WRIGHT DISTRIBUTION AND ITS APPLICATIONS ON UNIVALENT FUNCTIONS

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The purpose of the present paper is to introduce the Wright distribution and by using this distribution, we introduce a Wright distribution series and an integral operator associated with this series. Some sufficient conditions and inclusion relations have been obtained for this series and its integral operator belonging to certain classes of univalent functions.

Keywords: Analytic function, Wright function, Univalent function, Probability distribution.
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1. Introduction

In 1933, Wright [24] introduced a special function which is named as Wright function and defined in the following way

$$W_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\lambda n+\mu)}, \tag{1}$$

where $\lambda>-1,\ \mu\in\mathbb{C}$ and $\Gamma(\cdot)$ stands for the usual Gamma function. The series given by (1) is absolutely convergent for all $z\in\mathcal{C}$, while for $\lambda=-1$ this is absolutely convergent in $\mathbb{U}$. He also proved that it is an entire function for $\lambda>-1$. For more basic properties on Wright functions one may refer to Gorenflo et al. [10] and Mustafa [14]. It is easy to see that the series (1) is not in normalized form so we normalized it as

$$\mathbb{W}_{\lambda,\mu}(z)=\Gamma(\mu)zW_{\lambda,\mu}(z)$$

$$\mathbb{W}_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{\Gamma(\mu)z^{n+1}}{n!\Gamma\left(\lambda n+\mu\right)} \tag{2}$$

where $\lambda>-1,\ \mu>0,\ z\in\mathbb{U}$.

Now, we introduce Wright distribution in the following way, first we define the series

$$\mathbb{W}_{\lambda,\mu}(m)=\sum_{n=0}^{\infty}\frac{\Gamma(\mu)m^{n+1}}{n!\Gamma\left(\lambda n+\mu\right)} \tag{3}$$

which is convergent for all $\lambda,\mu,m>0$.

The probability mass function of Wright distribution is given by

$$p(n)=\frac{\Gamma(\mu)m^{n+1}}{n!\Gamma\left(\lambda n+\mu\right)\ \mathbb{W}_{\lambda,\mu}(m)},\ m,\mu,\lambda>0,\ n=0,1,2,3,\dots \tag{4}$$

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It is worthy to note that for $\lambda = 0$ it reduces to the Poisson distribution.

2. **Applications on Univalent functions**

Let $\mathcal{A}$ stand for the class of functions f of the form

$$f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \tag{5}$$

which are analytic in the open unit disk $\mathbb{U} = \{\xi : \xi \in \mathbb{C} \text{ and } |\xi| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Further, we let $\mathcal{S}$ represent the subclass of $\mathcal{A}$ consisting of functions f of the form (5) which are also univalent in $\mathbb{U}$.

Kanas and Wisniowska [11, 12] introduced the class of γ-uniformly convex functions denoted by $\gamma - \mathcal{UCV}$ in $\mathbb{U}$, consisting of functions f of the form (5) which satisfy the following condition

$$\Re\left\{1 + \frac{\xi f''(\xi)}{f'(\xi)}\right\} \geq \gamma \left|\frac{\xi f''(\xi)}{f'(\xi)}\right|, \ \gamma \geq 0, \tag{6}$$

Similarly, they introduced the class of γ-starlike functions denoted by $\gamma - \mathcal{ST}$ in $\mathbb{U}$, which consist of the functions f of the form (5) which satisfy the following analytic criteria

$$\Re\left\{\frac{\xi f'(\xi)}{f(\xi)}\right\} \geq \gamma \left|\frac{\xi f'(\xi)}{f(\xi)} - 1\right|, \ \gamma \geq 0. \tag{7}$$

The classes $\gamma - \mathcal{UCV}$ and $\gamma - \mathcal{ST}$ were further generalized and studied by Bharti et al. [3], Dixit and Porwal [6], (see also [8, 9, 13, 21]). A function $f(\xi) \in \mathcal{A}$ of the form (5) is said to be in the class $\mathcal{S}_\delta^*$ if it satisfied the condition

$$\left|\frac{\xi f'(\xi)}{f(\xi)} - 1\right| < \delta, \ \delta > 0.$$

Similarly, a function $f(\xi) \in \mathcal{A}$ of the form (5) is said to be in the class $\mathcal{C}_\delta$ if it satisfies the following analytic criteria

$$\left|\frac{\xi f''(\xi)}{f'(\xi)}\right| < \delta, \ \delta > 0.$$

The classes $\mathcal{S}_\delta^*$ and $\mathcal{C}_\delta$ were introduced and studied by Ponnusamy and Rønning [16]. In 1995, Dixit and Pal [5] introduced the class $\mathcal{R}^\tau(A, B)$ consisting of functions f of the form (5) which satisfy the following analytic criteria

$$\left|\frac{f'(\xi) - 1}{(A - B)\tau - B(f'(\xi) - 1)}\right| < 1,$$

where $\xi \in \mathbb{U}, \ \tau \in \mathbb{C}\backslash\{0\}, \ -1 \leq B < A \leq 1$.

