LINEAR DISCRETE MULTITIME MULTIPLE RECURRENCE

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The multitime multiple recurrences are common in analysis of algorithms, computational biology, information theory, queueing theory, filters theory, statistical physics etc. The theoretical part about them is little or not known. Therefore, the aim of our paper is to formulate and solve problems concerning nonautonomous multitime multiple recurrence equations. Among other things, we discuss in detail the cases of linear recurrences with constant coefficients, highlighting in particular the theorems of existence and uniqueness of solutions.

Keywords: multitime multiple recurrence, multiple linear recurrence, fundamental matrix, recurrences on a monoid.

MSC2010: 39A 06, 65Q 99.

1. Introduction

In this paper we shall refer to linear discrete multitime multiple recurrence, giving original results regarding generic properties and existence and uniqueness of solutions. Also, we seek to provide a fairly thorough and unified exposition of efficient recurrence relations in both univariate and multivariate settings. The scientific sources used in this paper are: filters theory [2], [8], general recurrence theory [7], [1], [13], our results regarding the diagonal multitime recurrence [3] - [5], and multitime dynamical systems [9]-[12].

Let $m \geq 1$ be an integer number. We denote $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{Z}^m$. Also, for each $\alpha \in \{1, 2, \ldots, m\}$, we denote $1_\alpha = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^m$, i.e., 1_α has 1 on the position α and 0 otherwise. We use the product order relation on $\mathbb{Z}^m$.

Let M be an arbitrary nonvoid set and $t_1 \in \mathbb{Z}^m$ be a fixed element. We consider the functions $F_\alpha \colon \{t \in \mathbb{Z}^m \,|\, t \geq t_1\} \times M \to M$, $\alpha \in \{1, 2, \ldots, m\}$. We fix $t_0 \in \mathbb{Z}^m$, $t_0 \geq t_1$. A first order discrete multitime recurrence of the type

$$x(t + 1_\alpha) = F_\alpha(t, x(t)), \quad \forall t \in \mathbb{Z}^m,\ t \geq t_0,\ \forall \alpha \in \{1, 2, \ldots, m\}, \tag{1}$$

is called a *discrete multitime multiple recurrence.*

This model of multiple recurrence can be justified by the fact that a completely integrable first order PDE system

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(t, x(t)),\ t \in \mathbb{R}^m,$$

can be discretized as: $x^i(t + 1_\alpha) - x^i(t) = X^i_\alpha(t, x(t)),\ t \in \mathbb{Z}^m$.

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The initial (Cauchy) condition, for the PDE system, is translated into initial condition for the multiple recurrence.

2. Linear discrete multitime multiple recurrence

Let K be a field. We denote by $\mathcal{Z}$ one of the sets $\mathbb{Z}^m$ or $\{t \in \mathbb{Z}^m \,|\, t \geq t_1\}$ (with $t_1 \in \mathbb{Z}^m$). For each $\alpha \in \{1, 2, \ldots, m\}$, we consider the functions $A_\alpha \colon \mathcal{Z} \to \mathcal{M}_n(K)$, $b_\alpha \colon \mathcal{Z} \to K^n = \mathcal{M}_{n,1}(K)$, which define the recurrence

$$x(t + 1_\alpha) = A_\alpha(t)x(t) + b_\alpha(t), \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{2}$$

with the unknown function $x \colon \{t \in \mathcal{Z} \,|\, t \geq t_0\} \to K^n = \mathcal{M}_{n,1}(K)$, $t_0 \in \mathcal{Z}$. This is a particular case of discrete multitime multiple recurrence (1), with $M = K^n$ and $F_\alpha(t, x) = A_\alpha(t)x + b_\alpha(t)$.

