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HOMOTOPY PERTURBATION METHOD FOR SOLVING GOVERNING EQUATION OF NONLINEAR FREE VIBRATION OF SYSTEMS WITH SERIAL LINEAR AND NONLINEAR STIFFNESS ON A FRICTIONLESS CONTACT SURFACE

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In this paper we take care of free vibration of a nonlinear system having combined linear and nonlinear springs in series. The conservative oscillation system is formulated as a nonlinear ordinary differential equation having linear and nonlinear stiffness components. Homotopy perturbation method (HPM) is used for solving governing equation of nonlinear free vibration of systems with serial linear and nonlinear stiffness. HPM deforms a difficult problem into a simple problem which can be easily solved. Result of this approach will be compared with numerical solution to see the validity and precision of the approach.

Keywords: nonlinear free vibration, Homotopy perturbation method

1. Introduction

Many physical phenomena are modeled by nonlinear deferential equations in order to have more opportunities to handle the real items in our real world. As an instance, vibration of mechanical systems associated with nonlinear properties is in this type. So, scientists tried to solve the problem and find some solutions which at last a number of approaches for solving nonlinear equations is emerged for the range of completely analytical to completely numerical ones. There have been many approaches for solving nonlinear oscillation systems, as well as the KBM method [1-4], the multiple scales method [1,2,3,5], which are applicable to nonlinear oscillation systems even for quite large amplitudes of oscillation. Recently, the weighted linearization method [6], the modified Lindstedt-Poincare method [7] and power-series approach [8] were proposed to solve inexact periods with large amplitude of oscillation.

Telli and Kompaz [9] tried to solve the motion of a mechanical system coupled with linear and nonlinear properties using analytical and numerical

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techniques. It dealt with vibration of a Conservative oscillation system with attached mass grounded by linear and nonlinear springs. The relation of the linear and nonlinear springs in series has been derived with cubic nonlinear characteristics in the equations of motion [9].

Juan Pena Miralles and others [10] had a study on the determination of periodic solutions of nonlinear oscillators in addition to on the qualitative of analysis of their stability. They improved an algorithm, based on the Galerkin method, using the fast fourier transform (FTT), to calculate the periodic solutions of governing equation.

Wu, Sun and Lim [11] applied an analytical approximation technique for large amplitude oscillators of a category of conservative single degree-of-freedom systems with unusual nonlinearities. The method incorporates with main features of both Newton's method and the harmonic balance method.

Recently Farzaneh and Tootoonchi [12] developed a modified variational approach called Global Error Minimization (GEM) method for obtaining an approximate closed-form analytical solution for nonlinear oscillator differential equations. This method converts the nonlinear differential equation to an equivalent minimization problem.

In order to achieve correct approximate analytical solution for the system with combined linear and nonlinear stiffness, this paper presents Homotopy perturbation method. HPM is the most useful ones for solving nonlinear equation. Developing the perturbation method for different usage is very difficult because this method has some restrictions and based on the existence of a small parameter. Therefore, many different new methods have recently been introduced to eliminate the small parameters. One of the semi-exact methods is HPM. The HPM is one of the familiar methods to solve nonlinear equations that are established in 1999 by He [13]. The references therein to handle a wide variety of scientific and engineering applications: linear and nonlinear, and homogeneous and inhomogeneous as well. It was shown by many authors that this method provides improvements over existing numerical methods [14].

2. Governing equation of motion

Take into account free vibration of a conservative, single-degree-of-freedom system with a mass attached to linear and nonlinear springs in series. This is shown in Fig.1. The motion is governed by a nonlinear differential equation [9] as

$$(1+3\varepsilon z v^2)\frac{d^2 v}{dt^2}+6\varepsilon z v\left(\frac{dv}{dt}\right)p^2 + {w_e}^2 v^3 = 0 \tag{1}$$

