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OPERATORS, FRAMES AND CONVERGENCE OF SEQUENCES OF BESSEL SEQUENCES

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Let $\mathcal{H}$ be a separable Hilbert space and let $\mathbf{B}$ be the set of all Bessel sequences in $\mathcal{H}$. We give a C^*-algebra structure to $\mathbf{B}$ and we study some properties of multiplication and adjoint that we define there. By introducing the notion of convergence of a sequence of elements in $\mathbf{B}$, we determine whether important properties of the sequence is preserved under the convergence. An interesting result in operator theory helps us to write a Bessel sequence as a multiple of a sum of arbitrary finite number of orthonormal bases for $\mathcal{H}$. Some characterization of Riesz bases and classification of frames and dual frames with respect to frame operators and positive operators are studied.

Keywords: Bessel sequence, Frame, Riesz basis, Positive operator, Banach algebra, C^*-algebra.

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1. Introduction

Frames were first introduced in 1952 by Duffin and Schaeffer [12] in the study of nonharmonic Fourier series. After 30 years, Young [21], Daubechies, Grossmann and Meyer [10] reintroduced frames and they used them as bases in Hilbert spaces and especially in $L^2(\mathbb{R})$. In 1989, Grochenig [13] generalized frames to Banach spaces. Recent researches show that frame theory has applications in pure and applied mathematics, harmonic analysis, engineering, differential and operation equations, and even quantum communication.

Frames have basis-like properties without being bases. In a Hilbert space, a frame can be used to find many different representations of a vector with respect to itself. In addition, the construction of frames is easier than the construction of orthonormal bases. These and many applications of frames demonstrate the interest of studying them. We will briefly recall some definitions and basic properties of frames. For more details, see [7] and [15].
A frame for a nonzero separable Hilbert space $\mathcal{H}$, is a sequence of elements $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$, for which there are positive constants A and B satisfying

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |< f, f_k >|^2 \leq B\|f\|^2,$$

for all $f \in \mathcal{H}$. The numbers A and B are called lower and upper frame bounds, respectively. If $A = B$, it is called a tight frame and for $A = B = 1$ it is a Parseval or a normalized tight frame. A sequence $F = \{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is called a Bessel sequence with Bessel bound B if the second part of the above inequality holds.

Proposition 1.1. *Let $F = \{f_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{H}$ and $T_F : \{c_k\}_{k=1}^{\infty} \longmapsto \sum_{k=1}^{\infty} c_k f_k$, be a relation from $\ell^2(\mathbb{N})$ into $\mathcal{H}$. Then*

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(i) F is a Bessel sequence if and only if T_F is a well-defined linear mapping from $\ell^2(\mathbb{N})$ into $\mathcal{H}$;

(ii) F is a frame for $\mathcal{H}$ if and only if T_F is a well-defined linear mapping from $\ell^2(\mathbb{N})$ onto $\mathcal{H}$.

Indeed in each case of the above proposition T_F is a bounded linear operator. The operator T_F in the Proposition 1.1 and its adjoint

$$T_F^* : \mathcal{H} \longrightarrow \ell^2(\mathbb{N}), \qquad T_F^* f = \{< f, f_k >\}_{k=1}^{\infty},$$

are called the pre-frame operator (or the synthesis operator) and the analysis operator of F, respectively. When F is a frame for $\mathcal{H}$, the operator

$$S_F : \mathcal{H} \longrightarrow \mathcal{H}, \qquad S_F f = T_F T_F^* f = \sum_{k=1}^{\infty} < f, f_k > f_k$$

is called the frame operator of F. The frame operator S_F is bounded, invertible, self-adjoint and positive. The sequence $\{S_F^{-1} f_k\}_{k=1}^{\infty}$ is a frame with frame operator S_F^{-1}. Clearly, $< S_F f, f >= \sum_{k=1}^{\infty} |< f, f_k >|^2$, and for a Parseval frame $F = \{f_k\}_{k=1}^{\infty}$, $S_F = I_{\mathcal{H}}$, where $\mathcal{H}$ is a complex Hilbert space [7, Section 5.1].

By definition a Riesz basis for $\mathcal{H}$ is a sequence of the form $\{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded bijective operator. A Reisz basis for $\mathcal{H}$ is a Schauder basis which is also a frame for $\mathcal{H}$, and then Proposition 1.1 implies that F is a Riesz basis for $\mathcal{H}$ if and only if T_F is an invertible operator from $\ell^2(\mathbb{N})$ onto $\mathcal{H}$. A frame that is not a Reisz basis is said to be overcomplete(or redundant) [7, Section 5.2].

Let $F = \{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with the frame operator S_F, then $f = \sum_{k=1}^{\infty} < f, S_F^{-1} f_k > f_k$, for all $f \in \mathcal{H}$. The frame $\{S_F^{-1} f_k\}_{k=1}^{\infty}$ is called the canonical dual frame of F. Whenever $F = \{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$, by definition a frame $G = \{g_k\}_{k=1}^{\infty}$ is a dual frame of F if $f = \sum_{k=1}^{\infty} < f, g_k > f_k$, for all $f \in \mathcal{H}$. It is well-known that if G is a dual frame of F, then $\sum_{k=1}^{\infty} < f, g_k > f_k = \sum_{k=1}^{\infty} < f, f_k > g_k$ for all $f \in \mathcal{H}$. If $F = \{f_k\}_{k=1}^{\infty}$ is an overcomplete frame for $\mathcal{H}$, then there exist frames $G = \{g_k\}_{k=1}^{\infty} \neq \{S^{-1} f_k\}_{k=1}^{\infty}$ for which $f = \sum_{k=1}^{\infty} < f, g_k > f_k$, for all $f \in \mathcal{H}$. A frame is a Riesz basis if and only if it has a unique dual frame (see [7, Chapter 5] and [15, Chapter 2]).

We denote the set of all Bessel sequences for $\mathcal{H}$ by $\mathbf{B}$ and the set of all frames for $\mathcal{H}$ by $\mathbf{F}$. It is clear that $\mathbf{B}$ is a vector space and F is a subset of $\mathbf{B}$, but F is not its subspace. Recently, we observed that the authors in [9], by giving a structure of Banach space to $\mathbf{B}$ in such a way that its norm topology induced by the operator norm topology on $B(\ell^2(\mathbb{N}), \mathcal{H})$, determined the connected components of the set $\mathbf{F}$ in $\mathbf{B}$, by an interesting technique. The authors in [2], by a kind of continuation of the work [9] and using a bijection between $\mathbf{B}$ and $B(\mathcal{H})$, characterized some subsets of $\mathbf{B}$ associated to different classes of operators in $B(\mathcal{H})$.

