A PROXIMAL ALGORITHM FOR SOLVING SPLIT MONOTONE VARIATIONAL INCLUSIONS

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In this paper, we study the split monotone variational inclusion problem in Hilbert spaces. We suggest a proximal algorithm for finding a solution of the split monotone variational inclusion problem. Strong convergence theorem is given under some mild conditions.

Keywords: maximal monotone operator, split monotone variational inclusion, resolvent operator, averaged operator.

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1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let $G : H_1 \to H_2$ be a bounded linear operator. Let $A_1 : H_1 \to H_1$ and $A_2 : H_2 \to H_2$ be two single-valued operators. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two multi-valued maximal monotone operators. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively.

Recall that the split feasibility problem is formulated as follows:

$$\text{find a point } x^\dagger \in C \quad \text{such that} \quad G(x^\dagger) \in Q. \tag{1}$$

The split feasibility problem (1) was originally formulated from medical image reconstruction by Censor and Elfving [3]. Consequently, some more prototypes were found in applications, for example, in signal processing, control theory, biomedical engineering and so on. Since then, the split problems were studied extensively by many scholars, for instance, the reader can refer to [5, 6, 7, 9, 10, 15, 20, 21, 22, 23, 24, 26, 30, 33, 34, 37] and related literature.

Recently, Moudafi [16] introduced the following split variational inclusion problem of finding $x^\dagger \in H_1$ verifying

$$x^\dagger \in (A_1 + B_1)^{-1}(0) \text{ and } G(x^\dagger) \in (A_2 + B_2)^{-1}(0). \tag{2}$$

Denote the solution set of split variational inclusion problem (2) by Γ.

Special cases:

(i) If $A_1 \equiv 0$ and $A_2 \equiv 0$, then problem (2) reduces to the following split variational inclusion problem of finding $u \in H_1$ verifying

$$u \in B_1^{-1}(0) \text{ and } G(u) \in B_2^{-1}(0). \tag{3}$$

Denote the solution set of split variational inclusion problem (3) by Γ_1.

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Many iterative algorithms have been presented for solving problem (3). In [2], Byrne suggested the following iteration with weak convergence for solving problem (3). Let $\lambda > 0$ and select an arbitrary starting point $x^0 \in H_1$. For given the current iterate x^k, compute

$$x^{k+1} = J_\lambda^{B_1}(x^k - \gamma G^*(I - J_\lambda^{B_2})G(x^k)), k \geq 0, \tag{4}$$

where $\gamma \in (0, 1/\|G\|^2)$.

(ii) If $B_1 = N_C$ and $B_2 = N_Q$ normal cones to closed convex sets C and Q, we have the following split variational inequality problem of finding $u \in C$ such that

$$\langle A_1(u), v - u\rangle \geq 0 \ (\forall v \in C) \text{ and } \langle A_2(G(u)), z - G(u)\rangle \geq 0 \ (\forall z \in Q). \tag{5}$$

Variational inequality problems are being used as mathematical programming tools and models to study a wide class of unrelated problems arising in mathematical, physical, regional, engineering and nonlinear optimization sciences. See, for instance, [8, 11, 13, 17, 18, 25, 27, 28, 29, 32, 35, 36]. To solve the (5), Censor, Gibali and Reich [4] proposed the following algorithm: Let $\lambda > 0$, select an arbitrary starting point $x^0 \in H_1$. Given the current iterate x^k, compute

$$x^{k+1} = proj_C(I - \lambda A_1)(x^k - \gamma G^*(I - proj_Q(I - \lambda A_2))G(x^k)), k \geq 0. \tag{6}$$

Motivated by the above work, in this paper, we further study the split monotone variational inclusion problem (2) in Hilbert spaces. We suggest a proximal algorithm for finding a solution of the split monotone variational inclusion problem (2). Strong convergence theorem is given under some mild conditions.

2. **Preliminaries**

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be an operator. We use $Fix(T)$ to denote the set of fixed points of T, that is, $Fix(T) = \{x | x = Tx, x \in C\}$.

