STANLEY DEPTH OF POWERS OF THE PATH IDEAL

Alin Ştefan[1]

The aim of this paper is to give a formula for the Stanley depth of quotients of powers of the path ideal. As a consequence, we establish that the behavior of the Stanley depth of the quotients of powers of the path ideal is the same as a classical result of Brodmann on depth.

Keywords: Characteristic poset, Monomial Ideal, Stanley depth, Stanley decomposition.

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1. Introduction

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K and M be a finitely generated $\mathbb{Z}^n$-graded S-module. Let $u \in M$ be homogeneous element and $Z \subset X = \{x_1, \dots, x_n\}$. Then the $K[Z]$-submodule $uK[Z]$ of M is called a *Stanley space* of M if $uK[Z]$ is a free $K[Z]$-submodule of M and $|Z|$ is called the *dimension* of $uK[Z]$, where $|Z|$ is the cardinality of Z. A *Stanley decomposition* $\mathcal{D}$ of M is a decomposition of M as a direct sum of $\mathbb{Z}^n$-graded K-vector space

$$\mathcal{D} : M = \bigoplus_{j=1}^{r} u_j K[Z_j],$$

where each $u_j K[Z_j]$ is a Stanley space of M.
The number

$$\text{sdepth}\,(D) = min\{|Z_i| : i = 1, \dots, r\}$$

is called the *Stanley depth of decomposition* $\mathcal{D}$ and the number

$$\text{sdepth}(M) := max\{\text{sdepth}(\mathcal{D}) :\ \mathcal{D} \textit{ is a Stanley decomposition of } M\}$$

is called *Stanley depth* of M. In 1982 Stanley conjectured in [19] that

$$\text{sdepth}(M) \geq \text{depth}(M)$$

for all $\mathbb{Z}^n$-graded S-module M. Apel [1], [2] proved the conjecture for a monomial ideal I over S and for the quotient S/I in at most three variables. Anwar and Popescu [3] and Popescu [14] proved the conjecture for S/I and $n = 4, 5$;

[1]Lecturer, Department of Computer Science, Information Technology, Mathematics and Physics, Petroleum and Gas University of Ploieşti, B-dul. Bucureşti, No. 39, Ploieşti 100680, Romania, nastefan@upg-ploiesti.ro, alin11_alg@yahoo.com

also for $n = 5$ Popescu proved the conjecture for square free monomial ideals. Herzog, Vlădoiu and Zheng [11] introduced a method to compute the Stanley depth of a factor of two monomial ideals which was later developed into an effective algorithm by Rinaldo [18] implemented in *CoCoA* [8]. Duval, Goeckner, Klivans and Martin proved that the conjecture is false [9]. They construct a non-partitionable Cohen-Macaulay simplicial complex and using a result of Herzog, Soleyman Jahan and Yassemi [12] deduce that the Stanley Reisner ring of this simplicial complex does not satisfy Stanley conjecture. The counterexample given in [9] is a quotient of squarefree monomial ideal. Thus, one can still ask whether Stanley conjecture holds for non-squarefree monomial ideals; in particular for high powers of monomial ideals. However, it is difficult to compute this invariant, even in some very particular cases. For instance in [5] Birό et al. proved that $\operatorname{sdepth}(\mathbf{m}) = \lceil \frac{n}{2} \rceil$ where $\mathbf{m} = (x_1, \ldots, x_n)$ is the graded maximal ideal of S and where for $x \in \mathbb{R}$, $\lceil x \rceil$ denote the smallest integer $\geq x$. For a friendly introduction on Stanley depth we refer the reader to [16] and, for a nice survey, to [10].