Here, by using a relatively different method with the works in [9] and [2], in Section 2, we introduce and study convergence of a sequence of Bessel sequences in $\mathcal{H}$. By using some interesting results in operator theory we study some topological properties of important subsets of $\mathbf{B}$ such as frames, Riesz bases and overcomplete frames and then we investigate the convergence of sequences in these subsets of $\mathbf{B}$. In Section 3 we give a structure of C^*-algebra to $\mathbf{B}$ and we study some properties of multiplication and adjoint in $\mathbf{B}$. By using an important theorem in operator theory in Section 4 we show that a Bessel sequence can be written as a multiple of a sum of arbitrary finite number of orthonormal bases in $\mathbf{B}$. Finally, in Section 5 according to invertible positive operators in $B(\mathcal{H})$ we give a partition of frames and we determine all classes of dual frames for frames with the same frame operator. Also some characterization of Riesz bases will be studied in Section 5.

2. **Convergence of Sequences of Frames**

For Hilbert spaces $\mathcal{H}$ and K, we denote by $B(K,\mathcal{H})$ the Banach space of all bounded operators from K into $\mathcal{H}$, $B(\mathcal{H}) = B(\mathcal{H},\mathcal{H})$, by $R(K,\mathcal{H})$ the set of all invertible operators in $B(K,\mathcal{H})$, $R(\mathcal{H}) = R(\mathcal{H},\mathcal{H})$, by $O(K,\mathcal{H})$ the set of all non-invertible surjective operators in $B(K,\mathcal{H})$, $O(\mathcal{H}) = O(\mathcal{H},\mathcal{H})$, by $F(K,\mathcal{H})$ the set of all surjective operators in $B(K,\mathcal{H})$, $F(\mathcal{H}) = F(\mathcal{H},\mathcal{H})$ and by $U(K,\mathcal{H})$ the set of all surjective isometry operators in $B(K,\mathcal{H})$ and $U(\mathcal{H}) = U(\mathcal{H},\mathcal{H})$.

Now, we recall some elementary results from operator theory that for instance can be found in [1] and [8].

Proposition 2.1. *Let T be an operator in $B(K,\mathcal{H})$, then*

(i) T is a co-isometry (i.e. T^* is an isometry) if and only if $TT^* = I_{\mathcal{H}}$.

(ii) T is a surjective isometry if and only if $TT^* = I_{\mathcal{H}}$ and $T^*T = I_K$.

Proposition 2.2. *$R(K,\mathcal{H})$, $O(K,\mathcal{H})$ and $F(K,\mathcal{H})$ are all open subsets of $B(K,\mathcal{H})$ with respect to the operator norm topology on $B(K,\mathcal{H})$.*

Let $E_0 = \{e_n^0\}_{n=1}^{\infty}$ be an arbitrary orthonormal basis for $\mathcal{H}$. From now on we consider E_0 as a fixed orthonormal basis for $\mathcal{H}$, and $T_{E_0} : \{c_k\}_{k=1}^{\infty} \longmapsto \sum_{k=1}^{\infty} c_k e_n^0$, which is clearly a surjective isometry from $\ell^2(\mathbb{N})$ onto $\mathcal{H}$.

Because of $\|T_{E_0}\| = \|T_{E_0}^{-1}\| = 1$, the assignment $T \longmapsto TT_{E_0}^{-1}$ suggests an isometric isomorphism from $B(\ell^2(\mathbb{N}),\mathcal{H})$ onto $B(\mathcal{H})$ with respect to their corresponding operator norms ($\|T\| = \|TT_{E_0}^{-1}T_{E_0}\| \leq \|TT_{E_0}^{-1}\| \leq \|T\|$).

For each $T \in B(\mathcal{H})$(or $B(\ell^2(\mathbb{N}),\mathcal{H})$) we denote by $nul(T)$ and $def(T)$ the cardinal numbers $\dim KerT$ and $\dim KerT^*$, respectively. Then it follows from the work of Bouldin [4] that:

Proposition 2.3. *Let T be in $B(\mathcal{H})$(or $B(\ell^2(\mathbb{N}),\mathcal{H})$). T belongs to the closure (operator norm closure) of $R(\mathcal{H})$(or $R(\ell^2(\mathbb{N}),\mathcal{H})$) if and only if $nul(T) = def(T)$ or the range of T is not closed.*

Now we define a norm on $\mathbf{B}$ that makes it into a Banach space. For each F in $\mathbf{B}$ we define $\|F\| := \|T_F\|$, where T_F is the pre-frame operator of F. This is well-defined because for each F in $\mathbf{B}$, T_F is unique. For each F and G in $\mathbf{B}$, we have $T_{F+G} = T_F + T_G$ and this together with the completeness of $B(\ell^2(\mathbb{N}),\mathcal{H})$ imply that $(\mathbf{B}, \|.\|)$ is a Banach space, and the mapping $\psi : \mathbf{B} \to B(\ell^2(\mathbb{N}),\mathcal{H})$ by $\psi(F) = T_F$ is an isometric isomorphism.

Proposition 2.4. *Let F be a Bessel sequence in $\mathbf{B}$.*

(i) F is a Parseval frame for $\mathcal{H}$ if and only if T_F is a co-isometry in $B(\ell^2(\mathbb{N}),\mathcal{H})$.

(ii) F is an orthonormal basis for $\mathcal{H}$ if and only if T_F is a surjective isometry in $B(\ell^2(\mathbb{N}),\mathcal{H})$.

We denote the set of all overcomplete frames for $\mathcal{H}$ by $\mathbf{O}$, the set of all Riesz bases for $\mathcal{H}$ by $\mathbf{R}$, the set of all Parseval frames for $\mathcal{H}$ by $\mathbf{P}$ and the set of all orthonormal bases by $\mathbf{U}$, then by Proposition 2.1 and Proposition 2.4, $\mathbf{P} = \{F \in \mathbf{B} : T_F T_F^* = I_{\mathcal{H}}\}$ and $\mathbf{U} = \{F \in \mathbf{B} : T_F T_F^* = I_{\mathcal{H}}, T_F^* T_F = I_{\ell^2(\mathbb{N})}\}$.

Theorem 2.1. *For Banach space $(\mathbf{B}, \|.\|)$ we have:*

(i) $\mathbf{F}$, $\mathbf{R}$ and $\mathbf{O}$ are all open subsets of $\mathbf{B}$.