In 2014, by using the definition of Poisson distribution, Porwal [17] introduced Poisson distribution series and gave a nice application of it on certain classes of univalent functions and opened up a new direction of research in the geometric function theory. After the investigation of this series several researchers investigated various distribution series like Hypergeometric distribution series [1], Pascal distribution series [4], Mittag-Leffler type Poisson distribution series [7], Binomial distribution series [15], generalized distribution series [18], Hypergeometric type distribution series [19], confluent hypergeometric distribution series [20], generalized hypergeometric distribution series [22], Borel distribution series [23] (see also [2]) and obtained various interesting results on certain classes of univalent functions for these series.

Now, using the definition of Wright distribution, we introduce the Wright distribution series as follows

$$K(\lambda, \mu, m, \xi) = \xi + \sum_{n=2}^{\infty} \frac{\Gamma(\mu) m^n}{(n-1)!\Gamma(\lambda(n-1) + \mu)\mathbb{W}_{\lambda,\mu}(m)} \xi^n. \tag{8}$$

The convolution of two power series $f(\xi)$ of the form (5) and $g(\xi) = \xi + \sum_{n=2}^{\infty} b_n \xi^n$ is defined as the power series

$$(f * g)(\xi) = \xi + \sum_{n=2}^{\infty} a_n b_n \xi^n. \tag{9}$$

Now, we introduce the linear operator $I(\lambda, \mu, m) : \mathcal{A} \to \mathcal{A}$ defined as

$$I(\lambda, \mu, m)f(\xi) = f(\xi) * K(\lambda, \mu, m, \xi) = \xi + \sum_{n=2}^{\infty} \frac{\Gamma(\mu) m^n}{\Gamma(\lambda(n-1) + \mu)(n-1)! \mathbb{W}_{\lambda,\mu}(m))} a_n \xi^n. \tag{10}$$

Motivating with the above mentioned work we obtain some sufficient conditions for the Wright distribution series $K(\lambda, \mu, m, \xi)$ belonging to the classes $\gamma - \mathcal{UCV}$, $\gamma - \mathcal{ST}$, S_δ^* and C_δ. Further we obtain some inclusion relations between the classes $\mathcal{R}^\tau(A, B)$, $\gamma - \mathcal{UCV}$ and $\mathcal{C}_\delta$ by applying certain convolution operator $I(\lambda, \mu, m)$ defined by (10).

3. Preliminary results

To establish our main results, we need to recall the following lemmas.

Lemma 3.1. *([12]) Let $f \in \mathcal{A}$ is of the form (5). If for some γ $(0 \le \gamma < \infty)$ the following inequality*

$$\sum_{n=2}^{\infty} [n(1+\gamma) - \gamma]|a_n| \le 1, \tag{11}$$

holds, then $f \in \gamma - \mathcal{ST}$

Lemma 3.2. *([11]) If $f \in \mathcal{A}$ is of the form (5) and satisfies the condition*

$$\sum_{n=2}^{\infty} n[n(1+\gamma) - \gamma]|a_n| \le 1$$

then $f \in \gamma - \mathcal{UCV}$, for some $\gamma(0 \le \gamma < 1)$.

Lemma 3.3. *([11]) If $f \in \mathcal{A}$ is of the form (5) and satisfies the condition*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \le \frac{1}{\gamma + 2}$$

for some $\gamma(0 \le \gamma < 1)$, then $f \in \gamma - \mathcal{UCV}$.

Lemma 3.4. *([16]) If $f \in \mathcal{A}$ is of the form (5) and satisfies the condition*

$$\sum_{n=2}^{\infty} (\delta + n - 1)|a_n| \le \delta, \ (\delta > 0)$$

then $f \in \mathcal{S}_\delta^*$.

Lemma 3.5. *([16]) If $f \in \mathcal{A}$ is of the form (5) and satisfies the condition*

$$\sum_{n=2}^{\infty} n(\delta + n - 1)|a_n| \le \delta \ (\delta > 0)$$

then $f \in \mathcal{C}_\delta$.