Theorem 2.1. *a) If, for any $(t_0, x_0) \in \mathcal{Z} \times K^n$, there exists at least one function $x \colon \{t \in \mathcal{Z} \,|\, t \geq t_0\} \to K^n$, which, for any $t \geq t_0$, verifies the recurrence (2) and the condition $x(t_0) = x_0$, then*

$$A_\alpha(t + 1_\beta)A_\beta(t) = A_\beta(t + 1_\alpha)A_\alpha(t), \tag{3}$$

$$A_\alpha(t + 1_\beta)b_\beta(t) + b_\alpha(t + 1_\beta) = A_\beta(t + 1_\alpha)b_\alpha(t) + b_\beta(t + 1_\alpha), \tag{4}$$

$$\forall t \in \mathcal{Z}, \quad \forall \alpha, \beta \in \{1, 2, \ldots, m\}.$$

b) If the relations (3), (4) are satisfied, then, for any $(t_0, x_0) \in \mathcal{Z} \times K^n$, there exists a unique function $x \colon \{t \in \mathcal{Z} \,|\, t \geq t_0\} \to K^n$, which, for any $t \geq t_0$ verifies the recurrence (2) and the initial condition $x(t_0) = x_0$.

Proof. *a)* One applies Proposition 1.1 from [6]. The relations (2) from [6] become

$$A_\alpha(t + 1_\beta)\big(A_\beta(t)x + b_\beta(t)\big) + b_\alpha(t + 1_\beta) = A_\beta(t + 1_\alpha)\big(A_\alpha(t)x + b_\alpha(t)\big) + b_\beta(t + 1_\alpha),$$

$$\forall x \in K^n, \ \forall t \geq t_1.$$

In the case $\mathcal{Z} = \mathbb{Z}^n$, the point t_1 is arbitrary. We deduce that the foregoing relations are true $\forall x \in K^n$, $\forall t \in \mathcal{Z}$. Setting $x = 0$, we obtain the relations (4). It follows that

$$A_\alpha(t + 1_\beta)A_\beta(t)x = A_\beta(t + 1_\alpha)A_\alpha(t)x, \quad \forall x \in K^n, \ \forall t \in \mathcal{Z}. \tag{5}$$

For $j \in \{1, 2, \ldots, n\}$, let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)^\top$ be the column of K^n which has 1 on the position j and 0 in rest. From (5) it follows:
$A_\alpha(t+1_\beta)A_\beta(t) \cdot \begin{pmatrix} e_1 & e_2 & \ldots & e_n \end{pmatrix} = A_\beta(t+1_\alpha)A_\alpha(t) \cdot \begin{pmatrix} e_1 & e_2 & \ldots & e_n \end{pmatrix}$, equivalent to $A_\alpha(t + 1_\beta)A_\beta(t)I_n = A_\beta(t + 1_\alpha)A_\alpha(t)I_n$, i.e., the relations (3).

From paper [6] – Theorem 3.1 it follows *b*). □

Theorem 2.2. *For each $\alpha \in \{1, 2, \ldots, m\}$, we consider the functions $A_\alpha \colon \mathbb{Z}^m \to \mathcal{M}_n(K)$, $b_\alpha \colon \mathbb{Z}^m \to K^n$, which define the recurrence (2).*

The following statements are equivalent:

i) For any $\alpha \in \{1, 2, \ldots, m\}$ and any $t \in \mathbb{Z}^m$, the matrix $A_\alpha(t)$ is invertible and $\forall t \in \mathbb{Z}^m$, $\forall \alpha, \beta \in \{1, 2, \ldots, m\}$ the relations (3), (4) hold.

ii) For any pair $(t_0, x_0) \in \mathbb{Z}^m \times K^n$, and any $\alpha_0 \in \{1, 2, \ldots, m\}$, there exists at least one function $x \colon \{t \in \mathbb{Z}^m \,|\, t \geq t_0 - 1_{\alpha_0}\} \to K^n$, which, for any $t \geq t_0 - 1_{\alpha_0}$, verifies the relations (2), and also the condition $x(t_0) = x_0$.