(ii) $\mathbf{P}$ is a closed subset of $\mathbf{B}$. Moreover, $\mathbf{P}$ is closed relative to $\mathbf{F}$.

Proof. (i) It is an immediate consequence of Proposition 2.2 and considering of the isometric isomorphism ψ from $\mathbf{B}$ onto $B(\ell^2(\mathbb{N}),\mathcal{H})$.

(ii) We define $\mathcal{S} : \mathbf{B} \to B(\mathcal{H})$ by $\mathcal{S}(F) = S_F = T_F T_F^*$, for each F in $\mathbf{B}$. We consider the operator norm topology on $B(\mathcal{H})$. Let $F_0 \in \mathbf{B}$ and $\varepsilon > 0$. Since $\|T_F\| = \|T_F^*\|$, for each F in $\mathbf{B}$, we have

$$\|T_F T_F^* - T_{F_0} T_{F_0}^*\| \le \|T_F - T_{F_0}\|(1 + 2\|T_{F_0}\|),$$

for each F in $\mathbf{B}$ with $\|F - F_0\| < 1$. Now by choosing $\delta = min\{1, \varepsilon/(1 + 2\|F_0\|)\}$, the continuouity of $\mathcal{S}$ at F_0 arises. Because of $\{I_{\mathcal{H}}\}$ is closed in $B(\mathcal{H})$, the continuouity of $\mathcal{S}$ on $\mathbf{B}$ implies $\mathcal{S}^{-1}(\{I_{\mathcal{H}}\})$ is closed in $\mathbf{B}$. But since $\mathcal{S}^{-1}(\{I_{\mathcal{H}}\}) = \{F \in \mathbf{B} : T_F T_F^* = I_{\mathcal{H}}\} = \mathbf{P}$, $\mathbf{P} \subset \mathbf{F}$ and $\mathbf{F}$ is open in $\mathbf{B}$, we conclude that $\mathbf{P}$ is a closed subset of $\mathbf{B}$ and it is closed relative to $\mathbf{F}$. □

Theorem 2.2. *(i) If F is in $\bar{\mathbf{R}}$, then F is not in $\mathbf{O}$.*

(ii) If F is in $\bar{\mathbf{O}}$, then F is not in $\mathbf{R}$.

(iii) The set $\mathbf{U} \subset \mathbf{R}$ is closed in $\mathbf{B}$.

(iv) The set $\mathbf{P} \setminus \mathbf{U} \subset \mathbf{O}$ is closed in $\mathbf{B}$.

Proof. (i) Let $F \in \bar{\mathbf{R}}$, then $T_F \in \overline{R(\ell^2(\mathbb{N}), \mathcal{H})}$. If T_F is surjective, then T_F^* is injective and since by Proposition 2.3, $\dim\mathrm{Ker}T_F = \dim\mathrm{Ker}T_F^*$, $T_F \in R(\ell^2(\mathbb{N}), \mathcal{H})$. Hence $F \in \mathbf{R}$ and F is not in $\mathbf{O}$. If T_F is not surjective, then F is not in $\mathbf{F} = \mathbf{R} \cup \mathbf{O}$ and so F is not in $\mathbf{O}$.

(ii) Let $F \in \bar{\mathbf{O}}$. If $F \in \mathbf{O}$, then F is not in $\mathbf{R}$. If F is not in $\mathbf{O}$, then F is a limit point of $\mathbf{O}$ and each neighborhood of F must intersect $\mathbf{O}$. Now by Theorem 2.1, $\mathbf{R}$ is open in $\mathbf{B}$, so F is not in $\mathbf{R}$.

We have $\bar{\mathbf{U}} \subset \bar{\mathbf{R}}$ and $\overline{\mathbf{P} \setminus \mathbf{U}} \subset \bar{O}$. Parts (i) and (ii) of the theorem imply $\bar{\mathbf{R}} \cap \mathbf{O} = \phi = \mathbf{R} \cap \bar{\mathbf{O}}$. Then $\bar{\mathbf{U}} \cap (\mathbf{P} \setminus \mathbf{U}) = \phi = \overline{(\mathbf{P} \setminus \mathbf{U})} \cap \mathbf{U}$. By Theorem 2.1(ii), $\mathbf{P} = \mathbf{U} \cup (\mathbf{P} \setminus \mathbf{U})$ is a closed set, so $\bar{\mathbf{U}}$ and $\overline{\mathbf{P} \setminus \mathbf{U}}$ are subsets of $\mathbf{P}$. Two previous phrases imply $\bar{\mathbf{U}} \subset \mathbf{U} \subset \mathbf{R}$ and $\overline{\mathbf{P} \setminus \mathbf{U}} \subset \mathbf{P} \setminus \mathbf{U} \subset \mathbf{O}$. Hence $\mathbf{U}$ and $\mathbf{P} \setminus \mathbf{U}$ are closed in $\mathbf{B}$, and we obtain (iii) and (iv). □

Now we give the definition of convergence of sequences in $\mathbf{B}$.

Definition 2.1. *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of Bessel sequences in $\mathbf{B}$.*

(i) We say that $\{F_n\}_{n=1}^{\infty}$ converges (uniformly) to some F in $\mathbf{B}$ and we write $F_n \longrightarrow F$ if $T_{F_n} \longrightarrow T_F$ with respect to operator norm topology in $B(\ell^2(\mathbb{N}), \mathcal{H})$.

(ii) We say that $\{F_n\}_{n=1}^{\infty}$ converges strongly to some F in $\mathbf{B}$ and we write $F_n \longrightarrow^{st} F$ if and only if $T_{F_n}(x) \longrightarrow T_F(x)$, for all $x \in \ell^2(\mathbb{N})$, the convergence occurring in the space $\mathcal{H}$, for any fixed x.

Now, a question which arises is to determine whether orthonormal basis, Riesz basis, Parseval frame, overcomplete frame are preserved under the convergences.

Theorem 2.3. *Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and let $\{F_n\}$ be a sequence of Bessel sequences in $\mathbf{B}$ such that $F_n \longrightarrow F$ for some $F \in \mathbf{B}$. Then:*

(i) If each F_n is in $\mathbf{R}$, then F is in $\mathbf{R}$ or F is in $\mathbf{B} \setminus \mathbf{F}$.

(ii) If each F_n is in $\mathbf{O}$, then F is in $\mathbf{O}$ or F is in $\mathbf{B} \setminus \mathbf{F}$.

(iii) If each F_n is in $\mathbf{P}$, then F is in $\mathbf{P}$.

(iv) If each F_n is in $\mathbf{U}$, then F is in $\mathbf{U}$.