The aim of the paper is to study the Stanley depth of S/I^t, where $t \geq 1$ and $I = I(P_n)$ is the edge ideal of the path graph of lenght $n-1$; see Definition 2.1. In general, if I is a squarefree monomial ideal, based on the behavior of the *limit depth* of I, Herzog [10, Conjecture 2.7] conjectured that the Stanley depth of S/I^k is constant for large k. This is clear if $\mathbf{m} = (x_1, \ldots, x_n)$ is the graded maximal ideal of S, since $S/\mathbf{m}^k$ is an artinian ring and thus we have $\operatorname{sdepth}(S/\mathbf{m}^k) = 0$, for every integer $k \geq 1$. In 2018 Cimpoeaş [6] proved that if I is a complete intersection monomial ideal which is minimally generated by t monomials we have

$$\operatorname{sdepth}(S/I^k) = \operatorname{sdepth}(I^k/I^{k+1}) = \dim(S/I) = n - t$$

for any integer $k \geq 1$. Our main result is Theorem 2.2, where we proved that $\operatorname{sdepth}(S/I^t) = max\{\lceil \frac{n-t+1}{3} \rceil, 1\}$. Moreover, $\operatorname{sdepth}(S/I^t)$ stabilizes for $t >> 0$. So, we obtain a similar result to [4] Brodmann' theorem on the Stanley depth.

2. **The Stanley depth of the path ideal**

Let $G = (V, E)$ be a simple graph on the vertex set $V = \{x_1, \ldots, x_n\}$ and the edge set E. The *edge ideal* $I = I(G)$ of the graph G is the ideal generated by all monomials of the form $x_i x_j$ such that $\{x_i, x_j\}$ is an edge of G.

Definition 2.1. *Suppose* $n \geq 2$. *A path* P_n *of lenght* $n-1$ *is the graph on the vertex set* $V = \{x_1, \ldots, x_n\}$ *and with* $n-1$ *edges* $e_i = \{x_i, x_{i+1}\}$ *for* $1 \leq i \leq n-1$. *The edge ideal of* P_n *is* $I = I(P_n) = (x_1x_2, x_2x_3, \ldots, x_{n-1}x_n) \subset S$.

For $I = I(P_n)$, Morey [13] proved that $\operatorname{depth}(S/I) = \lceil \frac{n}{3} \rceil$ and for the powers of I gave a lower bound, $\operatorname{depth}(S/I^t) \geq max\{\lceil \frac{n-t+1}{3} \rceil, 1\}$. The proof makes repeated use of applying the *Depth Lemma:*

Lemma 2.1. ([20], *Lemma* 1.3.9) *If*

$$0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$$

is a short exact sequence of modules over a local ring R, *then*
a) If depth $M <$ depth N, *then* depth $U =$ depth M.
b) If depth $M >$ depth N, *then* depth $U =$ depth $N + 1$.

Rauf [17] showed that most of the statments of the *Depth Lemma* are wrong if we replace depth by sdepth and prove the analog of Lemma 2.1.(a) for sdepth:

Lemma 2.2. *Let*

$$0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence of finitely generated $\mathbb{Z}^n$*-graded* S*-modules. Then*

$$\operatorname{sdepth} M \geq min\{\operatorname{sdepth} U,\ \operatorname{sdepth} N\}.$$

In [15], for $I = I(P_n)$, the authors, based on the proof from [13] of the fact that $\operatorname{depth}(S/I) \geq \lceil \frac{n}{3} \rceil$, showed that $\operatorname{sdepth}(S/I) \geq \lceil \frac{n}{3} \rceil$ and for the powers of I gave a lower bound, $\operatorname{sdepth}(S/I^t) \geq max\{\lceil \frac{n-t+1}{3} \rceil, 1\}$. See also [7] which generalizes this result beyond edge ideals.

Now, we present an algorithm, introduced in [11], in order to compute the Stanley depth of a module of the form I/J where $J \subset I \subset S$ are monomials ideals.

We define a natural partial order on $\mathbb{N}^n$ as follows: $a \leq b$ if and only if $a(i) \leq b(i)$ for $i = 1, \ldots, n$ and we will say that b *cover* a. Note that $x^a | x^b$ if and only if $a \leq b$. Here, for any $c \in \mathbb{N}^n$ we denote as usual by x^c the monomial $x_1^{c(1)} x_2^{c(2)} \cdots x_n^{c(n)}$. Observe that $\mathbb{N}^n$ with the partial order introduced is a distributive lattice with meet $a \wedge b$ and join $a \vee b$ defined as follows: $(a \wedge b)(i) = min\{a(i), b(i)\}$ and $(a \vee b)(i) = max\{a(i), b(i)\}$.