iii) For any pair $(t_0, x_0) \in \mathbb{Z}^m \times K^n$, and any $\alpha_0 \in \{1, 2, \ldots, m\}$, there exists a unique function $x \colon \{t \in \mathbb{Z}^m \,|\, t \geq t_0 - 1_{\alpha_0}\} \to K^n$, which, for any $t \geq t_0 - 1_{\alpha_0}$, verifies the relations (2), and also the condition $x(t_0) = x_0$.

iv) For any $t_0, t_1 \in \mathbb{Z}^m$, with $t_1 \leq t_0$, and for any $x_0 \in K^n$, there exists a unique function $x \colon \{t \in \mathbb{Z}^m \,|\, t \geq t_1\} \to K^n$, which, for any $t \geq t_1$, verifies the relations (2), and also the condition $x(t_0) = x_0$.

v) For any pair $(t_0, x_0) \in \mathbb{Z}^m \times K^n$*, there exists a unique function* $x\colon \mathbb{Z}^m \to K^n$*, which, for any* $t \in \mathbb{Z}^m$*, verifies the relations* (2)*, and also the condition* $x(t_0) = x_0$*.*

Proof. The equivalence of the statements *i*), *iii*), *iv*), *v*) follows from [6] – Theorem 3.2. Since the implication *iii*) $\Longrightarrow$ *ii*) is obvious, it is sufficient to prove the implication *ii*) $\Longrightarrow$ *i*).

ii) $\Longrightarrow$ *i*): The relations (3), (4) follow from Theorem 2.1.

Let $F_\alpha\colon \mathbb{Z}^m \times K^n \to K^n$, $F_\alpha(t,x) = A_\alpha(t)x + b_\alpha(t)$, $\forall (t,x) \in \mathbb{Z}^m \times K^n$. Let $\alpha_0 \in \{1, 2, \ldots, m\}$ and $t_0 \in \mathbb{Z}^m$. Let $y \in K^n$. There exists a function $x(\cdot)$ which verifies (2), $\forall t \geq t_0 - 1_{\alpha_0}$ and the condition $x(t_0) = y$.

For $t = t_0 - 1_{\alpha_0}$, one obtains $y = F_{\alpha_0}(t_0 - 1_{\alpha_0}, x(t_0 - 1_{\alpha_0}))$. Since y is arbitrary, it follows that $F_{\alpha_0}(t_0 - 1_{\alpha_0}, \cdot)$ is surjective, which is equivalent to that the matrix $A_{\alpha_0}(t_0 - 1_{\alpha_0})$ is invertible. Since t_0 is arbitrary, it follows that, for any $t \in \mathbb{Z}^m$, the matrices $A_{\alpha_0}(t)$ are invertible; here also $\alpha_0 \in \{1, 2, \ldots, m\}$ is arbitrary. □

Remark 2.1. *If the functions* $A_\alpha(\cdot)$ *and* $b_\alpha(\cdot)$ *are constants, then the relations* (3)*,* (4) *become*

$$A_\alpha A_\beta = A_\beta A_\alpha \tag{6}$$

$$(A_\alpha - I_n) b_\beta = (A_\beta - I_n) b_\alpha. \tag{7}$$

3. Fundamental (transition) matrix

We denote by $\mathcal{Z}$ one of the sets $\mathbb{Z}^m$ or $\{t \in \mathbb{Z}^m \mid t \geq t_1\}$ (with $t_1 \in \mathbb{Z}^m$).

Consider the functions $A_\alpha\colon \mathcal{Z} \to \mathcal{M}_n(K)$, $\alpha \in \{1, 2, \ldots, m\}$, which define the linear homogeneous recurrence

$$x(t + 1_\alpha) = A_\alpha(t) x(t), \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{8}$$

with the unknown function $x\colon \{t \in \mathcal{Z} \mid t \geq t_0\} \to K^n = \mathcal{M}_{n,1}(K)$, $t_0 \in \mathcal{Z}$.