(v) If each F_n is in $\mathbf{P} \setminus \mathbf{U}$, then F is in $\mathbf{P} \setminus \mathbf{U}$.

Proof. (i) By definition, since $\{F_n\}_{n=1}^{\infty}$ in $\mathbf{R}$ converges (uniformly) to some F in $\mathbf{B}$, then F is in $\bar{\mathbf{R}}$, the norm closure of $\mathbf{R}$. Hence Theorem 2.2(ii) implies that $F \in \mathbf{R}$ or $F \in \mathbf{B} \setminus \mathbf{F}$.

(ii) Similar to part (i), $F \in \bar{\mathbf{O}}$, and by Theorem 2.2(ii), $F \in \mathbf{O}$ or $F \in \mathbf{B} \setminus \mathbf{F}$.

Clearly, (iii) is a direct consequence of Theorem 2.1(ii). Moreover, parts (iv) and (v) are also immediate consequences of parts (iii) and (iv) of Theorem 2.2, respectively.

□

Proposition 2.5. *Let each* $F_n = \{f_{n_k}\}_{k=1}^{\infty}$ *and* $F = \{f_k\}_{k=1}^{\infty}$ *be Bessel sequences in* $\mathbf{B}$. *Then* $F_n \longrightarrow^{st} F$ *if and only if for each* $k \in \mathbb{N}$, $f_{n_k} \longrightarrow f_k$ *in* $\mathcal{H}$.

Proof. Since $\|T_{F_n}\delta_k - T_F\delta_k\| = \|f_{n_k} - f_k\|$, the result easily arises. □

While, the uniform limit of a sequence of overcomplete frames (Riesz basis) for $\mathcal{H}$ could not be a Riesz basis(an overcomplete frame) for $\mathcal{H}$, the strong limit of such sequences could be.

Example 2.1. *(i) Suppose* $F = \{e_k\}_{k=1}^{\infty}$ *be an orthonormal basis for* $\mathcal{H}$ *and* $F_n = \{e_1, ..., e_{n-1}, e_n, e_n, e_{n+1}, e_{n+2}, ...\}$ *for each* $n \in \mathbb{N}$. *Then each* F_n *is an overcomplete frame for* $\mathcal{H}$ *and for each* $k \in \mathbb{N}$,

$$\|T_{F_n}\delta_k - T_F\delta_k\| = \begin{cases} \sqrt{2} & if \quad n < k, \\ \\ 0 & if \quad n \geq k. \end{cases} \tag{1}$$

So by Proposition 2.5, $F_n \longrightarrow^{st} F$.

(ii) Let $F_1 = \{e_k\}_{k=1}^{\infty}$ *be an orthonormal basis for* $\mathcal{H}$, *and for each* $n \geq 2$, $F_n = \{f_{n_k}\}_{k=1}^{\infty}$, *where* $f_{n_1} = e_n$ *and*

$$f_{n_k} = \begin{cases} e_{k-1} & if \quad 2 \leq k < n+1, \\ \\ e_k & if \quad k \geq n+1. \end{cases} \tag{2}$$

Clearly, each F_n *is a Riesz basis for* $\mathcal{H}$. *Then in* $\mathcal{H}$,

$$f_k := \lim_{n\to\infty} f_{n_k} = \begin{cases} 0 & if \quad k = 1, \\ \\ e_{k-1} & if \quad k \geq 2. \end{cases} \tag{3}$$

We set $F = \{f_k\}_{k=1}^{\infty}$, *since* $f_{n_k} \longrightarrow f_k$ $(k \in \mathbb{N})$, *by Proposition 2.5,* $F_n \longrightarrow^{st} F$, *where* F *is an overcomplete frame for* $\mathcal{H}$.

3. Multiplication and Adjoint of Frames

Here we decide to define a multiplication on the Banach space $B(\ell^2(\mathbb{N}), \mathcal{H})$ with respect to the fixed orthonormal basis E_0 that makes it into a Banach algebra. Naturally, from the isometric isomorphism $\psi : B(\ell^2(\mathbb{N}), \mathcal{H}) \longrightarrow B(\mathcal{H})$, $T \longmapsto TT_{E_0}^{-1}$, we can define the multiplication of T and L in $B(\ell^2(\mathbb{N}), \mathcal{H})$ by

$$T * \mathrm{L} = \psi^{-1}((TT_{E_0}^{-1})(\mathrm{L}T_{E_0}^{-1})) = TT_{E_0}^{-1}\mathrm{L}.$$

Clearly the Banach space $B(\ell^2(\mathbb{N}), \mathcal{H})$ together with this multiplication is a complex algebra and in addition its operator norm satisfies the multiplicative inequality, $\|T * \mathrm{L}\| \leq \|T\|\|\mathrm{L}\|$ $(T, \mathrm{L} \in B(\ell^2(\mathbb{N}), \mathcal{H}))$.

Thus, $B(\ell^2(\mathbb{N}), \mathcal{H})$ is a Banach algebra. For $T \in B(\ell^2(\mathbb{N}), \mathcal{H})$, $T * T_{E_0} = T = T_{E_0} * T$ and $\|T_{E_0}\| = 1$ and it is clear that this unit element T_{E_0} is unique. Indeed $B(\ell^2(\mathbb{N}), \mathcal{H})$ is a non-commutative unital Banach algebra which is isometrically isomorphic to the Banach algebra $B(\mathcal{H})$.

Again, naturally from ψ, we can define an involution

$$\circledast : T \longmapsto \psi^{-1}((TT_{E_0}^{-1})^*) = T_{E_0}T^*T_{E_0} = T^{\circledast}$$

of $B(\ell^2(\mathbb{N}), \mathcal{H})$ into itself.

Proposition 3.1. $B(\ell^2(\mathbb{N}), \mathcal{H})$ *is a* C^**-algebra isometric* $*$*-isomorphic to* $B(\mathcal{H})$.

Definition 3.1. *Let $F = \{f_k\}_{k=1}^{\infty}$ and $G = \{g_k\}_{k=1}^{\infty}$ be two Bessel sequences in* $\mathbf{B}$ *with pre-frame operators T_F and T_G, respectively. We define multiplication of F and G with respect to E_0, or E_0-multiplication of F and G, to be the Bessel sequence $F * G$ with the pre-frame operator $T_{F*G} = T_F T_{E_0}^{-1} T_G$ and we denote it by $\{f_k * g_k\}_{k=1}^{\infty}$.*

Remark 3.1. *Consider two Bessel sequences $F = \{f_k\}_{k=1}^{\infty}$ and $G = \{g_k\}_{k=1}^{\infty}$ in* $\mathbf{B}$*, and the orthonormal basis $E_0 = \{e_k^0\}_{k=1}^{\infty}$ for $\mathcal{H}$. Then we have*
$g_k = \sum_{n=1}^{\infty} < g_k, e_n^0 > e_n^0$, *and* $f_k * g_k = \sum_{n=1}^{\infty} < g_k, e_n^0 > f_n$.