Suppose I is generated by the monomials $x^{a_1}, \ldots, x^{a_r}$ and J by the monomials $x^{b_1}, \ldots, x^{b_s}$. We choose $g \in \mathbb{N}^n$ such that $a_i \leq g$ and $b_j \leq g$ for all i and j. Let $P_{I/J}^g$ be the set of all $c \in \mathbb{N}^n$ with $c \leq g$ and such that $a_i \leq c$ for some i and $c \not\geq b_j$ for all j. The set $P_{I/J}^g$ viewed as a subposet of $\mathbb{N}^n$ is a finite poset and we call it the *characteristic poset* of I/J with respect to g. There is a natural choice of g, namely the join of all the a_i and b_j. For this g, the poset $P_{I/J}^g$ has the least number of elements, and we denote it simply by $P_{I/J}$.

Given any poset P and $a, b \in P$ we set $[a, b] = \{c \in P : a \leq c \leq b\}$ and call $[a, b]$ an *interval*. Of course, $[a, b] \neq \emptyset$ if and only if $a \leq b$. Suppose P is a finite poset. A *partition* of P is a disjoint union

$$\mathcal{P} : P = \bigcup_{i=1}^{r} [a_i, b_i]$$

of intervals.

In order to describe the Stanley decomposition of I/J coming from a partition of $P^g_{I/J}$ we shall need the following notation: for each $b \in P^g_{I/J}$, we set $Z_b = \{x_i : b(i) = g(i)\}$; we also introduce the function

$$\rho : P^g_{I/J} \longrightarrow \mathbb{Z}_{\geq 0}, \quad c \mapsto \rho(c),$$

where $\rho(c) = |\{x_i : c(i) = g(i)\}|(= |Z_c|)$. We then have

Theorem 2.1. ([11, Theorem 2.1.]) (a) *Let $\mathcal{P} : P^g_{I/J} = \bigcup_{i=1}^r [c_i, d_i]$ be a partition of $P^g_{I/J}$. Then*

$$\mathcal{D}(\mathcal{P}) : I/J = \bigoplus_{i=1}^{r} (\bigoplus_{c} x^c K[Z_{d_i}])$$

is a Stanley decomposition of I/J, where the inner direct sum is taken over all $c \in [c_i, d_i]$ for which $c(j) = c_i(j)$ for all j with $x_j \in Z_{d_i}$. Moreover,

$$\text{sdepth}(\mathcal{D}(\mathcal{P})) = min\{\rho(d_i) : i = 1, \dots, r\}.$$

(b) *One has*

$$\text{sdepth}(I/J) = max\{\text{sdepth}(\mathcal{D}(\mathcal{P})) : \mathcal{P} \textit{ is a partition of } P^g_{I/J}\}.$$

In particular, there exists a partition $\mathcal{P} : P^g_{I/J} = \bigcup_{i=1}^r [c_i, d_i]$ of $P^g_{I/J}$ such that

$$\text{sdepth}(I/J) = min\{\rho(d_i) : i = 1, \dots, r\}.$$

Lemma 2.3. *If $I = I(P_n)$, then* $\text{sdepth}(S/I) = \left\lceil \frac{n}{3} \right\rceil$.

Proof. The inequality $\text{sdepth}(S/I) \geq \left\lceil \frac{n}{3} \right\rceil$ is known, see ([15, Proposition 2.1.])

Now we prove the other inequality, $\text{sdepth}(S/I) \leq \left\lceil \frac{n}{3} \right\rceil$.

We denote by e_j the j^{th} canonical unit vector in $\mathbb{Z}^n$.

We identify S/I with the $\mathbb{Z}^n$-graded K-subvector space I^c of S which is generated by all monomials $u \in S \setminus I$.

The *characteristic poset* (see [11]) of S/I is

$$P = \{a \in \mathbb{N}^n \ : \ x^a \in I^c \textit{ and } x^a | x_1 x_2 \cdots x_n\},$$

where $x^a = x_1^{a(1)} x_2^{a(2)} \cdots x_n^{a(n)}$ and $a = (a(1), \dots, a(n)) \in \mathbb{N}^n$. Also, we introduce the function

$$\rho : P \longrightarrow \mathbb{Z}_{\geq 0}, \quad c \mapsto \rho(c),$$

where $\rho(c) = |\{i : c(i) = 1\}|$.