An operator $T : C \to C$ is said to be

(i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

(ii) averaged iff it can be written as $T = (1 - \mu)I + \mu S$, where $\mu \in (0, 1)$ and $S : C \to C$ is nonexpansive.

(iii) firmly nonexpansive if $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$ for all $x, y \in C$.

(iv) ρ-contractive with $\rho \in [0, 1)$ if $\|Tx - Ty\| \leq \rho\|x - y\|$ for all $x, y \in C$.

(v) α-inverse strongly monotone with $\alpha > 0$ if

$$\alpha\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \ \forall x, y \in C.$$

An operator T is said to be demiclosed if, for any sequence $\{x_n\}$ which weakly converges to $\tilde{x}$, and if the sequence $\{T(x_n)\}$ strongly converges to z, then $T(\tilde{x}) = z$.

An operator F is said to be a strongly positive bounded linear operator on H if there exists a constant ρ such that $\rho\|u\|^2 \leq \langle F(u), u\rangle, \forall u \in H$.

Recall that an operator with domain $D(B) := \{x \in H : B(x) \neq \emptyset\}$ and range $R(B) := \bigcup_{x\in D(B)} B(x)$ is said to be monotone iff $\langle u - v, x - y\rangle \geq 0$ whenever $u \in B(x)$ and $v \in B(y)$. It is said to be maximal monotone iff its graph is not properly contained in the graph of any other monotone operator. Let $B : H \to 2^H$ be a maximal monotone multi-valued operator. Let J_λ^B be the resolvent of B defined by $J_\lambda^B(x) := (I + \lambda B)^{-1}(x), \lambda > 0$ for all $x \in H$. It is known that the resolvent operator J_λ^B is single-valued and firmly nonexpansive.

The metric projection, denoted by $proj_C : H \to C$, assigns for each $x \in H$ the unique point $proj_C x \in C$ such that $\|x - proj_C x\| = \inf\{\|x - y\| : y \in C\}$. $proj_C : H \to C$ has the property $\langle v^\dagger - proj_C v^\dagger, v - proj_C v^\dagger\rangle \leq 0, \ v \in C, \ v^\dagger \in H$.

For all $x, y \in H$, the following conclusions hold:

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2,\ t \in [0,1], \tag{7}$$

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \tag{8}$$

and

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle. \tag{9}$$

Lemma 2.1 ([16])**.** *Let $\mu_1 > 0$, $\mu_2 > 0$ and B_1 and B_2 be two maximal monotone operators. Then, p solves (2) iff $p \in Fix(J_{\mu_1}^{B_1}(I - \mu_1 A_1))$ and $Gp \in Fix(J_{\mu_2}^{B_2}(I - \mu_2 A_2))$.*

Lemma 2.2 ([16])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\mu > 0$, A be an α-inverse strongly monotone operator and B be a maximal monotone operator. If $\mu \in (0, 2\alpha)$, then the operator $J_\mu^B(I - \mu A)$ is averaged.*

Lemma 2.3 ([31])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at 0.*

Lemma 2.4 ([19])**.** *Assume that $\{\delta_n, n \ge 1\}$ is a sequence of nonnegative real numbers such that*

$$\delta_{n+1} \le (1 - \xi_n)\delta_n + \xi_n \sigma_n,$$

where $\{\xi_n, n \ge 1\}$ is a sequence in $(0,1)$ and $\{\sigma_n, n \ge 1\}$ is a sequence in $\mathbb{R}$ such that
- $\sum_{n=1}^{\infty} \xi_n = \infty$;
- $\limsup_{n\to\infty} \sigma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\xi_n \sigma_n| < \infty$.

Then $\lim_{n\to\infty} \delta_n = 0$.

Lemma 2.5 ([12])**.** *Let $\{w_n\}$ be a sequence of real numbers. Assume there exists at least a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \le w_{n_k+1}$ for all $k \ge 0$. For every $n \ge N_0$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{i \le n : w_{n_i} < w_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \ge N_0$, we have $\max\{w_{\tau(n)}, w_n\} \le w_{\tau(n)+1}$.

