FUZZY IDEALS OF NEAR-RINGS BASED ON THE THEORY OF FALLING SHADOWS

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Based on the theory of falling shadows and fuzzy sets, the notion of falling fuzzy ideals of a near-ring is introduced. The relations between fuzzy ideals and falling fuzzy ideals are provided. Finally, we apply the concept of falling fuzzy inference relations to near-rings and obtain an important result.

Keywords: Falling shadow; ideal; fuzzy ideal; falling fuzzy ideal; falling fuzzy inference relation.
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1. Introduction

In the study of a unified treatment of uncertainty modelled by means of combining probability and fuzzy set theory, Goodman [4] pointed out the equivalence of a fuzzy set and a class of random sets. Wang and Sanchez [13] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. Falling shadow representation theory shows us the way of selection related on the joint degrees distributions. It is reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. The mathematical structure of the theory of falling shadows is formulated. Tan et al. [11, 12] established a theoretical approach to define a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows.

A near-ring satisfying all axioms of an associative ring, expect for commutativity of addition and one of the two distributive laws. Abou-Zaid [1] introduced the concepts of fuzzy subnear-rings (ideals) and studied some of their related properties in near-rings. Further, some properties were discussed by Hong and Kim et al. in [5] and [6], respectively. In [3], Davvaz introduced the concepts of $(\in \in \vee q)$-fuzzy subnear-rings(ideals) of near-rings. Further, Zhan et al. [14, 15, 16, 17] investigated some generalized fuzzy ideals of near-rings. The other important results can be found in [2, 10]. Fuzzy near-rings are in particular designed for situations in which natural-language expressions need to be modelled without artificial specifications about borderline cases. Numerous applications where fuzzy near-rings have been

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proven useful can for instance be found in information and automatic. For more details, see [9].

Recently, Jun et al. [7, 8] established a theoretical approach for defining fuzzy positive implicative ideals, fuzzy commutative ideals and fuzzy implicative ideals in a BCK-algebra based on the theory of falling shadows. Based on this theory, we apply it to near-rings. We introduce the notion of falling fuzzy ideals of a near-ring. The relations between fuzzy ideals and falling fuzzy ideals are provided. Finally, we apply the concept of falling fuzzy inference relations to near-rings and obtain an important result.

2. **Preliminaries**

A non-empty set R with two binary operation " + " and " · " is called a left near-ring if it satisfies the following conditions:

(1) $(R,+)$ is a group,

(2) $(R,\cdot)$ is a semigroup,

(3) $x\cdot(y+z)=x\cdot y+x\cdot z$, for all $x,y,z\in R$.

We will use the word "near-ring" to mean " left near-ring" and denote xy instead of $x\cdot y$.

An ideal I of a near-ring R is a subset of R such that

(i) $(I,+)$ is a normal subgroup of $(R,+)$,

(ii) $RI\subseteq I$,

(iii) $(x+a)y-xy\in I$ for any $a\in I$ and $x,y\in R$.

In what follows, let R be a near-ring unless otherwise specified.

Definition 2.1. [1] *A fuzzy set μ of R is called a fuzzy ideal of R if for any $x,y,a\in R$,*

(1) $\mu(x-y)\geq\mu(x)\wedge\mu(y)$,

(2) $\mu(xy)\geq\mu(x)\wedge\mu(y)$,

(3) $\mu(y+x-y)\geq\mu(x)$,

(4) $\mu(xy)\geq\mu(y)$,

(5) $\mu((x+a)y-xy)\geq\mu(a)$.