For $d \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$ let

$$P_d := \{a \in P \ : |a| = d\} \textit{ and } P_{d,\alpha} := \{a \in P_d \ : x^\alpha | x^a\},$$

where for $a = (a(1), \dots, a(n)) \in \mathbb{N}^n$, $|a| := \sum_{i=1}^n a(i)$.

Firstly, we note that if $\alpha \in P$ such that $P_{d,\alpha} = \emptyset$ then $\text{sdepth}(S/I) < d$. Indeed, let $\mathcal{P} : P = \bigcup_{i=1}^r [c_i, d_i]$ be a partition of P with

$$\text{sdepth}(S/I) = min\{\rho(d_i) : i = 1, \dots, r\}.$$

Since $\alpha \in P$ it follows that $\alpha \in [c_i, d_i]$ for some i. If $\rho(d_i) \geq d$ then it follows that $P_{d,\alpha} \neq \emptyset$, since there is $a \in [c_i, d_i]$ with $\rho(a) = d$ and $x^\alpha | x^a$, a contradiction. Thus, $\rho(d_i) < d$ and therefore $\mathrm{sdepth}(S/I) < d$.
We have three cases to study.

(1) If $n = 3k \geq 3$ and $\alpha = \sum_{i=1}^{k} e_{3i-1} \in P_k$, then $P_{k+1,\alpha} = \emptyset$. Indeed, if $u = x_2x_5 \cdots x_{3k-1}$, one can easily see that $x_j u \in I$ for all $j \notin \{2, 5, \ldots, 3k-1\}$. Therefore, by previous remark, $\mathrm{sdepth}(S/I) \leq k = \lceil \frac{n}{3} \rceil$, as required.
(2) If $n = 3k + 1 \geq 7$ and $\alpha = e_1 + \sum_{i=1}^{k} e_{3i} \in P_{k+1}$, then $P_{k+2,\alpha} = \emptyset$. As above, it follows that $\mathrm{sdepth}(S/I) \leq k + 1 = \lceil \frac{n}{3} \rceil$.
(3) If $n = 3k + 2 \geq 5$ and $\alpha = \sum_{i=1}^{k+1} e_{3i-2} \in P_{k+1}$, then $P_{k+2,\alpha} = \emptyset$ and therefore $\mathrm{sdepth}(S/I) \leq k + 1 = \lceil \frac{n}{3} \rceil$.

□

Theorem 2.2. *Let $I = I(P_n)$ be the path ideal. For $n \geq 3$ and $t \geq 1$ we have that* $\mathrm{sdepth}(S/I^t) = max\{\lceil \frac{n-t+1}{3} \rceil, 1\}$.

Proof. The inequality $\mathrm{sdepth}(S/I^t) \geq max\{\lceil \frac{n-t+1}{3} \rceil, 1\}$ is known (see [15, Proposition 2.5.]).
Now we prove the other inequality, $\mathrm{sdepth}(S/I^t) \leq max\{\lceil \frac{n-t+1}{3} \rceil, 1\}$ for any $t \geq 1$. By Lemma 2.3. the result holds for $t = 1$.
Let $t \geq 2$ fixed. We identify S/I^t with the $\mathbb{Z}^n$-graded K-subvector space $(I^t)^c$ of S which is generated by all monomials $u \in S \setminus I^t$.
The *characteristic poset* (see [11]) of S/I^t is

$$\mathcal{P} = \{a \in \mathbb{N}^n \ : \ x^a \in (I^t)^c \text{ and } x^a | (x_1x_2 \cdots x_n)^t\},$$

where $x^a = x_1^{a(1)} x_2^{a(2)} \cdots x_n^{a(n)}$ and $a = (a(1), \ldots, a(n)) \in \mathbb{N}^n$.