3. **Main results**

Let H_1 and H_2 be two real Hilbert spaces. Let $G : H_1 \to H_2$ be a bounded linear operator with its adjoint operator G^*. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two multi-valued maximal monotone operators. Let $A_1 : H_1 \to H_1$ and $A_2 : H_2 \to H_2$ be two inverse strongly monotone operators with coefficients α_1 and α_2, respectively. Let $f : H_1 \to H_1$ be a ρ_1-contractive operator and $F : H_1 \to H_1$ ba a strongly positive bounded linear operator with coefficient ρ_2. Assume that $\Gamma \neq \emptyset$.

Now, we present our algorithm for solving problem (2).

Algorithm 3.1. *Let $\{\alpha^k\}$ be a sequence in $[0,1]$. Let τ, μ_1, μ_2 and γ be four positive constants. Choose an arbitrary initial guess $x^0 \in H_1$. Compute the sequences $\{y^k\}$ and $\{x^k\}$ via the following iterations*

$$y^k = x^k + \tau G^*[J_{\mu_2}^{B_2}(I - \mu_2 A_2)G(x^k) - G(x^k)], k \ge 0, \tag{10}$$

and

$$x^{k+1} = J_{\mu_1}^{B_1}(I - \mu_1 A_1)[\gamma \alpha^k f(x^k) + (I - \alpha^k F)y^k], k \ge 0. \tag{11}$$

Theorem 3.1. *Assume that the following conditions are satisfied*
$(C1)$: $\lim_{k\to\infty} \alpha^k = 0$ *and* $\sum_{k=0}^{\infty} \alpha^k = \infty$;
$(C2)$: $\mu_1 \in (0, 2\alpha_1)$, $\mu_2 \in (0, 2\alpha_2)$, $\tau \in (0, 1/\|G\|^2)$ *and* $1 > \rho_2 > \gamma\rho_1$.
Then the sequence $\{x^k\}$ generated by (11) converges strongly to $x^* = proj_\Gamma(\gamma f + I - F)x^*$.

Proof. Denote by x^* the unique fixed point of the contractive operator $proj_\Gamma(I+\gamma f-F)$ with coefficient $1+\gamma\rho_1-\rho_2$ (see [14, 27]). By Lemma 2.1, we have $x^*=J^{B_1}_{\mu_1}(I-\mu_1A_1)x^*$ and $G(x^*)=J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^*)$. Then,

$$\begin{aligned}\|J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^*)\| &= \|J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^*)\| \\ &\le \|G(x^k)-G(x^*)\|.\end{aligned} \tag{12}$$

Set $z^k=\alpha^k\gamma f(x^k)+(I-\alpha^kF)y^k$ for all $k\ge 0$. It follows that

$$\begin{aligned}\|z^k-x^*\| &= \|\gamma\alpha^k f(x^k)+(I-\alpha^kF)y^k-x^*\| \\ &= \|\gamma\alpha^k(f(x^k)-f(x^*))+\alpha^k(\gamma f(x^*)-F(x^*)) \\ &\quad + (I-\alpha^kF)(y^k-x^*)\| \\ &\le \gamma\rho_1\alpha^k\|x^k-x^*\|+\alpha^k\|\gamma f(x^*)-F(x^*)\|+(1-\rho_2\alpha^k)\|y^k-x^*\|.\end{aligned} \tag{13}$$

From (11), we have

$$\begin{aligned}\|x^{k+1}-x^*\| &= \|J^{B_1}_{\mu_1}(I-\mu_1A_1)z^k-J^{B_1}_{\mu_1}(I-\mu_1A_1)x^*\| \\ &\le \|z^k-x^*\|.\end{aligned} \tag{14}$$

In terms of (10) and (8), we get

$$\begin{aligned}\|y^k-x^*\|^2 &= \|x^k-x^*+\tau G^*[J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^k)]\|^2 \\ &= \|x^k-x^*\|^2+\tau^2\|G^*[J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^k)]\|^2 \\ &\quad + 2\tau\langle G(x^k)-G(x^*), J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^k)\rangle.\end{aligned} \tag{15}$$