Proposition 3.1. *Suppose that the relations* (3) *hold true.*

For each $t_0 \in \mathcal{Z}$ *and* $X_0 \in \mathcal{M}_n(K)$ *there exists a unique matrix solution* $X\colon \{t \in \mathcal{Z} \mid t \geq t_0\} \to \mathcal{M}_n(K)$ *of the recurrence*

$$X(t + 1_\alpha) = A_\alpha(t) X(t), \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{9}$$

with the condition $X(t_0) = X_0$*.*

Proof. For the n recurrences which are equivalent to the matrix recurrence, we apply Theorem 2.1. □

For each $t_0 \in \mathcal{Z}$, we denote $\chi(\,\cdot\,, t_0)\colon \{t \in \mathcal{Z} \mid t \geq t_0\} \to \mathcal{M}_n(K)$, the unique matrix solution of the recurrence (9) which verifies $X(t_0) = I_n$.

Definition 3.1. *Suppose that the relations* (3) *hold true.*

The matrix function $\chi(\,\cdot\,, \cdot)\colon \{(t,s) \in \mathcal{Z} \times \mathcal{Z} \mid t \geq s\} \to \mathcal{M}_n(K)$ *is called fundamental (transition) matrix associated to the linear homogeneous recurrence* (8)*.*

For $\alpha \in \{1, \ldots, m\}$ and $k \in \mathbb{N}$, we define the function $C_{\alpha,k}\colon \mathcal{Z} \to \mathcal{M}_n(K)$,

$$C_{\alpha,k}(t) = \begin{cases} \displaystyle\prod_{j=1}^{k} A_\alpha(t + (k-j) \cdot 1_\alpha) & \text{if} \quad k \geq 1 \\ \qquad I_n & \text{if} \quad k = 0. \end{cases} \tag{10}$$

Proposition 3.2. *Suppose that the relations* (3) *hold true.*

The matrix functions $\chi(\cdot)$ *and* $C_{\alpha,k}(\cdot)$ *have the properties:*

a) $\chi(t,s)\chi(s,r) = \chi(t,r)$*,* $\forall t, s, r \in \mathcal{Z}$*, with* $t \geq s \geq r$*.*

b) $\chi(s,s) = I_n$*,* $\forall s \in \mathcal{Z}$*.*

c) $\chi(t + k \cdot 1_\alpha, s) = C_{\alpha,k}(t) \cdot \chi(t,s)$*,* $\forall k \in \mathbb{N}$*,* $\forall t, s \in \mathcal{Z}$*, with* $t \geq s$*.*

d) $C_{\alpha,k}(t) = \chi(t + k \cdot 1_\alpha, t), \ \forall k \in \mathbb{N}, \forall t \in \mathcal{Z}$.

e) $C_{\alpha,k}(t + p \cdot 1_\beta)C_{\beta,p}(t) = C_{\beta,p}(t + k \cdot 1_\alpha)C_{\alpha,k}(t), \ \forall k, p \in \mathbb{N}, \forall t \in \mathcal{Z}$.

f) For any $t, s \in \mathcal{Z}$ *with* $t \geq s$, *we have*

$$\chi(t,s) = \prod_{\alpha=1}^{m-1} C_{\alpha,t^\alpha - s^\alpha}(s^1, ..., s^\alpha, t^{\alpha+1}, ..., t^m) \cdot C_{m,t^m - s^m}(s^1, s^2, \dots, s^{m-1}, s^m).$$

g) For any $t, s \in \mathcal{Z}$ *with* $t \geq s$, *the fundamental matrix* $\chi(t,s)$ *is invertible if and only if, for any* $\alpha \in \{1, 2, \dots, m\}$ *and any* $t \in \mathcal{Z}$, *the matrix* $A_\alpha(t)$ *is invertible.*

h) For any $\alpha \in \{1, 2, \dots, m\}$, *any* $k \in \mathbb{N}$ *and for any* $t \in \mathcal{Z}$, $C_{\alpha,k}(t)$ *is invertible if and only if, for any* $\alpha \in \{1, 2, \dots, m\}$ *and any* $t \in \mathcal{Z}$, *the matrix* $A_\alpha(t)$ *is invertible.*

i) If $\forall \alpha \in \{1, 2, \dots, m\}$, $\forall t \in \mathcal{Z}$, *the matrix* $A_\alpha(t)$ *is invertible, then* $\forall t, s, t_0 \in \mathcal{Z}$, *with* $t \geq s \geq t_0$, *we have* $\chi(t,s) = \chi(t,t_0)\chi(s,t_0)^{-1}$.