Let us first show why $\mathrm{sdepth}(S/I^t) \leq 1$ for any $t \geq n - 2$. Assume $\mathrm{sdepth}(S/I^t) \geq 2$ for any $t \geq n - 2$. According to Theorem 2.1.(see [11]) there exists a partition of $\mathcal{P} = \bigcup_{i=1}^{r} [F_i, G_i]$ such that $min_{i=1}^{r} \rho(G_i) = 2$, where $\rho(G_i) = |\{j : t = G_i(j)\}|$ is the cardinality of $\{j : t = G_i(j)\}$.
For $t \geq n - 2$ fixed, let the sets:

$$[(t, t-1, t, 0, \ldots)] := \{(t, \alpha_2, t, \alpha_4, \beta) \in \mathcal{P} \mid 0 \leq \sum_{i=1}^{2} \alpha_{2i} \leq t - 1, \beta \in \mathbb{N}^{n-4}$$

$$\text{with } |\beta| = (t-1)(\left\lceil \frac{n}{2} \right\rceil - 2) - \sum_{i=1}^{2} \alpha_{2i}\},$$

$$[(t-1, t-1, t, 0, \ldots)] := \{(t-1, \alpha_2, t, \alpha_4, \beta) \in \mathcal{P} \mid 0 \leq \sum_{i=1}^{2} \alpha_{2i} \leq t-1, \beta \in \mathbb{N}^{n-4}$$

$$\text{with } |\beta| = (t-1)(\left\lceil \frac{n}{2} \right\rceil - 2) - \sum_{i=1}^{2} \alpha_{2i}\},$$

$$[(t,t-1,t-1,0,\ldots)] := \{(t,\alpha_2,t-1,\alpha_4,\beta) \in \mathcal{P} \mid 0 \leq \sum_{i=1}^{2}\alpha_{2i} \leq t-1,\ \beta \in \mathbb{N}^{n-4}$$

$$with\ |\beta| = (t-1)(\left\lceil \frac{n}{2} \right\rceil - 2) - \sum_{i=1}^{2}\alpha_{2i}\}.$$

The elements of $[(t,t-1,t,0,\ldots)]$ can only cover the elements of $[(t-1,t-1,t,0,\ldots)] \cup [(t,t-1,t-1,0,\ldots)]$ since for any $\gamma \in [(t,t-1,t,0,\ldots)]$, $\delta \in [(t-1,t-1,t,0,\ldots)]$ and $\eta \in [(t,t-1,t-1,0,\ldots)]$ we have $|\gamma| - 1 = |\delta| = |\eta|$, $\rho(\gamma) = 2$, $\rho(\delta) = \rho(\eta) = 1$. As long as there is an one to one corespondence between the sets $[(t,t-1,t,0,\ldots)]$ and $[(t-1,t-1,t,0,\ldots)]$, (respectively the sets $[(t,t-1,t,0,\ldots)]$ and $[(t,t-1,t-1,0,\ldots)]$) and $[(t-1,t-1,t,0,\ldots)] \cap [(t,t-1,t-1,0,\ldots)] = \emptyset$, then there exists elements from $[(t-1,t-1,t,0,\ldots)] \cup [(t,t-1,t-1,0,\ldots)]$ which can not be covered by elements of $[(t,t-1,t,0,\ldots)]$. Therefore $\mathrm{sdepth}(S/I^t) \leq 1$ for any $t \geq n-2$ and so $\mathrm{sdepth}(S/I^t) = 1$ for any $t \geq n-2$.

Using the same technique as above we show why $\mathrm{sdepth}(S/I^t) \leq \left\lceil \frac{n-t+1}{3} \right\rceil$ for any $2 \leq t \leq n-3$. Let $2 \leq t \leq n-3$ fixed and we denote by $a := \left\lceil \frac{n-t+1}{3} \right\rceil$. Assume $\mathrm{sdepth}(S/I^t) \geq a+1$. According to Theorem 2.1.([11]) there exists a partition of $\mathcal{P} = \bigcup_{i=1}^{r}[F_i,G_i]$ such that $min_{i=1}^{r}\rho(G_i) = a+1$.
We consider the sets:

$$[(t,t-1\underbrace{t,0,\ldots,t,0}_{a-times},\ldots)] := \{(t,\alpha_2,t,\alpha_4\ldots,t,\alpha_{2a+2},\beta) \in \mathcal{P} \mid 0 \leq \sum_{i=1}^{a+1}\alpha_{2i} \leq t-1,$$