Observe that

$$\begin{aligned}&\langle G(x^k)-G(x^*), J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^k)\rangle \\ &= \langle J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^*), J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^k)\rangle \\ &\quad - \|J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^k)\|^2.\end{aligned} \tag{16}$$

Applying (8), we obtain

$$\begin{aligned}&\langle J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^*), J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^k)\rangle \\ &= \frac{1}{2}(\|J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^*)\|^2-\|G(x^k)-G(x^*)\|^2 \\ &\quad + \|J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^k)\|^2).\end{aligned} \tag{17}$$

By virtue of (12), (16) and (17), we get

$$\begin{aligned}&\langle G(x^k)-G(x^*), J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^k)\rangle \\ &= \frac{1}{2}(\|J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^*)\|^2-\|G(x^k)-G(x^*)\|^2 \\ &\quad + \|J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^k)\|^2)-\|J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^k)\|^2 \\ &\le -\frac{1}{2}\|J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^k)\|^2.\end{aligned}$$

This together with (15) implies that

$$\begin{aligned}\|y^k-x^*\|^2 &\le \|x^k-x^*\|^2+\tau^2\|G\|^2\|J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^k)\|^2 \\ &\quad - \tau\|[J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^k)]\|^2 \\ &= \|x^k-x^*\|^2+(\tau^2\|G\|^2-\tau)\|J^{B_2}_{\mu_2}(I-\mu_2A_2)G(x^k)-G(x^k)\|^2 \\ &\le \|x^k-x^*\|^2.\end{aligned} \tag{18}$$

Substituting (18) into (13) to deduce

$$\begin{aligned}\|z^k - x^*\|^2 &\leq \gamma\rho_1\alpha^k\|x^k - x^*\| + \alpha^k\|\gamma f(x^*) - F(x^*)\| \\ &\quad + (1-\rho_2\alpha^k)\|x^k - x^*\| \\ &= \alpha^k\|\gamma f(x^*) - F(x^*)\| + [1-(\rho_2-\gamma\rho_1)\alpha^k]\|x^k - x^*\|.\end{aligned} \tag{19}$$

According to (14) and (19), we get

$$\begin{aligned}\|x^{k+1} - x^*\| &\leq \alpha^k\|\gamma f(x^*) - F(x^*)\| + [1-(\rho_2-\gamma\rho_1)\alpha^k]\|x^k - x^*\| \\ &\leq \max\{\|x^k - x^*\|, \frac{\|\gamma f(x^*) - F(x^*)\|}{\rho_2 - \gamma\rho_1}\}.\end{aligned}$$

The boundedness of the sequence $\{x^k\}$ yields. Consequently, the sequences $\{y^k\}$ and $\{z^k\}$ are all bounded.

There are two possible cases regarding the convergence analysis of the sequence $\{\|x^k - x^*\|\}_{k\geq 0}$. Case 1. There exists k_0 such that the sequence $\{\|x^k - x^*\|\}_{k\geq k_0}$ is decreasing. In this case, the limitation $\lim_{k\to\infty}\|x^k - x^*\|$ exists. In the light of (13), (14) and (18), we deduce

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\leq [\gamma\rho_1\alpha^k\|x^k - x^*\| + \alpha^k\|\gamma f(x^*) - F(x^*)\| \\ &\quad + (1-\rho_2\alpha^k)\|y^k - x^*\|]^2 \\ &= (\alpha^k)^2(\gamma\rho_1\|x^k - x^*\| + \|\gamma f(x^*) - F(x^*)\|)^2 + 2\alpha^k(1-\rho_2\alpha^k) \\ &\quad \times (\gamma\rho_1\|x^k - x^*\| + \|\gamma f(x^*) - F(x^*)\|)\|y^k - x^*\| \\ &\quad + (1-\rho_2\alpha^k)^2\|y^k - x^*\|^2 \\ &\leq (1-\rho_2\alpha^k)\|y^k - x^*\|^2 + M\alpha^k \\ &\leq (1-\rho_2\alpha^k)(\tau^2\|G\|^2 - \tau)\|J_{\mu_2}^{B_2}(I-\mu_2A_2)G(x^k) - G(x^k)\|^2 \\ &\quad + (1-\rho_2\alpha^k)\|x^k - x^*\|^2 + M\alpha^k,\end{aligned} \tag{20}$$