where

$$\varepsilon = \frac{\beta}{k_2} \tag{2}$$

$$\xi = \frac{k_2}{k_1} \tag{3}$$

$$z = \frac{\xi}{\xi + 1} \tag{4}$$

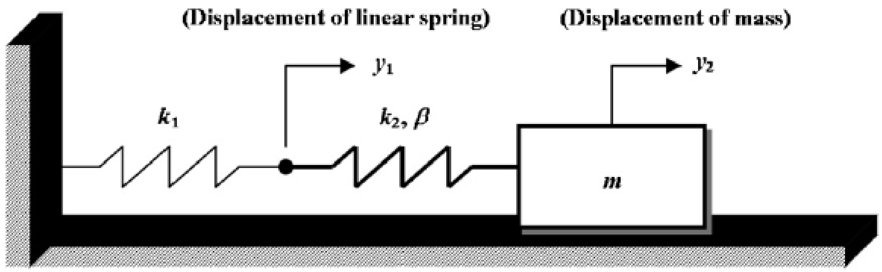

Fig.1. Nonlinear free vibration of a system of mass with serial linear and nonlinear stiffness on a frictionless contact surface [9].

$$w_e = \sqrt{\frac{k_2}{m(1+\xi)}} \tag{5}$$

with the initial conditions

$$v(0) = A, \quad \frac{dv}{dt}(0) = 0 \tag{6}$$

In which ε, β, v, w_e, m and ξ are perturbation parameter (not restricted to a small parameter), coefficient of nonlinear spring force, deflection of nonlinear spring, natural frequency, mass and the ratio of linear portion k_2 of the nonlinear spring constant to that of linear spring constant k_1, respectively.

The deflection of linear spring $y_1(t)$ and the displacement of attached mass $y_2(t)$ can be stand for by the deflection of nonlinear spring v in simple relationships [9] as

$$y_1(t) = \xi v(t) + \varepsilon \xi [v(t)]^3 \tag{7}$$

and

$$y_2(t) = v(t) + y_1(t) \tag{8}$$

Introducing a new independent temporary variable, $\tau = wt$, Eqs (1) and (6) become

$$w^2\left[(1+3\varepsilon z\nu^2)\ddot{\nu}+6\varepsilon z\nu\dot{\nu}^2\right]+w_e{}^2\nu+\varepsilon w_e{}^2\nu^3=0 \tag{9}$$

and

$$\nu(0)=0,\dot{\nu}(0)=0 \tag{10}$$

where a dot denotes differentiation with respect to τ. The deflection of nonlinear spring ν is a periodic function of τ of period 2π.

3. Homotopy perturbation method

The homotopy perturbation method is a combination of the classical perturbation technique and homotopy technique. To explain the basic idea of the HPM for solving nonlinear differential equations we consider the following nonlinear differential equation:

$$A(u)-f(r)=0, r\in\Omega \tag{11}$$

Subject to boundary condition

$$B(u,\frac{\partial u}{\partial n})=0, r\in\Gamma \tag{12}$$

where A is a general differential operator, B a boundary operator, f_r a known analytical function, Γ is the boundary of domain Ω and $\frac{\partial u}{\partial n}$ denotes differential along the normal drawn outwards from Ω. The operator A can, generally speaking, be divided into two parts: a linear part L and a nonlinear part N. Therefore Eq. (11) can be rewritten as follows:

$$L(u)+N(u)-f(r)=0 \tag{13}$$

In case that the nonlinear Eq. (11) has no “small parameter”, we can construct the following homotopy:

$$\begin{aligned} H(\nu,p)&=L(\nu)-L(u_0)\\ &+pL(u_0)+p(N(\nu)-f(r))=0 \end{aligned} \tag{14}$$

where

$$\nu(r,p):\Omega\times[0,1]\to R \tag{15}$$

In Eq. (7), $p\in[0,1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary conditions. We can assume that the solution of Eq. (4) can be written as a power series in p, as following:

$$\nu = \nu_0 + p\nu_1 + p^2\nu_2 + \cdots \tag{16}$$

And the best approximation for solution is:

$$u = \lim_{p\to 1} \nu = \nu_0 + \nu_1 + \nu_2 + \ldots \tag{17}$$

When, Eq. (4) correspond to Eq. (1) and Eq. (7) becomes the approximate solution of Eq.(1). Some fascinating results have been achieved using this method. Convergence and stability of this method is shown in reference [15]. The combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated the limitations of the traditional perturbation methods. In contrast, this technique can have full benefit of the usual perturbation techniques. The series (17) is convergent for most cases.

