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NON-LINEAR BEHAVIORS OF AIRY TYPE ON SCALES SPACE FROM A FRACTAL PERSPECTIVE

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Several non-linear behaviors in scales space, in the framework of Scale Relativity Theory, are highlighted. All these are possible through the employment of fractal-type Airy functions, which allow a revaluation of the wave/corpuscle duality, from the perspective of a fractal paradigm. In such a context, it is possible to distinguish the prevalence of either the corpuscle character, or of the wave character, in any experiment which implies microparticles.

Keywords: fractal object, scales space, Schrödinger representation, hydrodynamic representation, Airy function

1. Introduction

Regular models used to describe physical system dynamics are functional:

i) models developed on spaces with integer dimension – differentiable models (for example Classical Mechanics, Quantum Mechanics, etc.) [1-3];

ii) models developed on spaces with non–integer dimensions, which are explicitly written through fractional derivatives [4, 5] – non-differentiable models (for example fractal models).

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Expanding on both types of models, new developments have been made. These are based on Scale Relativity Theory, either in the monofractal dynamics as in the case of Nottale [6] in fractal dimension D_f=2, or in the multifractal dynamics as in the case of the Fractal Theory of Motion [7-17].

Regardless of the type of model in discussion, the fundamental hypothesis is the following: supposing that any physical system was assimilated both structurally and functionally to a fractal object, said dynamics can be described through motions of any physical system entity, dependent on the chosen scale resolution, on continuous and non – differentiable curves (fractal curves).

All these considerations imply that, in the description of any physical system dynamics, instead of "working" with a single variable (regardless of its nature, *i.e.*, velocity, density, etc.) described by a strict non – differentiable function, it is possible to "work" only with approximations of this mathematical function, obtained by averaging them on different scale resolutions. As a consequence, any variable purposed to describe any physical system dynamics will perform as the limit of a family of mathematical functions, this being non – differentiable for null scale resolutions and differentiable otherwise [6-8]. To put it differently, from a mathematical point of view, these variables could be explained through fractal functions, *i.e.* functions dependent not only on spatial and temporal coordinates, but also on the scale resolution.

All of the above specify the fact that any description of the physical system dynamics requires simultaneous dynamics descriptions, in the framework of the Scale Relativity Theory, on two manifolds: one on the usual space and the other one on the scales space. Thus, the fundamental assumption of the present proposed model is the one that the dynamics of any entity of the physical system will be described by continuous but non – differentiable motion curves (fractal motion curves), but only on the scale space, as the "classic" usual space models being commonly employed and generally well-known.

2. Mathematical Model

2.1 "Holographic implementations" of physical system dynamics on the scales space. A short reminder

Let it be considered a fractal function $F(x)$ where $x \in [a,b]$. With the help of this function, it is possible to describe any physical system dynamics. In such a context, let the sequences of values for x be:

$$x_a = x_0, x_1 = x_0 + \varepsilon, \dots, x_k = x_0 + k\varepsilon, \dots, x_n = x_0 + n\varepsilon = x_b \quad (1)$$

This sequence will correspond to $F(x,\varepsilon)$ as the broken line that connects the points:

$$F(x_0), \dots, F(x_k), \dots, F(x_n) \tag{2}$$

The so-called broken line can be defined as an ε-scale approximation of $F(x)$, i.e $F(x, \varepsilon)$.

In the same context, let another scale be considered with its $\bar{\varepsilon}$-scale approximation of $F(x, \bar{\varepsilon})$. Since $F(x)$ is a fractal function, it is self-similar almost everywhere, thus it can be translated into a property of holography (every part reflects the whole and vice versa) [6, 18-20]. To put it differently, the topic of "holographic implementations" of the physical system dynamics in the scale space becomes operational too. Let it be noted that the same result can be obtained if ε and $\bar{\varepsilon}$ are sufficiently small. Now, comparing the two approximations (ε and $\bar{\varepsilon}$), an infinitesimal increase/decrease $d\varepsilon$ of ε corresponds to an infinitesimal increase/decrease $d\bar{\varepsilon}$ of $\bar{\varepsilon}$. Consequently:

$$\frac{d\varepsilon}{\varepsilon} = \frac{d\bar{\varepsilon}}{\bar{\varepsilon}} = d\mu \tag{3}$$