Lemma 3.6. *([5]) If $f \in \mathcal{R}^\tau(A, B)$ is of the form (5) then*

$$|a_n| \le \frac{(A - B)|\tau|}{n}, \ n \ge 2.$$

The result is sharp.

4. **Main Results**

In our first theorem, we obtain a sufficient condition for the Wright distribution series $K(\lambda,\mu,m,\xi)$ in the class $\gamma-\mathcal{ST}$.

Theorem 4.1. *If $\lambda,\mu,m>0$ and for some γ $(0\le\gamma<1)$ the condition*

$$\Gamma(\mu)(\gamma+1)\mathbb{W}_{\lambda,\mu+\lambda}(m)\le\Gamma\left(\mu+\lambda\right) \tag{12}$$

is satisfied then $K(\lambda,\mu,m,\xi)\in\gamma-\mathcal{ST}$.

Proof. To prove that $K(\lambda,mu,m,\xi)\in\gamma-\mathcal{ST}$, from Lemma 3.1 it is sufficient to prove that

$$\sum_{n=2}^{\infty}[n(\gamma+1)-\gamma]\frac{\Gamma(\mu)m^n}{(n-1)!\Gamma\left(\lambda(n-1)+\mu\right)\mathbb{W}_{\lambda,\mu}(m)}\le 1.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(\gamma+1)-\gamma]\frac{\Gamma(\mu)m^n}{(n-1)!\Gamma\left(\lambda(n-1)+\mu\right)\mathbb{W}_{\lambda,\mu}(m)}\\
&=\sum_{n=2}^{\infty}\{(\gamma+1)(n-1)+1\}\frac{\Gamma(\mu)m^n}{(n-1)!\Gamma\left(\lambda(n-1)+\mu\right)\mathbb{W}_{\lambda,\mu}(m)}\\
&=\frac{1}{\mathbb{W}_{\lambda,\mu}(m)}\left[\sum_{n=2}^{\infty}(\gamma+1)\Gamma(\mu)\frac{1}{(n-2)!}\frac{m^n}{\Gamma(\lambda(n-1)+\mu)}+\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{(n-1)!}\frac{m^n}{\Gamma\left(\lambda(n-1)+\mu\right)}\right]\\
&=\frac{1}{\mathbb{W}_{\lambda,\mu}(m)}\left[(\gamma+1)\sum_{n=0}^{\infty}\frac{\Gamma(\mu)\ m^{n+2}}{n!\Gamma(\lambda(n+1)+\mu)}+\sum_{n=1}^{\infty}\frac{\Gamma(\mu)\ m^{n+1}}{n!\Gamma(\lambda n+\mu)}\right]\\
&=\frac{1}{\mathbb{W}_{\lambda,\mu}(m)}\left[(\gamma+1)m\frac{\Gamma(\mu)}{\Gamma\left(\mu+\lambda\right)}\mathbb{W}_{\lambda,\mu+\lambda}(m)+\mathbb{W}_{\lambda,\mu}(m)-m\right]\\
&\le 1,\ \text{ by the given hypothesis.}
\end{aligned}$$

Thus the proof of Theorem 4.1 is established. □

Theorem 4.2. *If $\lambda,\mu,m>0$ and for some γ $(0\le\gamma<1)$ the condition*

$$\frac{\Gamma(\mu)}{\Gamma\left(\mu+2\lambda\right)}(\gamma+1)m\mathbb{W}_{\lambda,\mu+2\lambda}(m)+(2\gamma+3)\frac{\Gamma(\mu)}{\Gamma\left(\mu+\lambda\right)}\mathbb{W}_{\lambda,\mu+\lambda}(m)\le 1 \tag{13}$$

is satisfied then $K(\lambda,\mu,m,\xi)\in\gamma-\mathcal{UCV}$.

Proof. The proof of above theorem is much similar to Theorem 4.1. Therefore, we omit the details involved. □

Theorem 4.3. *If $\lambda,\mu,m>0$ and for some δ $(\delta>0)$ the condition*

$$\Gamma\left(\mu\right)\mathbb{W}_{\lambda,\mu+\lambda}(m)\le\delta\Gamma\left(\mu+\lambda\right) \tag{14}$$

is satisfied then $K(\lambda,\mu,m,\xi)\in\mathcal{S}^*_\delta$.