$$\beta \in \mathbb{N}^{n-2a-2}\ with\ |\beta| = (t-1)(\left\lceil \frac{n}{2} \right\rceil - a) - \sum_{i=1}^{a+1}\alpha_{2i}\},$$

$$[(t-1,t-1\underbrace{t,0,\ldots,t,0}_{a-times},\ldots)] := \{(t-1,\alpha_2,t,\alpha_4,\ldots,t,\alpha_{2a+2},\beta) \in \mathcal{P} \mid 0 \leq \sum_{i=1}^{a+1}\alpha_{2i}$$

$$\sum_{i=1}^{a+1}\alpha_{2i} \leq t-1, \beta \in \mathbb{N}^{n-2a-2}\ with\ |\beta| = (t-1)(\left\lceil \frac{n}{2} \right\rceil - a) - \sum_{i=1}^{a+1}\alpha_{2i}\},$$

$$[(t,t-1\underbrace{t,0,\ldots,t,0}_{a-1-times},\ldots)]\{(t,\alpha_2,t,\alpha_4,\ldots,t,\alpha_{2a},\beta) \in \mathcal{P} \mid 0 \leq \sum_{i=1}^{a}\alpha_{2i} \leq t-1,$$

$$\beta \in \mathbb{N}^{n-2a}\ with\ |\beta| = (t-1)(\left\lceil \frac{n}{2} \right\rceil - a + 1) - \sum_{i=1}^{a}\alpha_{2i}\}.$$

The elements of $[(t,t-1\underbrace{t,0,\ldots,t,0}_{a-times},\ldots)]$ can only cover the elements of $[(t-1,t-1\underbrace{t,0,\ldots,t,0}_{a-times},\ldots)] \cup [(t,t-1\underbrace{t,0,\ldots,t,0}_{a-1-times},\ldots)]$ since for any $\gamma \in$

$[(t,t-1\underbrace{t,0,\ldots,t,0}_{a-times},\ldots)]$, $\delta\in[(t-1,t-1\underbrace{t,0,\ldots,t,0}_{a-times},\ldots)]$ and $\eta\in[(t,t-1\underbrace{t,0,\ldots,t,0}_{a-1-times},\ldots)]$ we have $|\gamma|-1=|\delta|=|\eta|$, $\rho(\gamma)=a+1$, $\rho(\delta)=\rho(\eta)=a$.
As long as there is an one to one corespondence between the sets $[(t,t-1\underbrace{t,0,\ldots,t,0}_{a-times},\ldots)]$ and $[(t-1,t-1\underbrace{t,0,\ldots,t,0}_{a-times},\ldots)]$, (respectively the sets $[(t,t-1\underbrace{t,0,\ldots,t,0}_{a-times},\ldots)]$ and $[(t,t-1\underbrace{t,0,\ldots,t,0}_{a-1-times},\ldots)]$) and $[(t-1,t-1\underbrace{t,0,\ldots,t,0}_{a-times},\ldots)]\cap[(t,t-1\underbrace{t,0,\ldots,t,0}_{a-1-times},\ldots)]=\emptyset$ then there exist elements from $[(t-1,t-1\underbrace{t,0,\ldots,t,0}_{a-times},\ldots)]\cup[(t,t-1\underbrace{t,0,\ldots,t,0}_{a-1-times},\ldots)]$ which can not be covered by elements of $[(t,t-1\underbrace{t,0,\ldots,t,0}_{a-times},\ldots)]$. Therefore $\operatorname{sdepth}(S/I^t)\leq\left\lceil\frac{n-t+1}{3}\right\rceil$ and so we have the equality $\operatorname{sdepth}(S/I^t)=\left\lceil\frac{n-t+1}{3}\right\rceil$ for any $2\leq t\leq n-3$.
Thus, we have $\operatorname{sdepth}(S/I^t)=max\{\left\lceil\frac{n-t+1}{3}\right\rceil,1\}$ for any $t\geq 1$. □

By a theorem of Brodmann ([4]), $\operatorname{depth}(S/I^t)$ is constant for $t>>0$. As a consequence of the previous theorem we obtain a similar result to Brodmann' theorem on the Stanley depth.