As an immediate consequence of the preceding definition, we can claim that the multiplication $*$ preserves Riesz basis, orthonormal basis, Parseval frame and overcomplete frame:

Theorem 3.1. *Let F and G be in* $\mathbf{B}$*. Then:*
(i) If F and G are Riesz bases, then $F * G$ and $G * F$ are Riesz bases.
(ii) If F and G are orthonormal bases, then $F * G$ and $G * F$ are orthonormal bases.
(iii) If F and G are Parseval frames, then $F * G$ and $G * F$ are Parseval frames.
(iv) If F and G are overcomplete frames, then $F * G$ and $G * F$ are overcomplete frames.

Proof. (i) Since T_F and T_G the pre-frame operators of F and G, respectively, are invertible in $B(\ell^2(\mathbb{N}), \mathcal{H})$, $T_{F*G} = T_F T_{E_0}^{-1} T_G$ the pre-frame operator of $F * G$ is also invertible in $B(\ell^2(\mathbb{N}), \mathcal{H})$.

(ii) By (i) $F * G$ is a Riesz basis. Clearly, the frame operator of $F * G$, $S_{F*G} = I_{\mathcal{H}}$. Hence $F * G$ and similarly $G * F$ are orthonormal bases.

(iii) Since $S_F = I_{\mathcal{H}}$ and $S_G = I_{\mathcal{H}}$, similar to the proof of (ii), we have $S_{F*G} = I_{\mathcal{H}}$ and hence $F * G$ and similarly $G * F$ are Parseval frames.

(iv) Clearly, $T_{F*G} = T_F T_{E_0}^{-1} T_G$ is surjective. But T_{F*G} is not injective, because of T_G is not. Hence $F * G$ is an overcomplete frame. □

Corollary 3.1. *If one of the frames F and G is overcomplete, then $F * G$ and $G * F$ are overcomplete.*

For each F in $\mathbf{B}$, $F * E_0 = E_0 * F = F$ and $\|E_0\| = \|T_{E_0}\| = 1$. It is clear that $\mathbf{B}$ can be considered as a unital Banach algebra with unit element E_0. Moreover, for F in $\mathbf{B}$ we define $F^{\circledast}$ to be the Bessel sequence in $\mathbf{B}$ with the pre-frame operator $T_{F^{\circledast}} = T_{E_0} T_F^* T_{E_0}$ and we denote it by $\{f_k^{\circledast}\}_{k=1}^{\infty}$. Obviously $\circledast : F \longmapsto F^{\circledast}$ defines an involution of $\mathbf{B}$ into itself and so we have:

Proposition 3.2. $\mathbf{B}$ *is a C^*-algebra isometric $*$-isomorphic to $B(\ell^2(\mathbb{N}), \mathcal{H})$.*

The following lemma which is the corollary of the Proposition 5.29 in [11] gives us an interesting property of the collection of invertible operators on a Hilbert space.
Lemma 3.1. [11, Corollary 5.30] *If $\mathcal{H}$ is a Hilbert space, then $R(\mathcal{H})$ the collection of invertible operators in $B(\mathcal{H})$ is arcwise connected.*

In particular, $R(\mathcal{H})$ is connected and we have the following result.

Proposition 3.3. *Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. The set* $\mathbf{F}$ *is disconnected and* $\mathbf{R}$ *is a clopen (closed and open) connected component of* $\mathbf{F}$ *with respect to the uniform topology on* $\mathbf{B}$*.*

Proof. By Theorem 2.1(i) $\mathbf{F}$, $\mathbf{R}$ and $\mathbf{O}$ are open in $\mathbf{B}$, so $\mathbf{R}$ and $\mathbf{O}$ are nonempty open relative to $\mathbf{F}$. Since $\mathbf{F}$ is the disjoint union of $\mathbf{R}$ and $\mathbf{O}$, so $\mathbf{F}$ is disconnected. By Theorem

2.2(i) (or directly from disconnectedness of **F**) $\bar{\mathbf{R}} \cap \mathbf{O} = \emptyset$. Then $\bar{\mathbf{R}} \cap \mathbf{F} = \mathbf{R}$ and $\mathbf{R}$ is closed relative to **F**. Since by the result after Lemma 3.1 $R(\mathcal{H}) \subset B(\mathcal{H})$ is connected, so Proposition 3.1 and Proposition 3.2 imply that **R** is connected. Moreover,the closure of **R** relative to **F** is **R**. Hence **R** is a component of **F** and the proof is complete. □

Remark 3.2. *For Bessel sequence $F = \{f_k\}_{k=1}^{\infty}$ in* **B**,

$$f_k^{\circledast} = T_{F^{\circledast}}\delta_k = T_{E_0}T_F^* e_k^0 = T_{E_0}(\sum_{n=1}^{\infty} < T_F^* e_k^0, \delta_n > \delta_n) \qquad = \sum_{n=1}^{\infty} < e_k^0, f_n > e_n^0.$$

An element T in the Banach algebra $B(\ell^2(\mathbb{N}), \mathcal{H})$ is invertible (or more precisely, E_0-invertible) if and only if there is an element $T^{'}$ in $B(\ell^2(\mathbb{N}), \mathcal{H})$ such that $T{*}T^{'} = T^{'}{*}T = T_{E_0}$.

Remark 3.3. *Obviously T is invertible in the Banach space $B(\ell^2(\mathbb{N}), \mathcal{H})$(i.e. there is an element $T^{'} \in B(\mathcal{H}, \ell^2(\mathbb{N}))$ such that $TT^{'} = I_{\mathcal{H}}$ and $T^{'}T = I_{\ell^2(\mathbb{N})}$) if and only if T is E_0-invertible in the Banach algebra $B(\ell^2(\mathbb{N}), \mathcal{H})$. It is clear that the E_0- inverse of an element T in the Banach algebra $B(\ell^2(\mathbb{N}), \mathcal{H})$ is unique and we denote it by $T^{\dashv}$. In fact $T^{\dashv} = T_{E_0}T^{-1}T_{E_0}$.*

Definition 3.2. *A Bessel sequence $F = \{f_k\}_{k=1}^{\infty}$ in* **B** *is called invertible with respect to E_0 or E_0-invertible and its inverse is denoted by $F^{\dashv} = \{f_k^{\dashv}\}_{k=1}^{\infty}$ if T_F is E_0- invertible in the Banach algebra $B(\ell^2(\mathbb{N}), \mathcal{H})$.*

Remark 3.4. *F is E_0- invertible in* **B** *if and only if $F \in$* **R**.