Proof. To prove $K(\lambda,\mu,m,\xi)\in\mathcal{S}^*_\delta$, from Lemma 3.2 it is sufficient to prove that

$$\sum_{n=2}^{\infty}(\delta+n-1)\frac{\Gamma(\mu)m^n}{(n-1)!\Gamma(\lambda(n-1)+\mu)}\frac{1}{\mathbb{W}_{\lambda,\mu}(m)}\le\delta.$$

Now

$$\begin{aligned}
&\frac{1}{\mathbb{W}_{\lambda,\mu}(m)}\sum_{n=2}^{\infty}(n-1+\delta)\frac{\Gamma(\mu)}{(n-1)!}\frac{m^n}{\Gamma(\lambda(n-1)+\mu)}\\
=&\frac{1}{\mathbb{W}_{\lambda,\mu}(m)}\left[\sum_{n=2}^{\infty}\frac{\Gamma(\mu)m^n}{(n-2)!\Gamma(\lambda(n-1)+\mu)}+\sum_{n=2}^{\infty}\frac{\delta\Gamma(\mu)}{(n-1)!}\frac{m^n}{\Gamma(\lambda(n-1)+\mu)}\right]\\
=&\frac{1}{\mathbb{W}_{\lambda,\mu}(m)}[m\frac{\Gamma(\mu)}{\Gamma(\mu+\lambda)}\mathbb{W}_{\lambda,\mu+\lambda}(m)+\delta(\mathbb{W}_{\lambda,\mu}(m)-m)]\\
\leq&\delta \text{ (from (14)).}
\end{aligned}$$

This completes the proof of Theorem 4.3. □

The proof of Theorems 4.4 and 4.5 is similar to that of Theorem 4.3, therefore we only state the results.

Theorem 4.4. *If $\lambda, \mu, m > 0$ and for some δ $(\delta > 0)$ the condition*

$$m\frac{\Gamma(\mu)}{\Gamma(\mu+2\lambda)}\mathbb{W}_{\lambda,\mu+2\lambda}(m)+\delta\frac{\Gamma(\mu)}{\Gamma(\mu+\lambda)}\mathbb{W}_{\lambda,\mu+\lambda}(m)\leq\delta \tag{15}$$

is satisfied then $K(\lambda,\mu,m,\xi)\in\mathcal{C}_\delta$.

Theorem 4.5. *If $\lambda, \mu, m > 0$ and for some γ $(0\leq\gamma<1)$ the condition*

$$m^2\frac{\Gamma(\mu)}{\Gamma(\mu+2\lambda)}\mathbb{W}_{\lambda,\mu+2\lambda}(m)+2m\frac{\Gamma(\mu)}{\Gamma(\mu+\lambda)}\mathbb{W}_{\lambda,\mu+\lambda}(m)\leq\frac{\mathbb{W}_{\lambda,\mu}(m)}{\gamma+2} \tag{16}$$

is satisfied then $K(\lambda,\mu,m,\xi)\in\gamma-\mathcal{UCV}$.

5. **Inclusion relation**

Theorem 5.1. *If $\lambda, \mu, m > 0$, $f\in\mathcal{R}^\tau(A,B)$ and the inequality*

$$\frac{(A-B)|\tau|}{\mathbb{W}_{\lambda,\mu}(m)}[(\gamma+1)m\frac{\Gamma(\mu)}{\Gamma(\mu+\lambda)}\mathbb{W}_{\lambda,\mu+\lambda}(m)+\mathbb{W}_{\lambda,\mu}(m)-m]\leq1 \tag{17}$$

is satisfied then $I(\lambda,\mu,m)f\in\gamma-\mathcal{UCV}$.

Proof. Let f be of the form (5) belonging to the class $\mathcal{R}^\tau(A,B)$. To show that $I(\lambda,\mu,m)f\in\gamma-\mathcal{UCV}$ it is sufficient to prove that

$$\sum_{n=2}^{\infty}n[n(\gamma+1)-\gamma]\frac{\Gamma(\mu)m^n|a_n|}{\mathbb{W}_{\lambda,\mu}(m)(n-1)!\Gamma(\lambda(n-1)+\mu)}\leq1.$$

Since $f\in R^\tau(A,B)$, from Lemma 3.6, we have $|a_n|\leq\frac{(A-B)|\tau|}{n}$.