In this approach, the scale transition from $\varepsilon + d\varepsilon$ to $d\varepsilon$ must be invariant. It results,

$$\varepsilon' = \varepsilon + d\varepsilon = \varepsilon + \varepsilon d\mu \tag{4}$$

Introducing (4) for the fractal function $F(x, \varepsilon)$, it results that:

$$F(x, \varepsilon') = F(x, \varepsilon + \varepsilon d\mu) \tag{5}$$

From here, in a first approximation,

$$F(x, \varepsilon') = F(x, \varepsilon) + \frac{\partial F}{\partial \varepsilon}(\varepsilon' - \varepsilon) \tag{6}$$

which implies:

$$F(x, \varepsilon') = F(x, \varepsilon) + \frac{\partial F}{\partial \varepsilon}\varepsilon d\mu \tag{7}$$

Let it be observed that for an arbitrary, but fixed ε_0, there will be:

$$\frac{\partial \ln\left(\frac{\varepsilon}{\varepsilon_0}\right)}{\partial \varepsilon} = \frac{\partial(\ln \varepsilon - \ln \varepsilon_0)}{\partial \varepsilon} = \frac{1}{\varepsilon} \tag{8}$$

As such, (6) can be written as:

$$F(x, \varepsilon') = F(x, \varepsilon) + \frac{\partial F(x, \varepsilon)}{\partial \ln\left(\frac{\varepsilon}{\varepsilon_0}\right)} d\mu \tag{9}$$

In the end,

$$F(x,\varepsilon') = \left(1 + \frac{\partial}{\partial \ln\left(\frac{\varepsilon}{\varepsilon_0}\right)} d\mu\right) F(x,\varepsilon) \tag{10}$$

so that the operator:

$$\widehat{D} = \frac{\partial}{\partial \ln\left(\frac{\varepsilon}{\varepsilon_0}\right)} \tag{11}$$

acts as a dilation/contraction operator, depending on the given process [21]. Thus, the invariance of equations that describe any physical system dynamics in the scales space is explicitly expressed, irrespective if one of these equations is changed if the operator is applied, while specifying that the intrinsic variation of the resolution is $\ln(\varepsilon/\varepsilon_0)$.

As a conclusion, in the scales space, any physical system dynamics can be described by means of two fundamental variables: first being logarithms of resolutions and the second one, the scale time.

Considering all of the above, since the scale space is now generalized to a non-differentiable and fractal geometry, the various elements of the new description also can be used [6]:

i) Infinity of trajectories, leading to the introduction of a scale velocity field $\mathbb{V} = \mathbb{V}(\ln \mathcal{L}(\tau), \tau)$, where $\mathcal{L}$ is the non-differential space scale coordinate, and τ is the time scale coordinate;

ii) Decomposition of the derivative of the fractal space scale coordinate in terms of a "classical part" and a "fractal part", described by a stochastic variable as in the case of the usual space such that

$$\langle d\xi_s^2 \rangle = 2\mathcal{D}_s d\tau \tag{12}$$

In (12), $\mathcal{D}_s$ is a constant coefficient assimilated to fractal-nonfractal transition in the scales space and ξ_s is the fractal part of the differential spatial coordinate in the scales space;

iii) Introduction of the two-valuedness of this derivative because of the symmetry breaking of the reflection invariance under the exchange $d\tau \leftrightarrow -d\tau$, leading to construct a complex scale velocity $\widetilde{\mathbb{V}}$ based on this two-valuedness;

iv) Construction of a new total covariant derivative with respect to the τ which can be written

$$\frac{\hat{d}}{d\tau} = \frac{\partial}{\partial \tau} + \widetilde{\mathbb{V}} \frac{\partial}{\partial \ln \mathcal{L}} - i\mathcal{D}_s \frac{\partial^2}{(\partial \ln \mathcal{L})^2} \tag{13}$$