Proposition 3.4. *(i)* $(\mathbf{R}, *)$ *is a non-abelian group in* **B**.

(ii) $(\mathbf{U}, *)$ *is a subgroup of* $(\mathbf{R}, *)$.

Proof. Let F, G be in **R**, then by Theorem 3.1(i) $F * G \in \mathbf{R}$. The binary operation $*$ is associative in **R**, because of **B** is a Banach algebra. Since E_0 is the unit element of **B**, it is the identity element of **R**. For F in **R**, by the Remark 3.3, F is E_0- invertible and so $F^{\dashv}$ is E_0- invertible. Hence $F^{\dashv}$ is in **R** and $(\mathbf{R}, *)$ is group in **B**.

(ii) Let E and F be in **U**. Then $E * F^{\dashv}$ corresponds to $T_{E*F^{\dashv}}$ in $B(\ell^2(\mathbb{N}), \mathcal{H})$. But

$$T_{E*F^{\dashv}} = T_E T_{E_0}^{-1} T_{F^{\dashv}} = T_E T_{E_0}^{-1}(T_{E_0}T_F^{-1}T_{E_0}) = T_E T_F^{-1} T_{E_0},$$

which is clearly a surjective isometry in $B(\ell^2(\mathbb{N}), \mathcal{H})$ and so $E * F^{\dashv}$ is in **U**. Hence **U** is a subgroup of **R**. □

Proposition 3.5. *Let $F = \{f_k\}_{k=1}^{\infty}$ be a Riesz basis for $\mathcal{H}$. Then we can write the E_0-inverse of F, $F^{\dashv} = \{f_k^{\dashv}\}_{k=1}^{\infty}$, with $f_k^{\dashv} = \sum_{n=1}^{\infty} < e_k^0, \tilde{f}_n > e_n^0$ $(k \in \mathbb{N})$, where, $\{\tilde{f}_n\}_{n=1}^{\infty}$ is the unique dual frame of the Riesz basis F.*

Proof. Let S_F be the frame operator of F. Then the unique dual frame of F is $\{S_F^{-1}f_k\}_{k=1}^{\infty} = \{\tilde{f}_k\}_{k=1}^{\infty}$, and since $e_k^0 = \sum_{n=1}^{\infty} < e_k^0, S^{-1}f_n > f_n$, we have

$$f_k^{\dashv} = \sum_{n=1}^{\infty} < e_k^0, \tilde{f}_n > e_n^0.$$

□

Corollary 3.2. *If $F = \{f_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$, then*

$$F^{\dashv} = \{\sum_{n=1}^{\infty} < e_k^0, f_n > e_n^0\}_{k=1}^{\infty},$$

and $E_0^{\dashv} = E_0$.

4. **Bessel Sequences and Orthonormal Bases**

About unitary operators, there exist interesting well-known results in operator theory. We present one of them in our manner that we want to use. It can be found in [14] and [16].

Theorem 4.1. *Let $\mathcal{H}$ be a separable Hilbert space. If $T \in B(\mathcal{H})$ and $\|T\| \leq 1 - \frac{2}{n}$ $(n = 3, 4, \ldots)$, then $T = \frac{1}{n}(U_1 + \ldots + U_n)$ with $U_1, \ldots, U_n$ in $U(\mathcal{H})$.*

Similar to the work of Casazza [6], we have the following immediate consequences of Theorem 4.1.

Proposition 4.1. *Let $F = \{f_k\}_{k=1}^{\infty}$ be a Bessel sequence for $\mathcal{H}$. Then for each $0 < \varepsilon < 1$, there are an integer $n \geq 3$ and orthonormal bases $E_i = \{e_k^i\}_{k=1}^{\infty}(i = 1, 2, 3, \ldots, n)$ such that $f_k = \frac{\|F\|}{\varepsilon n}(e_k^1 + \ldots + e_k^n)$ $(k \in \mathbb{N})$.*

Proof. We have $\|F\| = \|T_F\| = \|T_F T_{E_0}^{-1}\|$. If we set $T = \varepsilon \frac{T_F T_{E_0}^{-1}}{\|T_F T_{E_0}^{-1}\|}$, then $T \in B(\mathcal{H})$, $\|T\| < 1$ and so there is an integer $n \geq 3$ such that $\|T\| < 1 - \frac{2}{n}$. Now by Theorem 4.1, there are unitary operators $U_1, \ldots, U_n$ in $B(\mathcal{H})$ such that $T = \frac{1}{n}(U_1 + \ldots + U_n)$. Then

$$f_k = T_F \delta_k = T_F T_{E_0}^{-1} e_k^0 = \frac{\|F\|}{\varepsilon n}(U_1 e_k^0 + \ldots + U_n e_k^0) = \frac{\|F\|}{\varepsilon n}(e_k^1 + \ldots + e_k^n),$$

where $\{e_k^i\}_{k=1}^{\infty}$ $(i = 1, 2, \ldots, n)$ are orthonormal bases for $\mathcal{H}$. □

Proposition 4.2. *Let $F = \{f_k\}_{k=1}^{\infty}$ be a Bessel sequence in* **B**. *Then for each integer $n \geq 3$, we can write F as a multiple of a sum of n orthonormal bases in* **B**.

Proof. Let $n \geq 3$ be an integer. We can choose $0 < \varepsilon < 1$ such that $\varepsilon \leq 1 - \frac{2}{n}$. By setting $T = \varepsilon \frac{T_F T_{E_0}^{-1}}{\|T_F T_{E_0}^{-1}\|}$, we have $T \in B(\mathcal{H})$ and $\|T\| = \varepsilon \leq 1 - \frac{2}{n}$. Now by Theorem 4.1, there are unitary operators $U_1, \ldots, U_n$ such that $T = \frac{1}{n}(U_1 + \ldots + U_n)$, and since $\|F\| = \|T_F T_{E_0}^{-1}\|$,

$$f_k = T_F \delta_k = \frac{\|F\|}{\varepsilon n}(e_k^1 + e_k^2 + ... + e_k^n),$$

where, $\{U_i e_k^0\}_{k=1}^{\infty} = \{e_k^i\}_{k=1}^{\infty}$ $(i = 1, ..., n)$ are orthonormal bases for $\mathcal{H}$. □

5. **A Partition of Frames and some Characterization of Riesz Bases**

In this section we will concern to an equivalence relation on the set of all frames F for $\mathcal{H}$ and then we study some results about Riesz bases for $\mathcal{H}$.