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n[n(\gamma+1)-\gamma]\frac{\Gamma(\mu)m^n \qquad |a_n|}{\mathbb{W}_{\lambda,\mu}(m)(n-1)!\Gamma(\lambda(n-1)+\mu)}\\
=&\frac{(A-B)|\tau|}{\mathbb{W}_{\lambda,\mu}(m)}\sum_{n=2}^{\infty}[n(\gamma+1)-\gamma]\frac{\Gamma(\mu)m^n}{(n-1)!\Gamma(\lambda(n-1)+\mu)}\\
=&\frac{(A-B)|\tau|}{\mathbb{W}_{\lambda,\mu}(m)}\sum_{n=2}^{\infty}[(n-1)(\gamma+1)+1]\frac{\Gamma(\mu)m^n}{(n-1)!\Gamma(\lambda(n-1)+\mu)}\\
=&\frac{(A-B)|\tau|}{\mathbb{W}_{\lambda,\mu}(m)}\left[(\gamma+1)\sum_{n=2}^{\infty}\frac{\Gamma(\mu)m^n}{(n-2)!\Gamma(\lambda(n-1)+\mu)}+\sum_{n=2}^{\infty}\frac{\Gamma(\mu)m^n}{(n-1)!\Gamma(\lambda(n-1)+\mu)}\right]\\
=&\frac{(A-B)|\tau|}{\mathbb{W}_{\lambda,\mu}(m)}\left[(\gamma+1)m\frac{\Gamma(\mu)}{\Gamma(\mu+\lambda)}\mathbb{W}_{\lambda,\mu+\lambda}(m)+\mathbb{W}_{\lambda,\mu}(m)-m\right]\\
\leq&1, \text{ from (17).}
\end{aligned}$$

This completes the proof of Theorem 5.1. □

Theorem 5.2. *If $\lambda,\mu,m>0$, $f\in\mathcal{R}^{\tau}(A,B)$ and the inequality*

$$(A-B)|\tau|m\frac{\Gamma(\mu)}{\Gamma(\mu+\lambda)}\mathbb{W}_{\lambda,\mu+\lambda}(m)\leq\frac{\mathbb{W}_{\lambda,\mu}(m)}{\gamma+2}$$

is satisfied then $I(\lambda,\mu,m)f\in\gamma-\mathcal{UCV}$.

Proof. Let f be of the form (5) belonging to the class $\mathcal{R}^{\tau}(A,B)$. To show that $I(\lambda,\mu,m)f\in\gamma-\mathcal{UCV}$, from Lemma 3.3, it is sufficient to show that

$$\sum_{n=2}^{\infty}n(n-1)\frac{\Gamma(\mu)}{\mathbb{W}_{\lambda,\mu}(m)}\frac{m^n}{(n-1)!}\frac{|a_n|}{\Gamma(\lambda(n-1)+\mu)}\leq\frac{1}{\gamma+2}.$$

Since $f\in\mathcal{R}^{\tau}(A,B)$, from Lemma 3.6 we have

$$|a_n|\leq\frac{(A-B)|\tau|}{n}.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}n(n-1)\frac{\Gamma(\mu)m^n \qquad |a_n|}{\mathbb{W}_{\lambda,\mu}(m)(n-1)!\Gamma(n-1)+\mu}\\
&\leq\frac{(A-B)|\tau|}{\mathbb{W}_{\lambda,\mu}(m)}\left[\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{(n-2)!}\frac{m^n}{\Gamma(\lambda(n-1)+\mu)}\right]\\
=&\frac{(A-B)|\tau|}{\mathbb{W}_{\lambda,\mu}(m)}m\frac{\Gamma(\mu)}{\Gamma(\mu+\lambda)}\mathbb{W}_{\lambda,\mu+\lambda}(m)\\
\leq&\frac{1}{\gamma+2}, \quad \text{by given hypothesis.}
\end{aligned}$$

Thus the proof of Theorem 5.2 is established. □

Theorem 5.3. *If $\lambda,\mu,m>0$, $f\in\mathcal{R}^{\tau}(A,B)$ and the inequality*

$$\frac{(A-B)|\tau|}{\mathbb{W}_{\lambda,\mu}(m)}[m\frac{\Gamma(\mu)}{\Gamma(\mu+\lambda)}\mathbb{W}_{\lambda,\mu+\lambda}(m)+\delta(\mathbb{W}_{\lambda,\mu}(m)-m)]\leq\delta$$

is satisfied then $I(\lambda,\mu,m)f\in\mathcal{C}_{\delta}$.