v) Introduction of a wave function as a re-expression of the action, which is now complex

$$\Psi_s(\ln \mathcal{L}) = exp(iS_s/2\mathcal{D}_s) \tag{14}$$

In (14), Ψ_s represents the state function in the scale space, and S_s is the action in the scales space;

vi) Transformation and integration of the free Newtonian scale-dynamics equation

$$\frac{d^2 \ln \mathcal{L}}{d\tau^2} = 0 \tag{15}$$

under the form of a Schrödinger type equation now acting on scale variables:

$$\mathcal{D}_s^2 \frac{\partial^2 \Psi_s}{(\partial \ln \mathcal{L})^2} + i\mathcal{D}_s \frac{\partial \Psi_s}{\partial \tau} = 0 \tag{16}$$

2.2 Solutions of one-dimensional Schrödinger equations of fractal type in scales space

The solution of the one-dimensional Schrödinger equation of fractal type in the compact form, in the scales space, (i.e. a generalisation of (16) for an arbitrary fractal dimension)

$$\mu_s^2 \partial_{l_s} \partial^{l_s} \Psi(x_s, t_s) + i\mu_s \partial_{t_s} \Psi(x_s, t_s) = 0, \tag{17}$$

can be written in the form

$$\Psi(x_s, t_s) = \frac{1}{\sqrt{t_s}} \exp\left(i \frac{{x_s}^2}{4\mu_s t_s}\right),$$
$$\mu_s = \mathcal{D}_s [(d\tau)]^{\frac{2}{D_f}-1}, \quad x_s = \ln \mathcal{L}, \quad t_s = \tau \tag{18}$$

and is defined, of course, up to an arbitrary multiplicative constant. In the above relations, and also in the following ones, the indexation with "s" defines the variables and parameters of dynamics in the scales space, $d\tau$ is the scale resolution and D_f is the fractal dimension.

As such, the general solution of equation (17) can be written as a linear superposition of the form:

$$\Psi(x_s, t_s) = \frac{1}{\sqrt{t_s}} \int_{-\infty}^{+\infty} u(y_s) exp\left[i \frac{(x_s - y_s)^2}{4\mu_s t_s}\right] dy_s \tag{19}$$

Now, if $u(y_s)$ is an Airy function of fractal type, then $\Psi(x_s,t_s)$ retains this property, in the sense that its amplitude is an Airy function of fractal type. Indeed, in this case there will be:

$$u(y_s) \equiv Ai(y_s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} exp\left[i\left(\frac{\omega_s^3}{3} + \omega_s y_s\right)\right] d\omega_s \tag{20}$$

in such a way as the state function (29) will be written in the form:

$$\Psi(x_s,t_s) = \frac{1}{2\pi\sqrt{t_{u,s}}}\int_{-\infty}^{+\infty} exp\left\{i\left[\frac{\omega_s^3}{3} + \omega_s y_s + \frac{(x_s - y_s)^2}{4\mu_s t_s}\right]\right\} dy_s d\omega_s \tag{21}$$

If, at first, the integration will be carried out after y_s, up to a multiplicative constant, the results is:

$$\Psi(x_s,t_s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} exp\left[i\left(\frac{\omega_s^3}{3} + \omega x_s - \mu_s t_s \omega_s^2\right)\right] d\omega_s \tag{22}$$

The final result is obtained based on a special relation developed in [21] and it is:

$$\Psi(x_s,t_s) = [Ai(k_s x_s - {v_s}^2{t_s}^2)]exp\left[iv_s t_s\left(k_s x_s - \frac{2}{3}{v_s}^2{t_s}^2\right)\right] \tag{23}$$

with

$$v_s = {k_s}^2\mu_s \tag{24}$$

In these conditions, if Ψ is chosen in the form:

$$\Psi(x_s,t_s) = A(x_s,t_s)\exp\left[i\phi(x_s,t_s)\right] \tag{25}$$

where $A(x_s,t_s)$ is an amplitude and $\Phi(x_s,t_s)$ is a phase, by identifying in (23) the amplitude and the phase, there will be:

$$A(x_s,t_s) = Ai(k_s x_s - {v_s}^2{t_s}^2), \tag{26}$$

$$\phi(x_s,t_s) = v_s t_s\left(k_s x_s - \frac{2}{3}{v_s}^2{t_s}^2\right) \tag{27}$$