where $M > 0$ is a constant such that

$$\begin{aligned}&\sup_k\{(\gamma\rho_1\|x^k - x^*\| + \|\gamma f(x^*) - F(x^*)\|)^2 + 2(\gamma\rho_1\|x^k - x^*\| \\ &+\|\gamma f(x^*) - F(x^*)\|)\|x^k - x^*\|\} \leq M \text{ (by the boundedness of } \{x^k\}).\end{aligned}$$

By (20), we derive

$$\begin{aligned}&(1-\rho_2\alpha^k)(\tau - \tau^2\|G\|^2)\|J_{\mu_2}^{B_2}(I-\mu_2A_2)G(x^k) - G(x^k)\|^2 \\ &\leq (1-\rho_2\alpha^k)\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + M\alpha^k.\end{aligned} \tag{21}$$

Note that $\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 \to 0$ by the existence of $\lim_{k\to\infty}\|x^k - x^*\|$. This fact together with conditions $(C1)$ and $(C2)$ and (21) implies that

$$\lim_{k\to\infty}\|J_{\mu_2}^{B_2}(I-\mu_2A_2)G(x^k) - G(x^k)\| = 0. \tag{22}$$

From (10), we have

$$\begin{aligned}\|y^k - x^k\| &= \|\tau G^*[J_{\mu_2}^{B_2}(I-\mu_2A_2)G(x^k) - G(x^k)]\| \\ &\leq \tau\|G\|\|J_{\mu_2}^{B_2}(I-\mu_2A_2)G(x^k) - G(x^k)\|.\end{aligned}$$

It follows from (22) that

$$\lim_{k\to\infty}\|x^k - y^k\| = 0. \tag{23}$$

Note that $\|z^k - x^k\| \leq \alpha^k \|\gamma f(x^k) - F(y^k)\| + \|y^k - x^k\|$. This together with (23) implies that

$$\lim_{k\to\infty} \|z^k - x^k\| = 0. \tag{24}$$

Since $J_{\mu_1}^{B_1}(I - \mu_1 A_1)$ is averaged by Lemma 2.2, we can write $x^{k+1} = \zeta z^k + (1-\zeta)S(z^k)$ with S being nonexpansive and $\zeta \in (0,1)$. Thus, applying (7), we obtain

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|\zeta(z^k - x^*) + (1-\zeta)(S(z^k) - x^*)\|^2 \\
&= \zeta\|z^k - x^*\|^2 + (1-\zeta)\|S(z^k) - x^*\|^2 - \zeta(1-\zeta)\|z^k - S(z^k)\|^2 \\
&\leq \|z^k - x^*\|^2 - \zeta(1-\zeta)\|z^k - S(z^k)\|^2 \\
&\leq \alpha^k\|\gamma f(x^*) - F(x^*)\|^2/(\rho_2 - \gamma\rho_1) - \zeta(1-\zeta)\|z^k - S(z^k)\|^2 \\
&\quad + [1 - (\rho_2 - \gamma\rho_1)\alpha^k]\|x^k - x^*\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\zeta(1-\zeta)\|z^k - S(z^k)\|^2 &\leq \alpha^k\|\gamma f(x^*) - F(x^*)\|^2/(\rho_2 - \gamma\rho_1) - \|x^{k+1} - x^*\|^2 \\
&\quad + [1 - (\rho_2 - \gamma\rho_1)\alpha^k]\|x^k - x^*\|^2 \\
&\to 0.
\end{aligned}$$

Hence, $\lim_{k\to\infty} \|z^k - S(z^k)\| = 0$. Therefore,

$$\lim_{k\to\infty} \|x^{k+1} - z^k\| = \lim_{k\to\infty} (1-\zeta)\|z^k - S(z^k)\| = 0. \tag{25}$$