Theorem 2.1. [1] *A fuzzy set μ of R is a fuzzy ideal of R if and only if the non-empty subset μ_t is an ideal of R for all $t\in[0,1]$.*

We now display the basic theory on failing shadows. We refer the reader to the papers [4, 13, 11, 12] for further information regarding falling shadows. Given a universe of discourse U, let $\mathcal{P}(U)$ denote the power set of U. For each $u\in U$, let

$$\dot{u}=\{E|u\in E,\quad E\subseteq U\},$$

and for each $E\in\mathcal{P}(U)$, let

$$\dot{E}=\{\dot{u}|u\in E\}.$$

An ordered pair $(\mathcal{P}(U), \mathcal{B})$ is said to be a hyper-measurable structure on U if $\mathcal{B}$ is a σ-field in $\mathcal{P}$ and $\dot{U} \subseteq \mathcal{B}$. Given a probability space $(\Omega, \mathcal{A}, P)$ and a hyper-measurable structure $(\mathcal{P}(U), \mathcal{B})$ on U, a random set on U is defined to be a mapping $\xi : \Omega \to \mathcal{P}(U)$, which is $\mathcal{A} - \mathcal{B}$ measurable, that is,

$$\xi^{-1}(C) = \{\omega | \omega \in \Omega, \xi(\omega) \in C\} \in \mathcal{A}, \forall C \in \mathcal{B}.$$

Suppose that ξ is a random set on U. Let

$$\widetilde{H}(u) = P(\omega | u \in \xi(\omega)), \text{ for each } u \in U.$$

Then $\widetilde{H}$ is a kind of fuzzy set in U. We call $\widetilde{H}$ a falling shadow of the random set ξ, and ξ is called a cloud of $\widetilde{H}$.

For example, $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$, where $\mathcal{A}$ is a Borel field on [0,1] and m is the usual Lebesgue measure. Let $\widetilde{H}$ be a fuzzy set in U and $\widetilde{H}_t = \{u \in U | \widetilde{H}(u) \geq t\}$ be a t-cut of $\widetilde{H}$. Then

$$\xi : [0, 1] \to \mathcal{P}(U), t \longmapsto \widetilde{H}_t$$

is a random set and ξ is a cloud of $\widetilde{H}$. We shall call ξ defined above the cut-cloud of $\widetilde{H}$(see [4]).

3. **Falling fuzzy ideals**

In this section, we will introduce the notion of falling fuzzy ideals of a near-ring. The relations between fuzzy ideals and falling fuzzy ideals are provided.

Definition 3.1. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $\xi : \Omega \to \mathcal{P}(R)$ be a random set. If $\xi(\omega)$ is an ideal of R for any $w \in \Omega$, then the falling shadow of the random set ξ , i.e., $\widetilde{H}(u) = P(\omega | u \in \xi(\omega))$ is called a falling fuzzy ideal of R.*

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $F(R) = \{f | f : \Omega \to R\}$, where R is a near-ring.

Define an operation $\oplus$ and $\odot$ on $F(R)$ by

$(f \oplus g)(w) = f(w) + g(w)$

and $(f \odot g)(w) = f(w) \cdot g(w)$,

for all $w \in \Omega, f, g \in F(R)$.

Let $\theta \in F(R)$ be defined by $\theta(\omega) = 0$, for all $\omega \in \Omega$. Then we can check that $(F(R), \oplus, \odot, \theta)$ is a near-ring.

For any subset A of R and $f \in F(R)$, let $A_f = \{\omega \in \Omega | f(\omega) \in A\}$,

$$\xi : \Omega \to \mathcal{P}(F(R))$$
$$\omega \mapsto \{f \in F(R) | f(\omega) \in A\},$$

then $A_f \in \mathcal{A}$.

Proposition 3.1. *If A is an ideal of R, then $\xi(\omega) = \{f \in F(R) | f(\omega) \in A\}$ is an ideal of $F(R)$.*

Proof. Assume that A is an ideal of R and $\omega \in \Omega$. Let $f, g \in F(R)$ be such that $f, g \in \xi(\omega)$, then $f(\omega), g(\omega) \in A$. Since A is an ideal of R, then $f(\omega) - g(\omega) \in A$. Thus, $(f \ominus g)(\omega) = f(\omega) - g(\omega) \in A$, and so $f \ominus g \in \xi(\omega)$. Hence $\xi(\omega)$ is a subgroup of R. Similarly, we can prove the others of a near-ring. □

From the above proposition, we know $\widetilde{H}$ is a falling fuzzy ideal of $F(R)$, where $\widetilde{H}(f) = P(\omega | f(\omega) \in A\}$. In fact, since

$$\begin{aligned}\xi^{-1}(f) &= \{\omega \in \Omega | f \in \xi(\omega)\} \\ &= \{\omega \in \Omega | f(\omega) \in A\} \\ &= A_f \in \mathcal{A},\end{aligned}$$

We see that ξ is a random set on $F(R)$. By Proposition 3.1, we know $\widetilde{H}$ is a falling fuzzy ideal of R.