First we modify a result in [18] that we want to use it.

Proposition 5.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with bounds A, B and that S is its frame operator. Let $L : \mathcal{H} \to \mathcal{H}$ be a bounded operator. Then $\{Lf_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with bounds $\|L^{\dagger}\|A, \|L\|B$ if and only if L is surjective, where $L^{\dagger}$ is the pseudo-inverse of L. Moreover the frame operator for $\{Lf_k\}_{k=1}^{\infty}$ is LSL^*.*

Proof. First, suppose that $\{Lf_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ and S_L is its frame operator. Let $f \in \mathcal{H}$, then

$$f = \sum_{k=1}^{\infty} < f, S_L^{-1} L f_k > L f_k, \text{ and } \{c_k\}_{k=1}^{\infty} = \{< f, S_L^{-1} L f_k >\}_{k=1}^{\infty} \in \ell^2(\mathbb{N}).$$

But $T_L\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k L f_k \in \mathcal{H}$, where T_L is the pre-frame operator of $\{Lf_k\}_{k=1}^{\infty}$. Since $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$ and $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$, $\sum_{k=1}^{\infty} c_k f_k$ converges unconditionally in $\mathcal{H}$ and also the boundedness of L implies $f = L(\sum_{k=1}^{\infty} c_k f_k)$ and hence L is surjective. For the opposite implication, see [7, Corollary 5.3.2]. Now clearly, for all $f \in \mathcal{H}$, $S_L f = LSL^* f$. Thus $S_L = LSL^*$. $\square$

Here we will attend to frames that have the same frame operator. As an immediate consequence of Proposition 5.1 we have :

Proposition 5.2. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame operator S. If L is a bounded operator in which $\{Lf_k\}_{k=1}^{\infty}$ is also a frame for $\mathcal{H}$, then $\{f_k\}_{k=1}^{\infty}$ and $\{Lf_k\}_{k=1}^{\infty}$ have the same frame operator if and only if $S = LSL^*$.*

Proposition 5.3. *Let $F = \{f_k\}_{k=1}^{\infty}$ and $G = \{g_k\}_{k=1}^{\infty}$ be two frames for $\mathcal{H}$ with analysis operators T_F^* and T_G^*, respectively. Then F and G have the same frame operator if and only if $\|T_F^* f\| = \|T_G^* f\|$, for all $f \in \mathcal{H}$.*

Proof. The frame operators of F and G are $S_F = T_F T_F^*$ and $S_G = T_G T_G^*$, respectively. Since $\|T_F^* f\|^2 =< T_F^* f, T_F^* f >=< T_F T_F^* f, f >=< S_F f, f >$, and $\|T_G^* f\|^2 =< S_G f, f >$, for all $f \in \mathcal{H}$, the result is attained. $\square$

Note: Since the subtraction of S_F and S_G is a self-adjoint operator, the proof of the above proposition holds for the both cases of real and complex Hilbert spaces.

Definition 5.1. *Let F and G be two elements of the set of all frames for $\mathcal{H}$. We say that F is in frame operator relation to G and we write $F \sim G$ if F and G have the same frame operator S. Evidently, $\sim$ is an equivalence relation on $\mathbf{F}$. We say F is frame operator equivalent (or briefly f.o-equivalent) to G if $F \sim G$.*

It is well-known that the frame operator of any frame is a bounded, invertible, self-adjoint and positive operator. It is easily seen that every bounded, invertible, self-adjoint and positive operator on $\mathcal{H}$ is the frame operator of some frame for $\mathcal{H}$.

Proposition 5.4. *Let S be a bounded, invertible, self-adjoint and positive operator on a Hilbert space $\mathcal{H}$ and let $\{e_k\}_{k=1}^{\infty}$ be a Parseval frame for $\mathcal{H}$. Then $\{f_k\}_{k=1}^{\infty} = \{S^{1/2} e_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with frame operator S.*

Proof. The frame operator of the Parseval frame $\{e_k\}_{k=1}^{\infty}$ is the identity operator $I : \mathcal{H} \to \mathcal{H}$. By Proposition 5.1, $\{f_k\}_{k=1}^{\infty} = \{S^{1/2} e_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ and its frame operator is $S^{1/2} I (S^{1/2})^* = S$. $\square$

Let $F = \{f_k\}_{k=1}^{\infty}$ be an element of $\mathbf{F}$ with the frame operator S. Since an element of $\mathbf{F}$ is in the class $[F]$ if and only if its frame operator is S, we can represent $[F]$ by $[S]$, where $[S]$ is the set of all frames in $\mathbf{F}$ with the same frame operator S. Now by Proposition 5.4, $\mathbf{F} = \bigcup [S]$, where the union carry over all bounded, invertible, self-adjoint, and positive operators on $\mathcal{H}$.

Proposition 5.5. *Let S be a bounded, invertible, self-adjoint and positive operator on a Hilbert space $\mathcal{H}$. Then*

$$[S] = \{\{f_k\}_{k=1}^{\infty} : f_k = S^{1/2} e_k \ \ and \ \ \{e_k\}_{k=1}^{\infty} \ \ is \ \ a \ \ Parseval \ \ frame \ \ for \ \ \mathcal{H}\}$$

Proof. We must show that: (i) $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$; (ii) S is the frame operator of $\{f_k\}_{k=1}^{\infty}$; and (iii) every frame with frame operator S is of the form $\{S^{1/2} e_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is a Parseval frame.
By Proposition 5.4, (i) and (ii) hold. Now, let $\{g_k\}_{k=1}^{\infty}$ be any frame for $\mathcal{H}$ with frame operator S. Then $\{S^{-1/2} g_k\}_{k=1}^{\infty}$ is a Parseval frame (see, [7, Theorem 5.3.4]). For each $k \in \mathbb{N}$, set $e_k = S^{-1/2} g_k$, then $\{g_k\}_{k=1}^{\infty} = \{S^{1/2} e_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is a Parseval frame for $\mathcal{H}$ and the proof is complete. $\square$

Remark 5.1. *As an especial case of Proposition 5.5,*

$$[I] = \{\{e_k\}_{k=1}^{\infty} : \{e_k\}_{k=1}^{\infty} \text{ is a Parseval frame for } \mathcal{H}\},$$

where I is the identity operator on $\mathcal{H}$.