Corollary 2.1. *Stanley depth of factors of powers of path ideal stabilizes, i.e.* $\operatorname{sdepth}(S/(I(P_n))^t)=1$ *for any* $t\geq n-2$.

3. **Conclusions**

In this paper we computed the Stanley depth of $S/I(P_n)^t$ where $t\geq 1$ and $I(P_n)$ is the edge ideal of the path graph of lenght $n-1$. In particular, Herzog conjecture [10, Conjecture 2.7] holds for $I(P_n)$.

Future directions of research include the study of Stanley depth for powers of edge ideals, or, more generally, of m-path ideals associated to several classes of graphs.

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REFERENCES

[1] J. Apel, *On a conjecture of R.P. Stanley I. Monomial Ideals,* Journal of Algebraic Combinatorics, **17**(1) (2003), 36-56.

[2] J. Apel, *On a conjecture of R.P. Stanley II. Quotients modulo monomial ideals,* Journal of Algebraic Combinatorics, **17**(1) (2003), 57-74.

[3] I. Anwar, D. Popescu, *Stanley conjecture in small embedding dimension,* J. Algebra, **318**(2007), 1027-1031.

[4] M. Brodmann, *The asymptotic nature of the analitic spread*, Math. Proc. Cambridge Philos. Soc.**86** (1979), 35-39.

[5] C. Biró, D.M. Howard, M.T. Keller, W.T. Trotter, S.J. Young, *Interval partitions and Stanley depth,* Journal of Combinatorial Theory, Series A, **117** (2010), 475-482.

[6] M. Cimpoeaş, *On the Stanley depth of powers of some classes of monomial ideals,* Bull. Iran. Math. Soc. **44** (2018), 739-747.

[7] D. Campos, R. Gunderson, S. Morey, C. Paulsen, T. Polstra, *Depths and Stanley depth of Path Ideals of Spines*, Involve-a journal of mathematics **9, no.1**, (2016), 155-170.

[8] CoCoATeam, *CoCoA: a system for doing Computations in Commutative Algebra.* Avaible at http://cocoa.dima.unige.it

[9] A.M.Duval, B. Goeckneker, C. J. Klivans, J.L. Martine, *A non-partitionable Cohen-Macaulay simpliceal complex,* Advances in Mathematics **299** (2016), 381-395.

[10] J. Herzog, *A survey on Stanley Depth.* In "Monomial Ideals, Computations and Applications", A. Bigatti, P. Giménez, E. Sáenz-de-Cabezoń (Eds.), Proceedings of MONICA 2011. Lecture Notes in Mathematics **2083**, Springer 2013.

[11] J. Herzog, M. Vlădoiu, X. Zheng, *How to compute the Stanley depth of a monomial ideal,* J. Algebra, **322**(2009) 3151-3169.

[12] J.Herzog, A. Soleyman Jahan, S. Yassemi, *Stanley decompositions and partitionable simplicial complexes,* J. Algebraic Combin. **27**, (2008), 113-125.

[13] S. Morey, *Depths of powers of the edge ideal of a tree,* Comm. Algebra **38**(11) (2010) 4042-4055.

[14] D. Popescu, *Stanley depth of multigraded modules,* J. Algebra, **321** (2009) 2782-2797.

[15] M. R. Pournaki, S. A. Seyed Fakhari, S. Yassemi, *Stanley depth of powers of the edge ideal of forest* Proc. Amer. Math. Soc. **141** (2013), no. 10, 3327-3336.

[16] M. R. Pournaki, S. A. Seyed Fakhari, M. Tousi, S. Yassemi, *What is . . . Stanley depth?* Notices Amer. Math. Soc. **56**(2009), no. 9, 1106-1108.

[17] A. Rauf, *Depth and Stanley depth of multigraded modules,* Comm. Algebra **38**(2) (2010) 773-784.

[18] G. Rinaldo, *An algorithm to compute the Stanley depth of monomials ideals.* Le Matematiche Vol **LXIII** (2008), Fasc. II, 243-256.

[19] R. Stanley, *Linear Diophantine equations and local cohomology*, Invent. Math. **68**(1982) 175-193.

[20] R. H. Villareal, *Monomial Algebras,* New York: Marcel Dekker Inc, 2001.