Example 3.1. *Let $R = \{0, 1, 2, 3, 4, 5\}$ be a set with an addition operation and a multiplication operation as follow:*

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

·	0	1	2	3	4	5
0	0	3	0	3	0	3
1	0	1	2	3	4	5
2	0	5	4	3	2	1
3	0	3	0	3	0	3
4	0	1	2	3	4	5
5	0	5	4	3	2	1

Then $(R, +, \cdot)$ is a near-ring.

Let $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and $\xi : [0, 1] \to \mathcal{P}(R)$ be defined by

$$\xi(t) = \begin{cases} \{0\} & \text{if } t \in [0, 0.3), \\ \{0, 3\} & \text{if } t \in [0.3, 0.5), \\ \{0, 2, 4\} & \text{if } t \in [0.5, 0.9), \\ R & \text{if } t \in [0.9, 1]. \end{cases}$$

Then $\xi(t)$ is an ideal of R for all $t \in [0, 1]$. Hence $\widetilde{H} = P(t | x \in \xi(t))$ is a falling fuzzy ideal of R and it is represented as follows:

$$\widetilde{H}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.1 & \text{if } x = 1, 5, \\ 0.5 & \text{if } x = 2, 4, \\ 0.3 & \text{if } x = 3. \end{cases}$$

Then

$$\widetilde{H}_t = \begin{cases} \{0\} & \text{if } t \in (0.5, 1], \\ \{0, 2, 4\} & \text{if } t \in (0.3, 0.5], \\ \{0, 2, 3, 4\} & \text{if } t \in (0.1, 0.3], \\ R & \text{if } t \in [0, 0.1]. \end{cases}$$

If $t \in (0.1, 0.3]$, then $\widetilde{H}_t = \{0, 2, 3, 4\}$ is not an ideal of R since $3 + 4 = 1 \nsubseteq \{0, 2, 3, 4\}$. Thus, it follows from Theorem 2.1 that $\widetilde{H}$ is not a fuzzy ideal of R.

Theorem 3.1. *Every fuzzy ideal of R is a falling fuzzy ideal of R.*

Proof. Consider the probability space $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$, where $\mathcal{A}$ is a Bored field on [0,1] and m is the usual Lebesque measure. Let μ be a fuzzy ideal of R, then μ_t is an ideal of R for all $t \in [0, 1]$. Let $\xi : [0, 1] \to \mathcal{P}(R)$ be a random set and $\xi(t) = \mu_t$ for every $t \in [0, 1]$. Then μ is a falling fuzzy ideal of R. □

Remark 3.1. *Example 3.1 shows that the converse of Theorem 3.1 is not valid.*

Let $(\Omega, \mathcal{A}, P)$ be a probability space and a falling shadow of a random set $\xi : \Omega \to \mathcal{P}(R)$. For any $x \in R$, let $\Omega(x; \xi) = \{\omega \in \Omega | x \in \xi(\omega)\}$. Then $\Omega(x; \xi) \in \mathcal{A}$.

Lemma 3.1. *If a falling shadow $\widetilde{H}$ of a random set $\xi : \Omega \to \mathcal{P}(R)$ is a falling fuzzy ideal of R, then for all $x, y, a \in R$,*

(1) $\Omega(x; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x - y; \xi)$,
(2) $\Omega(x; \xi) \cap \Omega(y; \xi) \subseteq \Omega(xy; \xi)$,
(3) $\Omega(x; \xi) \subseteq \Omega(y + x - y; \xi)$,
(4) $\Omega(y; \xi) \subseteq \Omega(xy; \xi)$,
(5) $\Omega(a; \xi) \subseteq \Omega((x + a)y - xy; \xi)$.