j) If $\forall \alpha \in \{1, 2, \dots, m\}$, *the matrix functions* $A_\alpha(\cdot)$ *are constant, then*

$C_{\alpha,k}(t) = A_\alpha^k, \quad \forall k \in \mathbb{N}, \forall t \in \mathbb{Z}^m, \forall \alpha \in \{1, 2, \dots, m\}$,

$\chi(t,s) = A_1^{(t^1-s^1)} A_2^{(t^2-s^2)} \cdot \ldots \cdot A_m^{(t^m-s^m)}, \quad \forall t, s \in \mathbb{Z}^m$, *with* $t \geq s$.

Proof. *b)* It follows directly from the definition of the function $\chi(\cdot,\cdot)$.

a) We fix s, r, with $s \geq r$. Let $Y_1, Y_2 \colon \{t \in \mathcal{Z} \,|\, t \geq s\} \to \mathcal{M}_n(K)$,

$Y_1(t) = \chi(t,s)\chi(s,r), \quad Y_2(t) = \chi(t,r), \quad \forall t \geq s$.

Then $Y_1(t + 1_\alpha) = \chi(t + 1_\alpha, s)\chi(s,r) = A_\alpha(t)\chi(t,s)\chi(s,r) = A_\alpha(t)Y_1(t)$;

$Y_1(s) = \chi(s,s)\chi(s,r) = I_n\chi(s,r) = \chi(s,r) = Y_2(s)$.

It follows that the functions $Y_1(\cdot)$ and $Y_2(\cdot)$ are both solutions of the recurrence (9) and coincide for $t = s$. From uniqueness property, it follows that $Y_1(\cdot)$ and $Y_2(\cdot)$ coincide; hence $\chi(t,s)\chi(s,r) = \chi(t,r)$, $\forall t \geq s$.

c) Induction after k. For $k = 0$, the statement is obvious.

For $k = 1$: the equality $\chi(t + 1_\alpha, s) = C_{\alpha,1}(t) \cdot \chi(t,s)$ is equivalent to $\chi(t + 1_\alpha, s) = A_\alpha(t)\chi(t,s)$, that is true, according to the definition of $\chi(\cdot, s)$. Let $k \geq 2$. Suppose the statement is true for $k - 1$ and we shall prove for k.

$\chi(t + k \cdot 1_\alpha, s) = A_\alpha(t + (k-1) \cdot 1_\alpha)\chi(t + (k-1) \cdot 1_\alpha, s)$

$= A_\alpha(t + (k-1) \cdot 1_\alpha)C_{\alpha,k-1}(t) \cdot \chi(t,s)$

$= A_\alpha(t + (k-1) \cdot 1_\alpha) \cdot A_\alpha(t + (k-2) \cdot 1_\alpha) \cdot \ldots \cdot A_\alpha(t + 1_\alpha)A_\alpha(t) \cdot \chi(t,s)$

$= C_{\alpha,k}(t)\chi(t,s)$.

d) In the equality from the step *c)*, we set $s = t$. We obtain $\chi(t + k \cdot 1_\alpha, t) = C_{\alpha,k}(t)\chi(t,t) = C_{\alpha,k}(t)$.

e) We use the step *d)*. $C_{\alpha,k}(t + p \cdot 1_\beta)C_{\beta,p}(t) =$

$= \chi(t + p \cdot 1_\beta + k \cdot 1_\alpha, t + p \cdot 1_\beta)\chi(t + p \cdot 1_\beta, t) = \chi(t + p \cdot 1_\beta + k \cdot 1_\alpha, t)$.