However the convergent rate date on nonlinear operator $A(u)$, the following opinions are recommended by He [16]:

(1) The second derivative of $N(u)$ with respect to u must be small because the parameter may be relatively large, i.e., $p \to 1$

(2) The norm of $L^{-1}\dfrac{\partial N}{\partial V}$ must be smaller than one so that the series converges.

4. Method of solution

In this study, the governing equation is in the form of:

$$(1+3\varepsilon z\nu^2)\frac{d^2\nu}{dt^2} + 6\varepsilon z\nu\left(\frac{d\nu}{dt}\right)^2 + w_e{}^2\nu + \varepsilon w_e{}^2\nu^3 = 0 \tag{18}$$

With the initial condition of

$$\nu(0) = A, \frac{d\nu}{dt}(0) = 0 \tag{19}$$

Substituting Eq. (6) into Eq. (4) and rearranging based on powers of p-terms, we have:

$$p^0 = w^2\left[\frac{d^2}{d\tau^2}\nu_0(\tau)\right] + w_e{}^2\nu_0(\tau) = 0 \tag{20}$$

The initial condition is defined as:

$$\nu_0(0) = 0.5, \dot{\nu}_0(0) = 0 \tag{21}$$

In the same way we have:

$$p^1: 3w^2 \varepsilon z v_0{}^2(\tau)\left[\frac{d^2}{d\tau^2}v_0(\tau)\right] + w_e{}^2 v_0{}^2(\tau) + 6w^2 \varepsilon z v_0(\tau)\left[\frac{d}{d\tau}v_0(\tau)\right]^2$$
$$+\varepsilon w_e{}^2 v_0{}^3(\tau) + w^2\left[\frac{d^2}{d\tau^2}v_1(\tau)\right] = 0 \quad (22)$$

The initial condition is defined as

$$v_1(0) = 0, \dot{v}_1(0) = 0 \quad (23)$$

Solving the above equations (Eqs. (10) - (17)) and when $p \to 1$, the result may be written in the type:

$$v(\tau) = \frac{1}{2}\cos\left(\frac{w_e \tau}{w}\right) + \frac{w_e\left[(9zw - w)\cos\left(\frac{w_e\tau}{w}\right) + (-9zw + w)\cos\left(\frac{3w_e\tau}{w}\right) + 12\sin\left(\frac{w_e\tau}{w}\right)(zw_e\tau - w_e\tau)\right]}{256w} \quad (24)$$

5. Results and discussion

We demonstrate accuracy and efficiency of HPM by applying the method to governing equation and comparing the HPM solution with the numerical solution. The parameter ε is linearly dependant on the coefficient of nonlinear force β as given in Eq. (2). The latter can be positive or negative depending on whether the nonlinear spring has hard or soft-spring properties. In Fig.2 we have the diagram of the deflection of nonlinear spring for small parameters and then in Fig.3 it is compared with the numerical solution. As we see in this diagram, HPM solution has a good precision for small parameters. In table 1, we extracted the error of HPM method in comparison with numerical solution for small parameters. In accordance with the HPM results in Table 1, one can see that the approximate solutions by HPM are quite closed to their numerical solution for small parameters.

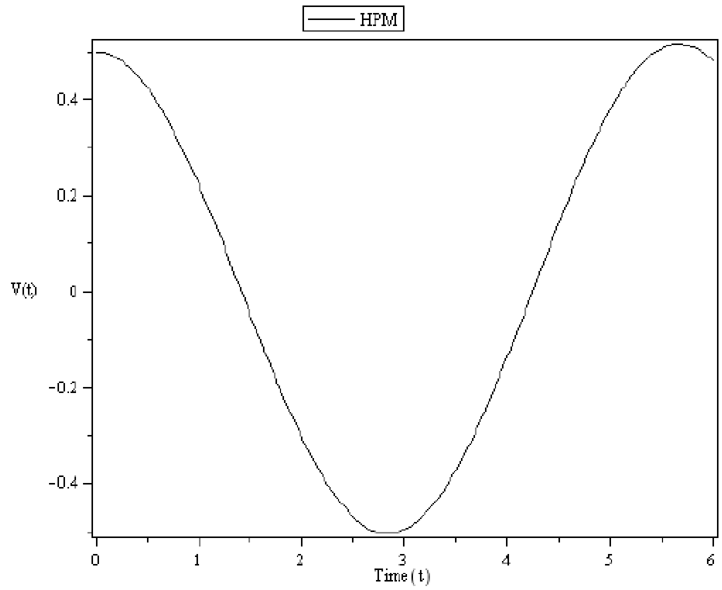

Fig. 2. Deflection of nonlinear spring $v(t)$ for $m=1$, $A=0.5$, $\varepsilon=0.5$ and $\xi=0.1$ ($k_1=50,\ k_2=5$) with HPM solution

