(X, Y)Z = \overline{g}(Y, Z)X - \overline{g}(X, Z)Y \\ &+\overline{g}(JY, Z)JX - \overline{g}(JX, Z)JY - 2\overline{g}(JX, Y)JZ \\ &+\overline{g}(AY, Z)AX - \overline{g}(AX, Z)AY + \overline{g}(JAY, Z)JAX - \overline{g}(JAX, Z)JAY \end{aligned} \tag{36}$$

for $X, Y, Z \in \Gamma(T\overline{M})$. Given orthonormal vectors $X, Y \in T_p\overline{M}$, $p \in \overline{M}$, we obtain

$$\begin{aligned} &\overline{K}(X \wedge Y) = \overline{R}(X, Y, Y, X) = 1 + 3\overline{g}(JX, Y)^2 \\ &+\overline{g}(AY, Y)\overline{g}(AX, X) - \overline{g}(AX, Y)^2 + \overline{g}(JAY, Y)\overline{g}(JAX, X) - \overline{g}(JAX, Y)^2. \end{aligned} \tag{37}$$

Since A is an involution (i.e., $A^2 = id$), we get the following decompositions

$$\begin{aligned} X &= a\overline{X}_1 + b\overline{X}_2 \\ Y &= c\overline{Y}_1 + d\overline{Y}_2, \end{aligned}$$

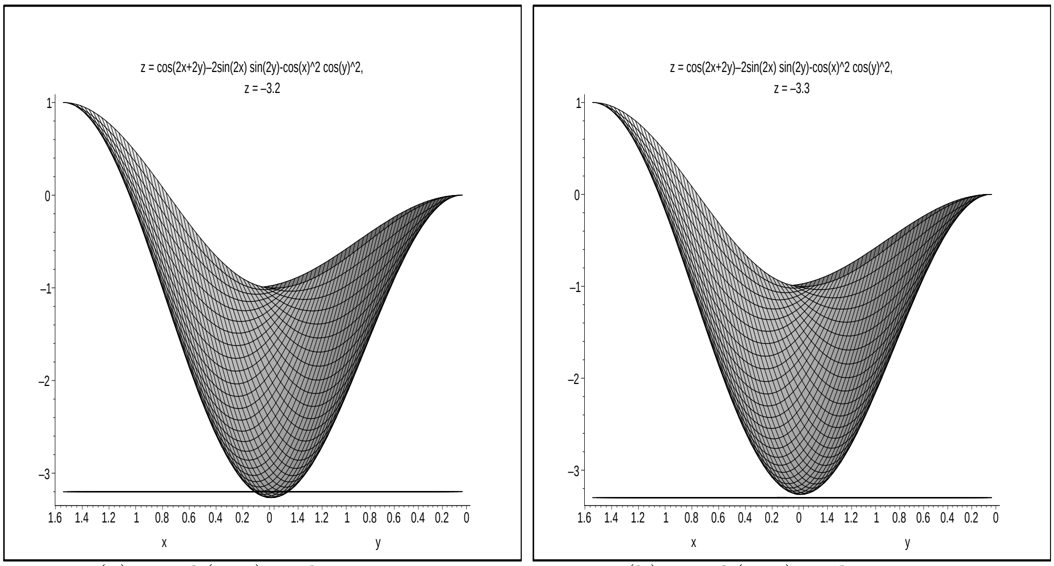
where $|\overline{X}_1| = |\overline{X}_2| = |\overline{Y}_1| = |\overline{Y}_2| = 1$, $\overline{X}_1, \overline{Y}_1 \in V(A) = \{Z \in T_p\overline{M} \mid AZ = Z\}$, $\overline{X}_2, \overline{Y}_2 \in JV(A)$ (See [13]) so that

$$\begin{aligned} 1 &= |X|^2 = a^2 + b^2, \\ 1 &= |Y|^2 = c^2 + d^2, \\ 0 &= \overline{g}(X, Y) = ac\overline{g}(\overline{X}_1, \overline{Y}_1) + bd\overline{g}(\overline{X}_2, \overline{Y}_2). \end{aligned}$$

Conveniently, let $(a, b) = (\cos\alpha, \sin\alpha)$ and $(c, d) = (\cos\beta, \sin\beta)$.

If necessary, by replacing $\overline{X}_1, \overline{X}_2, \overline{Y}_1, \overline{Y}_2$ with $-\overline{X}_1, -\overline{X}_2, -\overline{Y}_1, -\overline{Y}_2$, respectively, we may assume

$$0 \leq \alpha, \beta \leq \frac{\pi}{2}. \tag{38}$$



(a) $z = h(x, y)$ and $z = -3.2$ (b) $z = h(x, y)$ and $z = -3.3$

FIGURE 1. The lower bound of $h(x, y)$

Thus, with a simple calculation, we have

$$\overline{K}(X \wedge Y) = 1 + 2\overline{a}^2 \cos^2 \alpha \sin^2 \beta + 2\overline{b}^2 \sin^2 \alpha \cos^2 \beta + \cos 2\alpha \cos 2\beta \\ +2\overline{a}\overline{b} \sin 2\alpha \sin 2\beta + \overline{c}\overline{d} \sin 2\alpha \sin 2\beta - \overline{e}^2 \cos^2 \alpha \cos^2 \beta, \tag{39}$$

where

$$\begin{aligned} \overline{a} &= \overline{g}(\overline{X}_1, J\overline{Y}_2) \\ \overline{b} &= \overline{g}(\overline{X}_2, J\overline{Y}_1) \\ \overline{c} &= \overline{g}(J\overline{Y}_1, \overline{Y}_2) \\ \overline{d} &= \overline{g}(J\overline{X}_1, \overline{X}_2) \\ \overline{e} &= \overline{g}(\overline{X}_1, \overline{Y}_1). \end{aligned}$$