Analogously, one shows that $C_{\beta,p}(t + k \cdot 1_\alpha)C_{\alpha,k}(t) = \chi(t + k \cdot 1_\alpha + p \cdot 1_\beta, t)$.

f) One uses the step *c)*: $\chi(t,s) = \chi(t - (t^1 - s^1) \cdot 1_1 + (t^1 - s^1) \cdot 1_1, s) =$

$= C_{1,t^1-s^1}(t - (t^1 - s^1) \cdot 1_1) \cdot \chi(t - (t^1 - s^1) \cdot 1_1, s) =$

$= C_{1,t^1-s^1}(s^1, t^2, \dots, t^m)\chi\big((s^1, t^2, \dots, t^m), s\big) =$

$= C_{1,t^1-s^1}(s^1, t^2, \dots, t^m)\chi\big((s^1, t^2, \dots, t^m) - (t^2 - s^2) \cdot 1_2 + (t^2 - s^2) \cdot 1_2, s\big) =$

$= C_{1,t^1-s^1}(s^1, t^2, \dots, t^m)C_{2,t^2-s^2}\big((s^1, t^2, \dots, t^m) - (t^2 - s^2) \cdot 1_2\big) \cdot$

$\cdot \chi\big((s^1, t^2, \dots, t^m) - (t^2 - s^2) \cdot 1_2, s\big) =$

$= C_{1,t^1-s^1}(s^1, t^2, \dots, t^m)C_{2,t^2-s^2}(s^1, s^2, t^3, \dots, t^m)\chi\big((s^1, s^2, t^3 \dots, t^m), s\big)$ etc.

g) and *h)* If all the matrices $\chi(\cdot,\cdot)$ are invertible, then from the equality $\chi(t + 1_\alpha, s) = A_\alpha(t)\chi(t,s)$ it follows that $A_\alpha(t)$ is invertible.

If all matrices $A_\alpha(t)$ are invertible, then $C_{\alpha,k}(t)$ is invertible since $C_{\alpha,k}(t)$ is either I_n, or a product of the matrices $A_\alpha(\cdot)$.

If all the matrices $C_{\alpha,k}(t)$ are invertible, then $\chi(t,s)$ is invertible since $\chi(t,s)$ is a product of matrices $C_{\alpha,k}(\cdot)$, according to the step f).

i) The matrix $\chi(s,t_0)$ is invertible. From the relation
$\chi(t,s)\chi(s,t_0)=\chi(t,t_0)$, we obtain $\chi(t,s)=\chi(t,t_0)\chi(s,t_0)^{-1}$.

j) The relation $C_{\alpha,k}(t)=A_\alpha^k$ follows directly from the definition of $C_{\alpha,k}(t)$. The second equality required is obtained using the step f). □

The following result can be proved easily by direct computation.

Proposition 3.3. *We denote by $\mathcal{Z}$ one of the sets $\mathbb{Z}^m$ or $\{t\in\mathbb{Z}^m \mid t\ge t_1\}$ (with $t_1\in\mathbb{Z}^m$).*

We consider the functions $A_\alpha\colon\mathcal{Z}\to\mathcal{M}_n(K)$, $\alpha\in\{1,2,\dots,m\}$, for which the relations (3) *are satisfied. Let $(t_0,x_0)\in\mathcal{Z}\times K^n$. Then, the unique function $x\colon\{t\in\mathcal{Z}\mid t\ge t_0\}\to K^n=\mathcal{M}_{n,1}(K)$, which, for any $t\ge t_0$, verifies the recurrence* (8) *and the initial condition $x(t_0)=x_0$, is*

$$x(t)=\chi(t,t_0)x_0,\quad \forall t\ge t_0.$$

If $\forall\alpha\in\{1,2,\dots,m\}$, the matrix functions $A_\alpha(\cdot)$ are constant, then

$$x(t)=A_1^{(t^1-t_0^1)}A_2^{(t^2-t_0^2)}\cdot\ldots\cdot A_m^{(t^m-t_0^m)}x_0,\quad \forall t\ge t_0. \tag{11}$$

Remark 3.1. *Let us suppose that $A_\alpha(\cdot)$, $b_\alpha(\cdot)$ verify the relations* (3) *and* (4). *Let $y\colon\{t\in\mathcal{Z}\mid t\ge t_0\}\to K^n$ be a particular solution of the recurrence* (2).