Note that $x^{k+1} = J_{\mu_1}^{B_1}(I - \mu_1 A_1)z^k$. So, by (24) and (25), we deduce

$$\lim_{k\to\infty} \|z^k - J_{\mu_1}^{B_1}(I - \mu_1 A_1)z^k\| = 0. \tag{26}$$

Next, we show that $\limsup_{k\to\infty}\langle \gamma f(x^*) - F(x^*), z^k - x^*\rangle \leq 0$. Since $\{z^k\}$ is bounded, without loss of generality, we can choose a subsequence $\{z^{k_i}\}$ of $\{z^k\}$ such that $z^{k_i} \rightharpoonup z$ and

$$\limsup_{k\to\infty}\langle \gamma f(x^*) - F(x^*), z^k - x^*\rangle = \lim_{i\to\infty}\langle \gamma f(x^*) - F(x^*), z^{k_i} - x^*\rangle. \tag{27}$$

Consequently, $x^{k_i} \rightharpoonup z$ and $Gx^{k_i} \rightharpoonup Gz$. By Lemma 2.3 and (26), we deduce $z \in Fix(J_{\mu_1}^{B_1}(I - \mu_1 A_1))$. By Lemma 2.3 and (22), we deduce $Gz \in Fix(J_{\mu_2}^{B_2}(I - \mu_2 A_2))$. Thus, $z \in \Gamma$. By (27), we deduce

$$\begin{aligned}
\limsup_{k\to\infty}\langle \gamma f(x^*) - F(x^*), z^k - x^*\rangle &= \lim_{i\to\infty}\langle \gamma f(x^*) - F(x^*), z^{k_i} - x^*\rangle \\
&= \langle \gamma f(x^*) - F(x^*), z - x^*\rangle \leq 0.
\end{aligned} \tag{28}$$

Applying (9), we have

$$\begin{aligned}
\|z^k - x^*\|^2 &= \|(I - \alpha^k F)(y^k - x^*) + \alpha^k(\gamma f(x^k) - F(x^*))\|^2 \\
&\leq (1 - \rho_2\alpha^k)^2\|y^k - x^*\|^2 + 2\alpha^k\langle \gamma f(x^k) - F(x^*), z^k - x^*\rangle \\
&\leq (1 - \rho_2\alpha^k)^2\|x^k - x^*\|^2 + 2\gamma\alpha^k\langle f(x^k) - f(x^*), z^k - x^*\rangle \\
&\quad + 2\alpha^k\langle \gamma f(x^*) - F(x^*), z^k - x^*\rangle \\
&\leq (1 - \rho_2\alpha^k)^2\|x^k - x^*\|^2 + 2\gamma\rho_1\alpha^k\|x^k - x^*\|\|z^k - x^*\| \\
&\quad + 2\alpha^k\langle \gamma f(x^*) - F(x^*), z^k - x^*\rangle \\
&\leq (1 - \rho_2\alpha^k)^2\|x^k - x^*\|^2 + \gamma\rho_1\alpha^k\|x^k - x^*\|^2 \\
&\quad + \gamma\rho_1\alpha^k\|z^k - x^*\|^2 + 2\alpha^k\langle \gamma f(x^*) - F(x^*), z^k - x^*\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}\|z^k - x^*\|^2 &\le \left[1 - \frac{2(\rho_2 - \gamma\rho_1)\alpha^k}{1 - \gamma\rho_1\alpha^k}\right]\|x^k - x^*\|^2 + \frac{(\rho_2\alpha^k)^2}{1 - \gamma\rho_1\alpha^k}\|x^k - x^*\|^2 \\ &\quad + \frac{2\alpha^k}{1 - \gamma\rho_1\alpha^k}\langle \gamma f(x^*) - F(x^*), z^k - x^*\rangle \\ &\le \left[1 - \frac{2(\rho_2 - \gamma\rho_1)\alpha^k}{1 - \gamma\rho_1\alpha^k}\right]\|x^k - x^*\|^2 + \frac{(\rho_2\alpha^k)^2}{1 - \gamma\rho_1\alpha^k}M \\ &\quad + \frac{2\alpha^k}{1 - \gamma\rho_1\alpha^k}\langle \gamma f(x^*) - F(x^*), z^k - x^*\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\le \left[1 - \frac{2(\rho_2 - \gamma\rho_1)\alpha^k}{1 - \gamma\rho_1\alpha^k}\right]\|x^k - x^*\|^2 + \frac{(\rho_2\alpha^k)^2}{1 - \gamma\rho_1\alpha^k}M \\ &\quad + \frac{2\alpha^k}{1 - \gamma\rho_1\alpha^k}\langle \gamma f(x^*) - F(x^*), z^k - x^*\rangle.\end{aligned} \tag{29}$$