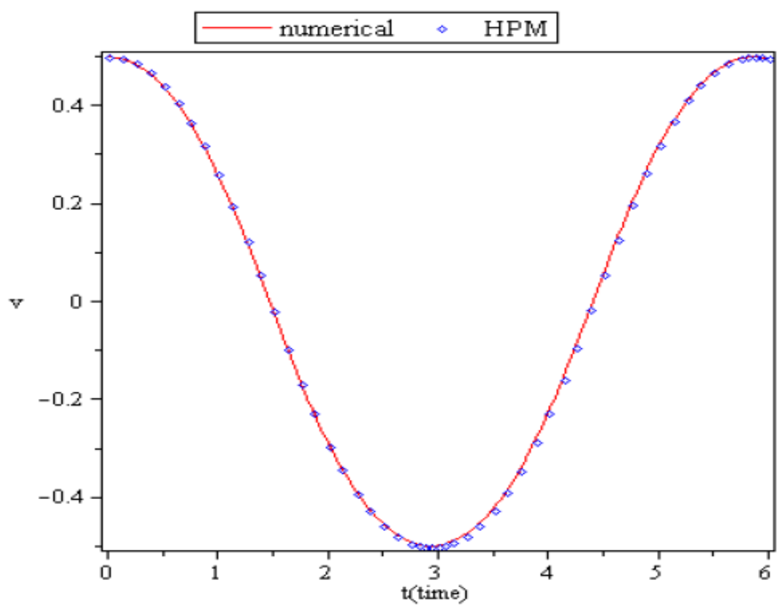

Fig. 3. Comparison of the deflection of nonlinear spring $v(t)$ for $m=1$, $A=0.5$, $\varepsilon=0.5$ and $\xi=0.1$ ($k_1=50,\ k_2=5$) for HPM and numerical solution

Table1

Error percentage of HPM method for $m=1$, $A=0.5$, $\varepsilon=0.5$ **and** $\xi=0.1$ **(**$k_1=50$, $k_2=5$**)**

time	VHPM	$V_{numerical}$	(VHPM - Vnumerical)/ Vnumerical
0	0.5	0.5	0
0.5	0.4245562811	0.4247466930	0.0004482950
1	0.2214058720	0.2219909820	0.0026357380
1.5	-0.0481198533	-0.0470439154	0.0228709260
2	-0.3040235756	-0.3020612840	0.0064963360
2.5	-0.4700676214	-0.4667343800	0.0071416240
3	-0.4953572825	-0.4910917550	0.0086858060
3.5	-0.3710749256	-0.3676750550	0.0092469440
4	-0.1343758157	-0.1340900150	0.0021314090
4.5	0.1439903975	0.1394766430	0.0323620820
5	0.3815925281	0.3714530130	0.0272968980
5.5	0.5072529006	0.4921163700	0.0307580310
6	0.4814562646	0.4646914490	0.0360773060

For showing the accuracy and validity of HPM method, we did it for different values of parameters in Fig.4, Fig.5 and Fig.6. The parameters m and A increased step by step until they get the values $m=4$ and $A=10$ in Fig.6. As we see in these figures, the said method still has a good accuracy.

To extend applicability and to show flexibility and accuracy of this method for large parameters, an example for $m=8$, $A=200$, $\varepsilon=2$, $\xi=20$, $k_1=5$ and $k_2=100$ is offered in Fig. 7. In table 2 deflection of nonlinear spring for HPM and numerical solution is compared. As we see, this method is applicable for such large parameters and the error percentage is still small.