Proof. The proof of above theorem is much akin to that of the previous theorem. Hence, we omit the details involved. □

6. An integral operator

In this section, we obtain some sufficient conditions for a particular integral operator $G(\lambda,\mu,m,\xi)$ which is defined in the following way

$$G(\lambda,\mu,m,\xi)=\int_0^z \frac{k(\lambda,\mu,m,t)dt}{t}$$

or

$$G(\lambda,\mu,m,\xi)=\xi+\sum_{n=2}^{\infty}\frac{\Gamma(\mu)m^n}{n!(\Gamma(\lambda(n-1)+\mu))}\frac{\xi^n}{\mathbb{W}_{\lambda,\mu}(m)}. \tag{18}$$

Theorem 6.1. *If $\lambda,\mu,m>0$ and for some γ $(0\le\gamma<1)$ the condition* (12) *is satisfied then $G(\lambda,\mu,m,\xi)\in\gamma-\mathcal{UCV}$.*

Proof. To prove that $G(\lambda,\mu,m,\xi)\in\gamma-\mathcal{UCV}$ from Lemma 3.2, we have to prove that

$$\sum_{n=2}^{\infty}n[n(1+\gamma)-\gamma]\frac{\Gamma(\mu)m^n}{n!\mathbb{W}_{\lambda,\mu}(m)\Gamma(\lambda(n-1)+\mu)}\le 1.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}n[n(1+\gamma)-\gamma]\frac{\Gamma(\mu)m^n}{n!\mathbb{W}_{\lambda,\mu}(m)\Gamma(\lambda(n-1)+\mu)}\\
=&\frac{1}{\mathbb{W}_{\lambda,\mu}(m)}\sum_{n=2}^{\infty}[(\gamma+1)(n-1)+1]\frac{\Gamma(\mu)m^n}{(n-1)!\mathbb{W}_{\lambda,\mu}(m)\Gamma(\lambda(n-1)+\mu)}\\
=&\frac{1}{\mathbb{W}_{\lambda,\mu}(m)}\left[(\gamma+1)\sum_{n=2}^{\infty}\frac{\Gamma(\mu)m^n}{(n-2)!\Gamma(\lambda(n-1)+\mu)}+\sum_{n=2}^{\infty}\frac{\Gamma(\mu)m^n}{(n-1)!\Gamma(\lambda(n-1)+\mu)}\right]\\
=&\frac{1}{\mathbb{W}_{\lambda,\mu}(m)}[(\gamma+1)m\frac{\Gamma(\mu)}{\Gamma(\mu+\lambda)}\mathbb{W}_{\lambda,\mu+\lambda}(m)+\mathbb{W}_{\lambda,\mu}(m)-m]\\
\le& 1,\ \text{from (12)}.
\end{aligned}$$

Thus, the proof of Theorem 6.1 is complete. □

Theorem 6.2. *Let $\lambda,\mu,m>0$ and for some γ $(0\le\gamma<1)$ the inequality*

$$\Gamma(\mu)(\gamma+2)m\mathbb{W}_{\lambda,\mu+\lambda}(m)\le\mathbb{W}_{\lambda,\mu}(m)\Gamma(\mu+\lambda)$$

is satisfied then $G(\lambda,\mu,m,\xi)\in\gamma-\mathcal{UCV}$.

Proof. To prove that $G(\lambda,\mu,m,\xi)\in\gamma-\mathcal{UCV}$ it is sufficient to prove that

$$\sum_{n=2}^{\infty}n(n-1)\frac{\Gamma(\mu)m^n}{n!\mathbb{W}_{\lambda,\mu}(m)\Gamma(\lambda(n-1)+\mu)}\le\frac{1}{\gamma+2}.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}n(n-1)\frac{\Gamma(\mu)m^n}{n!\mathbb{W}_{\lambda,\mu}(m)\Gamma(\lambda(n-1)+\mu)}\\
=&\sum_{n=2}^{\infty}\frac{\Gamma(\mu)m^n}{(n-2)!\Gamma(\lambda(n-1)+\mu)}\\
=&\frac{1}{\mathbb{W}_{\lambda,\mu}(m)}m\frac{\Gamma(\mu)}{\Gamma(\mu+\lambda)}\mathbb{W}_{\lambda,\mu+\lambda}(m),\\
\le&\frac{1}{\gamma+2},\ \text{by the given hypothesis.}
\end{aligned}$$

Thus the proof of Theorem 6.2 is established. □

Theorem 6.3. *If* $\lambda, \mu, m > 0$ *and for some* $\delta > 0$ *the inequality* (14) *is satisfied then* $G(\lambda, \mu, m, \xi) \in \mathcal{C}_\delta$.

Proof. The proof of above theorem is similar to that of Theorem 6.1. Therefore, we omit the details involved. □

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