Applying Lemma 2.4 and (28) to (29), we deduce $x^k \to x^*$.

Case 2. For any k^0, there exist integer $m \ge k^0$ such that $\|x^m - x^*\| \le \|x^{m+1} - x^*\|$. Set $\Omega_k = \{\|x^k - x^*\|\}$. Then, we have $\Omega_{k^0} \le \Omega_{k^0+1}$. Define an integer sequence $\{\tau_n\}$ for all $k \ge k^0$ by $\tau(k) = \max\{i \in \mathbb{N} | k^0 \le i \le k, \Omega_i \le \Omega_{i+1}\}$. It is clear that $\tau(k)$ is a non-decreasing sequence satisfying $\lim_{k\to\infty} \tau(k) = \infty$ and $\Omega_{\tau(k)} \le \Omega_{\tau(k)+1}$, for all $k \ge k^0$. By the similar argument as that of Case 1, we can obtain $\lim_{k\to\infty} \|x^{\tau(k)} - J_{\mu_1}^{B_1}(I - \mu_1 A_1)x^{\tau(k)}\| = 0$ and $\lim_{k\to\infty} \|J_{\mu_2}^{B_2}(I - \mu_2 A_2)G(x^{\tau(k)}) - G(x^{\tau(k)})\| = 0$. Thus, all weak cluster points $\omega_w(x^{\tau(k)}) \subset \Gamma$. Consequently,

$$\limsup_{k\to\infty}\langle \gamma f(x^*) - F(x^*), z^{\tau(k)} - x^*\rangle \le 0. \tag{30}$$

Since $\Omega_{\tau(k)} \le \Omega_{\tau(k)+1}$, we have from (29) that

$$\begin{aligned}\Omega^2_{\tau(k)} &\le \Omega^2_{\tau(k)+1} \\ &\le \left[1 - \frac{2(\rho_2 - \gamma\rho_1)\alpha^{\tau(k)}}{1 - \gamma\rho_1\alpha^{\tau(k)}}\right]\Omega^2_{\tau(k)} + \frac{(\rho_2\alpha^{\tau(k)})^2}{1 - \gamma\rho_1\alpha^{\tau(k)}}M \\ &\quad + \frac{2\alpha^{\tau(k)}}{1 - \gamma\rho_1\alpha^{\tau(k)}}\langle \gamma f(x^*) - F(x^*), z^{\tau(k)} - x^*\rangle.\end{aligned} \tag{31}$$

It follows that

$$\Omega^2_{\tau(k)} \le \frac{1}{\rho_2 - \gamma\rho_1}\langle \gamma f(x^*) - F(x^*), z^{\tau(n)} - x^*\rangle + \frac{(\rho_2)^2\alpha^{\tau(k)}}{2(\rho_2 - \gamma\rho_1)}M. \tag{32}$$

Combining (30) and (32), we have $\limsup_{k\to\infty} \Omega_{\tau(k)} \le 0$ and hence

$$\lim_{k\to\infty} \Omega_{\tau(k)} = 0. \tag{33}$$

From (31), we deduce that $\limsup_{k\to\infty} \Omega^2_{\tau(k)+1} \le \limsup_{k\to\infty} \Omega^2_{\tau(k)}$. This together with (33) implies that $\lim_{k\to\infty} \Omega_{\tau(k)+1} = 0$. Applying Lemma 2.5 to get $0 \le \Omega_k \le \max\{\Omega_{\tau(k)}, \Omega_{\tau(k)+1}\}$. Therefore, $\Omega_k \to 0$. That is, $x^k \to x^*$. This completes the proof. □