Proof. We only prove (1), and the others are similar. Let $\omega \in \Omega(x; \xi) \cap \Omega(y; \xi)$, then $x, y \in \xi(\omega)$. Since $\xi(\omega)$ is an ideal of R by Definition 3.1, then $x - y \in \xi(\omega)$, and so $\omega \in \Omega(x - y; \xi)$. This completes the proof. □

Theorem 3.2. *If $\widetilde{H}$ is a falling fuzzy ideal of R, then for all $x, y, a \in R$,*

(1) $\widetilde{H}(x - y) \geq T_m(\widetilde{H}(x), \widetilde{H}(y))$,
(2) $\widetilde{H}(xy) \geq T_m(\widetilde{H}(x), \widetilde{H}(y))$,
(3) $\widetilde{H}(y + x - y) \geq \widetilde{H}(x)$,
(4) $\widetilde{H}(xy) \geq \widetilde{H}(x)$,
(5) $\widetilde{H}((x + a)y - xy) \geq \widetilde{H}(a)$,
where $T_m(s, t) = \max\{s + t - 1, 0\}$, *for all* $s, t \in [0, 1]$.

Proof. We only prove the conclusion (1) and the others are similar. By Definition 3.1, $\xi(\omega)$ is an ideal of R for any $\omega \in \Omega$. Hence

$\{\omega \in \Omega | x \in \xi(\omega)\} \cap \{\omega \in \Omega | y \in \xi(\omega)\} \subseteq \{\omega \in \Omega | x - y \in \xi(\omega)\}$,

and so

$$
\begin{aligned}
\widetilde{H}(x - y) &= P(\omega | x - y \in \xi(\omega)) \\
&\geq P(\{\omega | x \in \xi(\omega)\} \cap \{\omega | y \in \xi(\omega)\}) \\
&\geq P(\omega | x \in \xi(\omega)) + P(\omega | y \in \xi(\omega)) - P(\omega | x \in \xi(\omega) \text{ or } y \in \xi(\omega)) \\
&\geq \widetilde{H}(x) + \widetilde{H}(y) - 1.
\end{aligned}
$$

□

Remark 3.2. *Theorem 3.2 means that every falling fuzzy ideal of R is a T_m-fuzzy ideal of R.*

4. **Falling fuzzy inference relations**

Based on the theory of falling shadows, Tan et al. [12] establish a theoretical approach to define a fuzzy inference relation. Let A and B be fuzzy sets in the universes U and V, respectively, ξ and η be cut-clouds of A and B, respectively. Then the fuzzy inference relation $I_{A\to B}$ of the implication $A \to B$ is defined to be

$$I_{A\to B}(u,v) = P((\lambda,\mu)|(u,v) \in I_{A_\lambda \to B_\mu}) = P((\lambda,\mu)|(u,v) \in (A_\lambda \times B_\mu) \cup (A^c_\lambda \times V)),$$

where P is a joint probability on $[0,1]^2$. So different probability distribution P will generate different formula for the fuzzy inference relation. The following three basic cases are considered.

Theorem 4.1. [12] *(1)If the whole probability P of (λ,μ) on $[0,1]^2$ is concentrated and uniformly distributed on the main diagonal $\{(\lambda,\lambda)|\lambda \in [0,1]\}$ of the unit square $[0,1]^2$, then P is the diagonal distribution and $I_{A\to B}(u,v) = \min(1-A(u)+B(v),1)$.*

(2) If the whole probability P of (λ,μ) on $[0,1]^2$ is concentrated and uniformly distributed on the anti-diagonal $\{(\lambda,1-\lambda)|\lambda \in [0,1]\}$ of the unit square $[0,1]^2$, then P is the anti-diagonal distribution and $I_{A\to B}(u,v) = \max(1-A(u),B(v))$.