Taking into account the asymptotic behavior of the function $Ai(z)$ in its general form:

$$Ai(z_s)\sim\begin{cases}\frac{1}{2\pi^{1/2}}{z_s}^{-1/4}exp\left(-\frac{2}{3}{z_s}^{3/2}\right), & z_s\to+\infty\\ \frac{1}{\pi^{1/2}}|z_s|^{-1/4}\sin\left(\frac{2}{3}|z_s|^{3/2}+\frac{\pi}{4}\right), & z_s\to-\infty\end{cases}\qquad(28)$$

the state (25) with (26), (27) function in the asymptotic limit $\Psi\to\Psi_A$ becomes:

$$\Psi_A\sim\begin{cases}\frac{1}{2\pi^{1/2}}(k_sx_s-{v_s}^2{t_s}^2)^{-1/4}exp\begin{bmatrix}-\frac{2}{3}(k_sx_s-{v_s}^2{t_s}^2)^{\frac{3}{2}}\\+iv_st_s\left(k_sx_s-\frac{2}{3}{v_s}^2{t_s}^2\right)\end{bmatrix}\\ \frac{1}{\pi^{1/2}}|k_sx_s-{v_s}^2{t_s}^2|^{-1/4}\sin\begin{matrix}\left[\frac{2}{3}|k_sx_s-{v_s}^2{t_s}^2|^{\frac{3}{2}}+\frac{\pi}{4}\right]\\ exp\left[iv_st_s\left(k_sx_s-\frac{2}{3}{v_s}^2{t_s}^2\right)\right]\end{matrix}\end{cases}\qquad(29)$$

3. Results and Discussion

In Figure 1 it is represented the 3D and contour plot representation of the wave function as defined through (25). It can be observed that the wave function can be influenced by a wide series of factors, including time. For relative low values of the control parameters (*t, v, k*) the function follows closely the Airy type representation which is dominant in the (25). When the system evolved at longer moment of time, we observe a self-modulation of the system and a change in frequency as both time and spatial coordinate are varied. The modulation appears on the temporal axis and defines a complex behavior on the spatial coordinate.

This means that, by simply selecting the appropriate scale (defined by: *z,t,v,k*), one can better investigate particular dynamics of the system. The self-modulation is better seen when the properties of the wave are modified (through *k*) where it is possible to see that for $k = 6$, $v = 3$ and $t =1$, the wave-function defined wave like structure in the scales space coordinate while in time it complicates features with multiple oscillation frequencies. This means that by tailoring the resolution scale at which a system is investigated it is possible to transition from a time or space modulated structure which characterizes particular phenomena. Further understanding of the space-time modulation of the wave function could become important when investigating transient phenomena like laser produced plasma, or complex fluid flows where often the there are reports of temporal analysis for a fixed spatial volume or spatial analysis for a fixed moment of time [22-25].