Algorithm 3.2. *Let $\{\alpha^k\}$ be a sequence in $[0,1]$. Let τ, μ_1 and μ_2 be three positive constants. Choose an arbitrary initial guess $x^0 \in H_1$. Compute the sequences $\{y^k\}$ and $\{x^k\}$ via the following iterations*

$$y^k = x^k + \tau G^*[J_{\mu_2}^{B_2}(I - \mu_2 A_2)G(x^k) - G(x^k)], k \ge 0,$$

and

$$x^{k+1} = J_{\mu_1}^{B_1}(I - \mu_1 A_1)[(1-\alpha^k)y^k], k \geq 0. \quad (34)$$

Corollary 3.1. *Assume that the following conditions are satisfied*
$(C1)$: $\lim_{k\to\infty} \alpha^k = 0$ *and* $\sum_{k=0}^{\infty} \alpha^k = \infty$*;*
$(C3)$: $\mu_1 \in (0, 2\alpha_1)$*,* $\mu_2 \in (0, 2\alpha_2)$ *and* $\tau \in (0, 1/\|G\|^2)$*.*
Then the sequence $\{x^k\}$ *generated by* (34) *converges strongly to* $proj_\Gamma(0)$*, the minimum norm solution in* Γ*.*

Algorithm 3.3. *Let* $\{\alpha^k\}$ *be a sequence in* $[0, 1]$*. Let* τ*,* μ_1*,* μ_2 *and* γ *be four positive constants. Choose an arbitrary initial guess* $x^0 \in H_1$*. Compute the sequences* $\{y^k\}$ *and* $\{x^k\}$ *via the following iterations*

$$y^k = x^k + \tau G^*[J_{\mu_2}^{B_2} G(x^k) - G(x^k)], k \geq 0,$$

and

$$x^{k+1} = J_{\mu_1}^{B_1}[\gamma\alpha^k f(x^k) + (I - \alpha^k F)y^k], k \geq 0. \quad (35)$$

Corollary 3.2. *Suppose* $\Gamma_1 \neq \emptyset$*. Assume that the following conditions are satisfied*
$(C1)$: $\lim_{k\to\infty} \alpha^k = 0$ *and* $\sum_{k=0}^{\infty} \alpha^k = \infty$*;*
$(C4)$: $\mu_1 \in (0, \infty)$*,* $\mu_2 \in (0, \infty)$*,* $\tau \in (0, 1/\|G\|^2)$ *and* $1 > \rho_2 > \gamma\rho_1$*.*
Then the sequence $\{x^k\}$ *generated by* (35) *converges strongly to* $x^* = proj_{\Gamma_1}(\gamma f + I - F)x^*$*.*

Algorithm 3.4. *Let* $\{\alpha^k\}$ *be a sequence in* $[0, 1]$*. Let* τ*,* μ_1 *and* μ_2 *be three positive constants. Choose an arbitrary initial guess* $x^0 \in H_1$*. Compute the sequences* $\{y^k\}$ *and* $\{x^k\}$ *via the following iterations*

$$y^k = x^k + \tau G^*[J_{\mu_2}^{B_2} G(x^k) - G(x^k)], k \geq 0,$$

and

$$x^{k+1} = J_{\mu_1}^{B_1}[(1-\alpha^k)y^k], k \geq 0. \quad (36)$$

Corollary 3.3. *Suppose* $\Gamma_1 \neq \emptyset$*. Assume that the following conditions are satisfied*
$(C1)$: $\lim_{k\to\infty} \alpha^k = 0$ *and* $\sum_{k=0}^{\infty} \alpha^k = \infty$*;*
$(C5)$: $\mu_1 \in (0, \infty)$ *and* $\mu_2 \in (0, \infty)$*,* $\tau \in (0, 1/\|G\|^2)$*.*
Then the sequence $\{x^k\}$ *generated by* (36) *converges strongly to* $proj_{\Gamma_1}(0)$*, the minimum norm solution in* Γ_1*.*

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