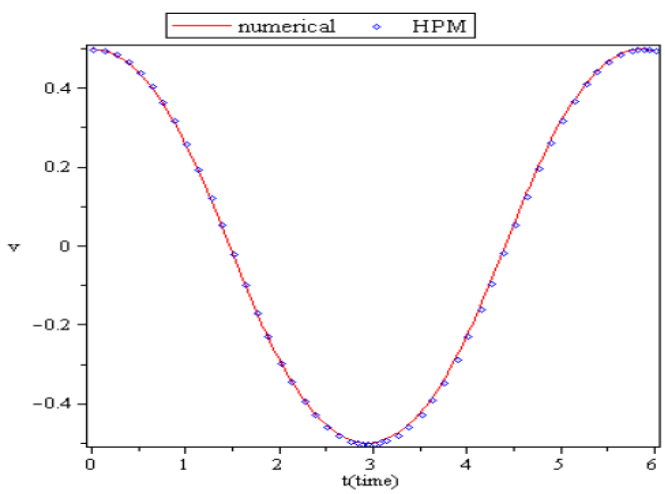

Fig.4. Comparison of the deflection of nonlinear spring $v(t)$ for $m=1$, $A=2$, $\varepsilon=0.5$ and $\xi=10$ ($k_1=5, k_2=50$) for HPM and numerical solution.

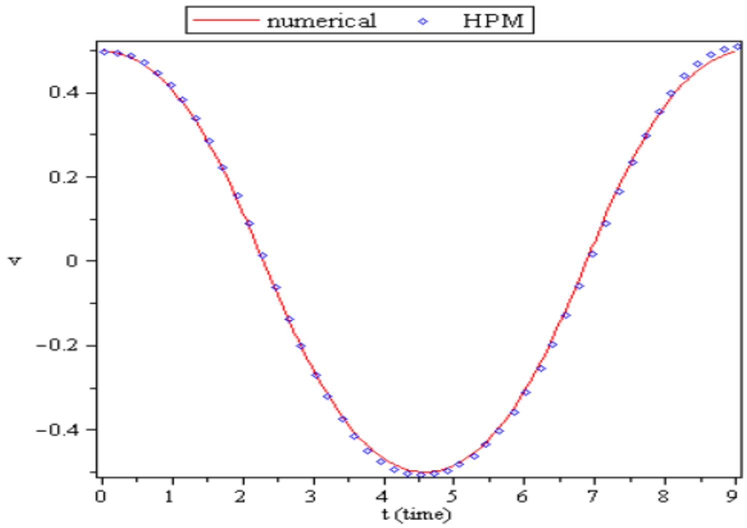

Fig.5. Comparison of the deflection of nonlinear spring $v(t)$ for $m=3$, $A=5$, $\varepsilon=1$ and $\xi=2$ ($k_1=8, k_2=16$) for HPM and numerical solution

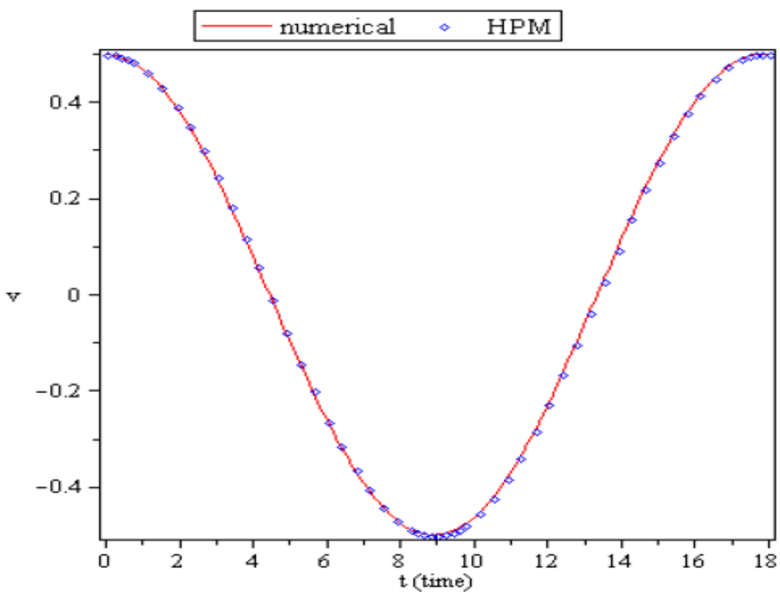

Fig.6. Comparison of the deflection of nonlinear spring $v(t)$ for $m=4$, $A=10$, $\varepsilon=0.008$ and $\xi=0.5$ ($k_1=6, k_2=3$) for HPM and numerical solution.