We see that

$$-1 \leq \overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e} \leq 1. \tag{40}$$

Consider the function

$$S(x, y) = 2\overline{a}^2 \cos^2 x \sin^2 y + 2\overline{b}^2 \sin^2 x \cos^2 y + \cos 2x \cos 2y \\ +2\overline{a}\overline{b} \sin 2x \sin 2y + \overline{c}\overline{d} \sin 2x \sin 2y - \overline{e}^2 \cos^2 x \cos^2 y \tag{41}$$

for $(x, y) \in [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$.

Since $\sin 2x \sin 2y \geq 0$, by (40), we obtain

$$\begin{aligned} S(x, y) &\leq 2 \cos^2 x \sin^2 y + 2 \sin^2 x \cos^2 y \\ &+ \cos 2x \cos 2y + 2 \sin 2x \sin 2y + \sin 2x \sin 2y \\ &= 2(\cos x \sin y + \sin x \cos y)^2 + \cos(2x - 2y) + \sin 2x \sin 2y \\ &= 2 \sin^2(x + y) + \cos(2x - 2y) + \sin 2x \sin 2y \\ &\leq 4 \end{aligned} \tag{42}$$

and

$$S(x,y) \geq \cos 2x \cos 2y - 2\sin 2x \sin 2y \qquad (43)$$
$$- \sin 2x \sin 2y - \cos^2 x \cos^2 y$$
$$= \cos(2x+2y) - 2\sin 2x \sin 2y - \cos^2 x \cos^2 y.$$

Consider the function $h(x,y) = \cos(2x+2y) - 2\sin 2x \sin 2y - \cos^2 x \cos^2 y$ for $(x,y) \in [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$.

We see that (See Figure 1)

$$h(x,y) \geq -3.3\ . \qquad (44)$$

From Lemma 2.1, by using (39), (41), (42), (43), and (44), the result follows. □

Remark 4.3. *We get* $h(\frac{\pi}{4}, \frac{\pi}{4}) = -3.25$. *But* $h_x(\frac{\pi}{4}, \frac{\pi}{4}) = \frac{1}{2} \neq 0$ *and* $h_y(\frac{\pi}{4}, \frac{\pi}{4}) = \frac{1}{2} \neq 0$, *which implies that* $(\frac{\pi}{4}, \frac{\pi}{4})$ *is not a critical point of* $h(x,y)$.

Corollary 4.3. *Let* $(M,g) = (B \times_f F, g_B + f^2 g_F)$ *be a warped product manifold and* $(\overline{M}, \overline{g}) = Q^m = SO_{m+2}/SO_m SO_2$ *the complex quadric. Let* $\psi : (M,g) \mapsto (\overline{M}, \overline{g})$ *be an isometric immersion. Assume that the manifold* (M,g) *is a 2-dimensional totally geodesic* J*-invariant submanifold of* $(\overline{M}, \overline{g})$ *with a non-vanishing vector field* $X \in \Gamma(TM) \cap V(A)$.

Then we get

$$m_1 \cdot 2 \leq \frac{\triangle f}{f} \leq m_1 \cdot 2.$$

Proof. By Lemma 2.1 and (37), we obtain the result. □

Lemma 4.3. *Let* $(M,g) = (B \times_f F, g_B + f^2 g_F)$ *be a warped product manifold and* $(\overline{M}, \overline{g}) = Q^m = SO_{m+2}/SO_m SO_2$ *the complex quadric. Let* $\psi : (M,g) \mapsto (\overline{M}, \overline{g})$ *be an isometric immersion.*

There does not exist a 2-dimensional totally geodesic J*-invariant submanifold* (M,g) *of* $(\overline{M}, \overline{g})$ *with a non-vanishing vector field* $X \in \Gamma(TM) \cap V(A)$ *such that either the warping function* f *is not an eigen-function or the eigenvalue of* f *is not equal to* $2m_1$.

Corollary 4.4. *Let* $(M,g) = (B \times_f F, g_B + f^2 g_F)$ *be a warped product manifold and* $(\overline{M}, \overline{g}) = Q^m = SO_{m+2}/SO_m SO_2$ *the complex quadric. Let* $\psi : (M,g) \mapsto (\overline{M}, \overline{g})$ *be an isometric immersion. Assume that the manifold* (M,g) *is a 2-dimensional totally geodesic submanifold of* $(\overline{M}, \overline{g})$ *with* $TM \subset V(A)$.