(3)If the whole probability P of (λ,μ) on $[0,1]^2$ is uniformly distributed on the unit square $[0,1]^2$, then P is the independent distribution and $I_{A\to B}(u,v) = 1-A(u)+A(u)B(v)$.

We call the three fuzzy inference relations falling implication operators on [0, 1].

Definition 4.1. *Let μ be a fuzzy set of R, I be a falling implication operator over [0, 1] and $\lambda \in (0,1]$. Then μ is called an I-fuzzy ideal of R, if for all $x,y,a \in R$, the following conditions are satisfied:*

(1) $I(\min\{\mu(x),\mu(y)\},\mu(x-y)) \geq \lambda$;
(2) $I(\min\{\mu(x),\mu(y)\},\mu(xy)) \geq \lambda$;
(3) $I(\mu(x),\mu(y+x-y)) \geq \lambda$;
(4) $I(\mu(y),\mu(xy)) \geq \lambda$;
(5) $I(\mu(a),\mu((x+a)y-xy)) \geq \lambda$.

Clearly, if $\lambda = 1$ and P is the diagonal distribution, then Definition 4.1 is equivalent to Definition 2.1.

From Theorem 4.1 and Definition 4.1, we can immediately get the following results:

Theorem 4.2. *Let μ be a fuzzy set of R and $\lambda = 0.5$, then for all $x,y,a \in R$,*

(1) if P is the diagonal distribution, then μ is an I-fuzzy ideal of R, if and only if it satisfies the following conditions:

(a) $\min\{\mu(x),\mu(y)\} \leq \mu(x-y)$ *or* $0 < \min\{\mu(x),\mu(y)\} - \mu(x-y) \leq 0.5$,

(b) $\min\{\mu(x), \mu(y)\} \leq \mu(xy)$ *or* $0 < \min\{\mu(x), \mu(y)\} - \mu(xy) \leq 0.5$,
(c) $\mu(x) \leq \mu(y + x - y)$ *or* $0 < \mu(x) - \mu(y + x - y) \leq 0.5$,
(d) $\mu(y) \leq \mu(xy)$ *or* $0 < \mu(y) - \mu(xy) \leq 0.5$,
(e) $\mu(a) \leq \mu((x + a)y - xy)$ *or* $0 < \mu(a) - \mu((x + a)y - xy) \leq 0.5$.

(2) if P is the anti-diagonal distribution, then μ is an I-fuzzy ideal of R, if and only if it satisfies the following conditions:
(a) $\min\{\mu(x), \mu(y)\} \leq \max\{\mu(x - y), 0.5\}$ *or* $\min\{\mu(x), \mu(y), 0.5\} \leq \mu(x - y)$,
(b) $\min\{\mu(x), \mu(y)\} \leq \max\{\mu(xy), 0.5\}$ *or* $\min\{\mu(x), \mu(y), 0.5\} \leq \mu(xy)$,
(c) $\mu(x) \leq \max\{\mu(y + x - y), 0.5\}$ *or* $\min\{\mu(x), 0.5\} \leq \mu(y + x - y)$,
(d) $\mu(y) \leq \max\{\mu(xy), 0.5\}$ *or* $\min\{\mu(y), 0.5\} \leq \mu(xy)$,
(e) $\mu(a) \leq \{\max\{\mu((x+a)y-xy), 0.5\}\}$ *or* $\min\{\mu(a), 0.5\} \leq \mu((x+a)y-xy)$.

(3) if P is the independent distribution, then μ is an I-fuzzy ideal of R, if and only if it satisfies the following conditions:
(a) $\min\{\mu(x), \mu(y)\}(1 - \mu(x - y)) \leq 0.5$,
(b) $\min\{\mu(x), \mu(y)\}(1 - \mu(xy)) \leq 0.5$,
(c) $\mu(x)(1 - \mu(y + x - y)) \leq 0.5$,
(d) $\mu(y)(1 - \mu(xy)) \leq 0.5$,
(e) $\mu(a)(1 - \mu((x + a)y - xy)) \leq 0.5$.

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