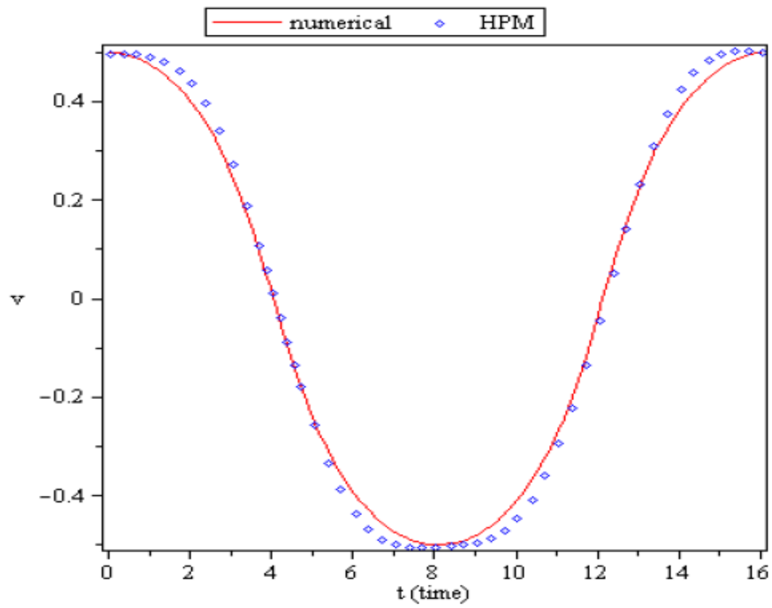

Fig.7. Comparison of the deflection of nonlinear spring $v(t)$ for $m = 8$, $A = 200$, $\varepsilon = 2$ and $\xi = 20$ ($k_1 = 5, k_2 = 100$) for HPM and numerical solution.

Table 2

Error percentage of HPM method for
$m = 8$, $A = 200$, $\varepsilon = 2$ and $\xi = 20$ ($k_1 = 5$, $k_2 = 100$)

time	VHPM	Vnumerical	(VHPM - Vnumerical)/ Vnumerical
0	0.5	0.5	0
2	0.4384794342	0.4111818671	0.0663880620
4	0.0135665712	0.0117159289	0.1579594940
6	-0.4328875008	-0.4042821428	0.0707559270
8	-0.5007219584	-0.4998983574	0.0016475370
10	-0.4428290759	-0.4177663123	0.0599923040
12	-0.0405791079	-0.0350321883	0.1583377980
14	0.4258454412	0.3970534745	0.0725140780
16	0.5028630640	0.4995916056	0.0065482650

For all cases illustrated in the figures, only one period of oscillation are presented. This is because only conservative, nonlinear free vibration of mass-spring system is considered.

6. Conclusions

In this article homotopy perturbation method has been successfully applied to find the solution of governing equation of nonlinear free vibration of systems with serial linear and nonlinear stiffness. Homotopy perturbation method is useful to nonlinear oscillators which are practical in so many branches of sciences such as: electromagnetic and waves, telecommunication, civil and its structures and all supposed majors' application, etc. we verified the accuracy and efficiency of presenting method with some strong nonlinear problems. We can advise HPM method as strongly nonlinear method for oscillation systems which provide easy and direct process for determining approximations to the periodic solutions. But there are limitations in applying the proposed method. It could not be applied for coupled and higher degree of freedom systems. One another problem is that if the governing equation of motion contains a damping term, HPM could not be applied in that form and so, the researchers restricted to non periodic forms of HPM (see [15-27]). In this paper the approximate solution obtained by the HPM method are compared with numerical solution. One can see that the approximate solutions which we obtained have three terms, and they are already quite accurate. The reliability of HPM gives it a wider applicability, especially in engineering. In our work we used MAPLE 12 package to carry the computations.

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