Then we get

$$m_1 \cdot 2 \leq \frac{\triangle f}{f} \leq m_1 \cdot 2.$$

Lemma 4.4. *Let* $(M,g) = (B \times_f F, g_B + f^2 g_F)$ *be a warped product manifold and* $(\overline{M}, \overline{g}) = Q^m = SO_{m+2}/SO_m SO_2$ *the complex quadric. Let* $\psi : (M,g) \mapsto (\overline{M}, \overline{g})$ *be an isometric immersion.*

There does not exist a 2-dimensional totally geodesic submanifold (M,g) *of* $(\overline{M}, \overline{g})$ *with* $TM \subset V(A)$ *such that either the warping function* f *is not an eigen-function or the eigenvalue of* f *is not equal to* $2m_1$.

Corollary 4.5. *Let* $(M,g) = (B \times_f F, g_B + f^2 g_F)$ *be a warped product manifold and* $(\overline{M}, \overline{g}) = Q^m = SO_{m+2}/SO_m SO_2$ *the complex quadric. Let* $\psi : (M,g) \mapsto (\overline{M}, \overline{g})$ *be an isometric immersion. Assume that the manifold* (M,g) *is a 2-dimensional totally geodesic submanifold of* $(\overline{M}, \overline{g})$ *with* $TM \perp J(TM)$ *and* $\dim(TM \cap V(A)) = \dim(TM \cap JV(A)) = 1$.

Then we get

$$m_1 \cdot 0 \leq \frac{\triangle f}{f} \leq m_1 \cdot 0.$$

Lemma 4.5. *Let $(M, g) = (B \times_f F, g_B + f^2 g_F)$ be a warped product manifold and $(\overline{M}, \overline{g}) = Q^m = SO_{m+2}/SO_m SO_2$ the complex quadric. Let $\psi : (M, g) \mapsto (\overline{M}, \overline{g})$ be an isometric immersion.*

There does not exist a 2-dimensional totally geodesic submanifold (M, g) of $(\overline{M}, \overline{g})$ with $TM \perp J(TM)$ and $\dim(TM \cap V(A)) = \dim(TM \cap JV(A)) = 1$ such that the warping function f is not a harmonic function.

5. **Open questions**

In section 3 and section 4, we deal with estimates of the functions $\frac{\triangle f}{f}$ for isometric immersions $\psi : (M, g) = (B \times_f F, g_B + f^2 g_F) \mapsto (\overline{M}, \overline{g})$. And we also consider equality cases and their applications. As future projects, we can use these results to study the properties of base manifolds and target manifolds and investigate other equality cases and their applications. We will also estimate the functions $\frac{\triangle f}{f}$ by changing target manifolds.

Questions

1. What kind of eigenvalues of the warping functions f can we get?
(We obtained the following eigenvalues:
$m_1 c, \frac{m_1 c}{4}, \frac{m_1(c+3)}{4}, m_1, \frac{m_1(c-3)}{4}, 8m_1, -4m_1, 2m_1, 0.$)
2. If the warping function f is an eigen-function with eigenvalue d, then what can we say about M and $\overline{M}$?

REFERENCES

[1] J. Berndt, Y. J. Suh, *Real hypersurfaces in complex two-plane Grassmannians* , Monatsh. Math. (1999), **127**, 1-14.

[2] J. Berndt, Y. J. Suh, *Hypersurfaces in noncompact complex Grassmannians of rank two* , Internat. J. Math. (2012), **23**, 1250103(35 pages).

[3] B. Y. Chen, *On isometric minimal immersions from warped products into real space forms*, Proc. Edinburgh Math. Soc., (2002), **45**, No. 3, 579-587.

[4] B. Y. Chen, *Warped products in real space forms*, Rocky Mountain J. Math., (2004), **34**, No. 2, 551-563.

[5] B. Y. Chen, *Geometry of warped product and CR-warped product submanifolds in Kähler manifolds: modified version*, arXiv:1806.11102 [math.DG].

[6] S. Ishihara, *Quaternion Kählerian manifolds*, J. Diff. Geometry, (1974), **9**, 483-500.

[7] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Mathematical Journal (1972), **24**, No. 1, 93-103.

[8] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, John Wiley & Sons, New York, 1996.

[9] J. F. Nash, *The imbedding problem for Riemannian manifolds*, Ann. of Math., (1956), **63**, 20-63.

[10] K. Ogiue, *On almost contact manifolds admitting axiom of planes or axiom of free mobility*, Kodai Math. Sem. Rep., (1964), **16**, No. 4, 223-232.

[11] K. S. Park, *Warped products in Riemannian manifolds*, Bull. Aust. Math. Soc., (2014), **90**, 510-520.

[12] K. S. Park, *Inequalities for the Casorati curvatures of real hypersurfaces in some Grassmannians*, Taiwan. J. Math., (2018), **22**, No. 1, 63-77.

[13] Y. J. Suh, *Real hypersurfaces in the complex quadric with parallel Ricci tensor*, Advances in Math., (2015), **281**, 886-905.