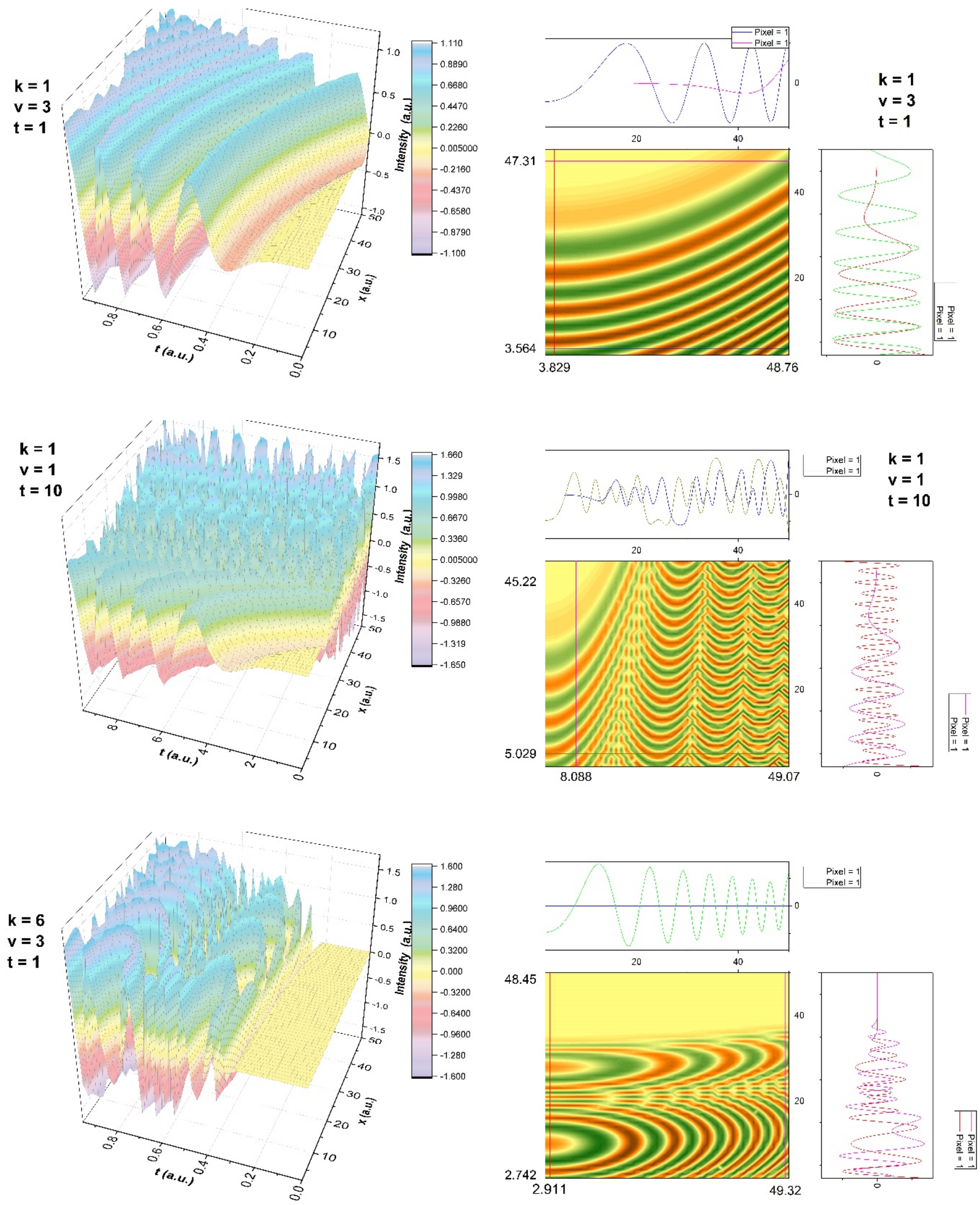

Fig. 1. 3D and contour plot representation of the wave function given through (25)

By substituting (25) in (17), by means of direct calculation, the following relation is checked:

$$\begin{aligned} & i\partial_{t_s}\Psi + \mu_s \partial_{l_s}\partial^{l_s}\Psi \\ & = -\left[\partial_{t_s}\phi + \mu_s(\partial_{l_s}\phi)^2 - \mu_s \frac{\partial_{l_s}\partial^{l_s}\mathrm{A}}{A}\right] \\ & + \frac{i}{2A^2}\left[\partial_{t_s}A^2 + 2\mu_s \partial_{l_s}(A^2\partial^{l_s}\phi)\right] \end{aligned} \tag{30}$$

Now, the "specific constraints" necessary for Ψ to be a solution of the non-stationary differential equation (30) will be reducible to the differential equations:

$$\begin{aligned} \partial_{t_s}\phi + \mu_s\left(\partial_{l_s}\phi\partial^{l_s}\phi\right) = \mu_s \frac{\partial_{l_s}\partial^{l_s}A}{A} \\ \partial_{t_s}A^2 + 2\mu_s\left(A^2\partial_{l_s}\phi\right) = 0 \end{aligned} \tag{31}$$