Remark 5.2. *Let S and $S^{'}$ be two bounded, invertible, self-adjoint and positive operators on a Hilbert space $\mathcal{H}$. Then by definition of $[S]$ and $[S^{'}]$ and the uniqueness of the frame operator of a frame for $\mathcal{H}$ we have $[S] = [S^{'}]$ if and only if $S = S^{'}$.*

Now we give some characterizations of Riesz bases.

Theorem 5.1. *A Riesz basis for $\mathcal{H}$ is a sequence of the form $\{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ is a bounded, invertible, self-adjoint and positive operator.*

Proposition 5.6. *Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be two Riesz bases for $\mathcal{H}$ such that $\{f_k\}_{k=1}^{\infty} \in [S]$ and $\{g_k\}_{k=1}^{\infty} \in [P]$. Then there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that $g_k = P^{1/2}US^{-1/2}f_k,\ \forall\, k \in \mathbb{N}$.*

Proof. Suppose that $\{f_k\}_{k=1}^{\infty} \in [S]$ and $\{g_k\}_{k=1}^{\infty} \in [P]$ are Riesz bases for $\mathcal{H}$. By the proof of Theorem 5.1 , there exist orthonormal bases $\{e_k\}_{k=1}^{\infty}$ and $\{e'_k\}_{k=1}^{\infty}$ such that $f_k = S^{1/2}e_k$ and $g_k = P^{1/2}e'_k,\ \forall\, k \in \mathbb{N}$. Now there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that $Ue_k = e'_k,\ \forall\, k \in \mathbb{N}$ (see [7, Theorem 3.2.7]). But $e_k = S^{-1/2}f_k$, $e'_k = US^{-1/2}f_k$, and so $P^{1/2}US^{-1/2}f_k = g_k,\ \forall\, k \in \mathbb{N}$. □

Corollary 5.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a Riesz basis for $\mathcal{H}$ with frame operator S. Then the Riesz bases for $\mathcal{H}$ with the same frame operator S are precisely the family $\{S^{1/2}US^{-1/2}f_k\}_{k=1}^{\infty}$ where $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator.*

Let $F = \{f_k\}_{k=1}^{\infty}$ be a frame in the class $[S]$. We decide to find all of the classes of its dual frames. To this end, first we state the following two lemmas [7, Lemma 5.7.2 and Lemma 5.7.3] and then we give a theorem that is another characterization of Riesz bases in terms of their pre-frame and frame operators.

Lemma 5.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with pre-frame operator T and $\{\delta_k\}_{k=1}^{\infty}$ be the canonical orthonormal basis for $\ell^2(\mathbb{N})$. The dual frames for $\{f_k\}_{k=1}^{\infty}$ are precisely the families $\{g_k\}_{k=1}^{\infty} = \{V\delta_k\}_{k=1}^{\infty}$, where $V : \ell^2(\mathbb{N}) \to \mathcal{H}$ is a bounded left-inverse of T^*.*

Lemma 5.2. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with pre-frame operator T.Then the bounded left-inverses of T^* are precisely the operators having the form*

$$S^{-1}T + W(I - T^*S^{-1}T),$$

where $W : \ell^2(\mathbb{N}) \to \mathcal{H}$ is a bounded operator and I denotes the identity operator on $\ell^2(\mathbb{N})$.

Theorem 5.2. *Let $F = \{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame operator S and pre-frame operator T. Then F is a Riesz basis for $\mathcal{H}$ if and only if*

$$T^*S^{-1}T = I.$$

Proof. Let $G = \{g_k\}_{k=1}^{\infty}$ be a dual frame for F with pre-frame operator $T^{'}$ and frame operator $S^{'}$. Then we have $f = \sum_{k=1}^{\infty} < f, g_k > f_k = \sum_{k=1}^{\infty} < f, f_k > g_k$, and $T^{'}T^*f = T^{'}(\{< f, f_k >\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} < f, f_k > g_k = f,\ \ \forall f \in \mathcal{H}$. Thus $T^{'}T^* = I$ and $T^{'}$ is a left-inverse of T^*. Now, Lemma 5.2 implies that

$$T^{'} = S^{-1}T + W(I - T^*S^{-1}T),$$

for some bounded operator $W : \ell^2(\mathbb{N}) \to \mathcal{H}$, and here I is the identity operator on $\ell^2(\mathbb{N})$. But since I, S^{-1} are self-adjoint and $TT^* = S$, straightforward calculation gives that

$$\begin{aligned} S^{'} &= T^{'}T^{'^*} = (S^{-1}T + W(I - T^*S^{-1}T)(T^*S^{-1} + (I - T^*S^{-1}T)W^*) \\ &= S^{-1} + W(I - T^*S^{-1}T)W^* \qquad (5.1). \end{aligned}$$

From Lemma 5.1, we conclude that the set of all pre-frame operators of all dual frames for F is the set of all bounded left inverses of T^*. Hence the set of all frame operators of all dual frames for F is

$$\mathbf{D} = \{S^{-1} + W(I - T^*S^{-1}T)W^* : W \in B(\ell^2(\mathbb{N}), \mathcal{H})\}.$$

Let F be a Riesz basis for $\mathcal{H}$. Then the unique dual frame for F is the canonical dual frame $\{S^{-1}f_k\}_{k=1}^{\infty}$ with frame operator S^{-1} (see [15, Corollary 2.26]). Therefore, in this case $\mathbf{D} = \{S^{-1}\}$, and $W(I-T^*S^{-1}T)W^* = 0$, for all bounded operators $W : \ell^2(\mathbb{N}) \to \mathcal{H}$ (in particular for the surjective isometry $W : \ell^2(\mathbb{N}) \to \mathcal{H}$, $\{c_k\}_{k=1}^{\infty} \mapsto \sum_{k=1}^{\infty} c_k e_k$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$). Then we have $T^*S^{-1}T = I$. The converse implication is an immediate consequence of (5.1) and the well-known fact that a Riesz basis is a frame which has precisely one dual frame. □

By considering the proof of Theorem 5.2, we have the following corollary.

Corollary 5.2. *Let $F = \{f_k\}_{k=1}^{\infty}$ be in $[S]$ with pre-frame operator T. Then the classes of dual frames for F are precisely the classes having the form $[S^{-1} + W(I - T^*S^{-1}T)W^*]$, where $W : \ell^2(\mathbb{N}) \to \mathcal{H}$ is a bounded operator.*

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