The first of these equations is the Hamilton – Jacobi equation of fractal type, while the second equation is the continuity equation of fractal type. From here, the correspondence with the hydrodynamic model of fractal type pertaining to Scale Relativity [6-8], becomes evident based on the substitutions:

$$V_D^{i_s} = \mu_s \partial^{i_s}\phi, \ \ \rho = A^2 \tag{32}$$

where $V_D^{i_s}$ is the differential component of the velocity field and ρ is the density of states. The conservation law of fractal type of the specific momentum can be found:

$$\partial_{t_s}V_D^{i_s} + V_D^{l_s}\partial_{l_s}V_D^{i_s} = -\partial^{i_s}Q \tag{33}$$

and respectively, the conservation law of the density of states of fractal type:

$$\partial_{t_s}\rho + \partial^{l_s}\left(\rho V_D^{l_s}\right) = 0 \tag{34}$$

The specific potential of fractal type:

$$Q = -{\mu_s}^2 \frac{\partial_{l_s}\partial^{l_s}\sqrt{\rho}}{\sqrt{\rho}} \tag{35}$$

through the induced specific force of fractal type:

$$f^{i_s} = -\partial^{i_s}Q = -{\mu_s}^2\partial^{i_s}\left(\frac{\partial_{l_s}\partial^{l_s}\sqrt{\rho}}{\sqrt{\rho}}\right) \tag{36}$$

becomes a measure of the fractal degree pertaining to the motion curves. In such a motion, the "specific constraints" (31) are also checked in detail, with a specific potential of a fractal type:

$$Q(x_s, t_s) = \mu_s\nu_s(k_s x_s - {\nu_s}^2{t_s}^2) \tag{37}$$

suppressing, in the Scale Relativity sense, dynamics with a constant force of fractal type.

As such, the non-stationary Schrödinger equation of fractal type in the scales space, in its “universal” instance given through (17), can lead to a wide range of interpretation. The “fractal object”/particle is in a uniformly accelerated motion of a fractal type (see both the argument of the Airy function of fractal type, as well as the expression of the specific potential of fractal type). This, evidently, holds with the condition of accepting the functionality of the de Broglie theory of fractal type, linked to “wave phenomena named fractal object”. The generation of probability densities is given by the square of the Airy function of fractal type. As this cannot be integrated on the entire real straight line, the state/wave package of fractal type can have a center which is achievable in the sense of de Broglie theory of fractal type. Then, the state/wave package of fractal type represents an ensemble of fractal objects/particles which all have a uniform rectilinear motion, but each with a different velocity, and the argument of the Airy function of fractal type represents a caustic in the dynamics space (the envelope of the ensemble of geodesics which represents the corresponding trajectories). Such an interpretation is linked with the nature of the invention of the Airy function: the behavior of light in the proximity of caustics. The functionality of an equivalence principle of fractal type, which implies the fact that the Airy state/wave package of fractal type will not scatter, because it represents a fractal object/particle in an enclosure analogous to Einstein’s Elevator, the uniform field of gravitational forces being thus suppressed.

4. Conclusions

The conclusions of the present paper are the following:

i) A short round-up of the main results of the Scale Relativity Theory on the scales space was presented. In such a context, it was shown that the dynamics of any physical system can be described through a fractal-type Schrödinger equation, i.e. a Schrödinger equation in space-time scale coordinates, at various fractal degrees;

ii) For the one-dimensional case of a fractal-type Schrödinger equation in the space scale, an Airy solution of fractal type was obtained;

iii) A link between the fractal-type Schrödinger model and fractal-type hydrodynamic model was established. In such a context, the existence of the Airy solution of fractal-type for a fractal-type Schrödinger equation, allowed the explaining of the scalar potential of fractal type;

iv) Several interpretations of the obtained results were given, in terms of the presented solution, which highlights the non-linear behaviors of the wave/corpuscle duality. In such a conjecture, it is possible to distinguish the prevalence of either the corpuscle character, or of the wave character, in any experiment which implies microparticles.

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