Note that for any other solution of the recurrence (2), *$x\colon\{t\in\mathcal{Z}\mid t\ge t_0\}\to K^n$, the function $x(\cdot)-y(\cdot)$ is a solution of the recurrence* (8). *Hence, according to Proposition* 3.3, *we find $x(t)-y(t)=\chi(t,t_0)\big(x(t_0)-y(t_0)\big)$, $\forall t\ge t_0$.*

It follows that the unique function $x\colon\{t\in\mathcal{Z}\mid t\ge t_0\}\to K^n$, which, for any $t\ge t_0$, verifies the recurrence (2) *and the initial condition $x(t_0)=x_0$, is*

$$x(t)=\chi(t,t_0)\big(x_0-y(t_0)\big)+y(t),\quad \forall t\ge t_0.$$

Theorem 3.1. *We consider the functions $A_\alpha\colon\{t\in\mathbb{Z}^m\mid t\ge t_0\}\to\mathcal{M}_n(K)$, $\alpha\in\{1,2,\dots,m\}$ (with $t_0\in\mathbb{Z}^m$), for which the relations* (3) *are satisfied. We denote*

$$V(t_0)=\Big\{x\colon\{t\in\mathbb{Z}^m\mid t\ge t_0\}\to K^n\ \Big|\ x \text{ is solution of the recurrence } (8)\Big\}.$$

a) *The set $V(t_0)$ is a K - vector space of dimension n.*
b) *Let $\{v_1,v_2,\dots,v_n\}$ be a basis of K^n. For $j\in\{1,2,\dots,n\}$, we consider $y_j\colon\{t\in\mathbb{Z}^m\mid t\ge t_0\}\to K^n$ as solution of the recurrence* (8) *which verifies $y_j(t_0)=v_j$. Then the set $\{y_1(\cdot),y_2(\cdot),\dots,y_n(\cdot)\}$ is a basis of the vector space $V(t_0)$.*
c) *If $z_1(t)$, $z_2(t)$, …, $z_n(t)$ are the columns of the matrix $\chi(t,t_0)$ (for $t\ge t_0$), then $\{z_1(\cdot),z_2(\cdot),\dots,z_n(\cdot)\}$ is a basis of the vector space $V(t_0)$.*

Proof. *a*) and *b*): One verifies automatically that $V(t_0)$ is a vector space.

Let $x(\cdot)\in V(t_0)$ and let $x_0=x(t_0)$. There exist a_1, a_2, …, $a_n\in K$ such that $x_0=a_1v_1+a_2v_2+\ldots+a_nv_n$.

Let $y(\cdot)=a_1y_1(\cdot)+a_2y_2(\cdot)+\ldots+a_ny_n(\cdot)$. Obvious that $y(\cdot)\in V(t_0)$. We find: $y(t_0)=a_1y_1(t_0)+a_2y_2(t_0)+\ldots+a_ny_n(t_0)=a_1v_1+a_2v_2+\ldots+a_nv_n=x_0$. Since $x(\cdot)$ and $y(\cdot)$ are solutions of the recurrence (8) and $x(t_0)=y(t_0)=x_0$, by uniqueness property (Theorem 2.1), it follows that $x(\cdot)=y(\cdot)$; consequently

$x(\cdot) = a_1 y_1(\cdot) + a_2 y_2(\cdot) + \ldots + a_n y_n(\cdot)$. We have proved that $\{y_1(\cdot), y_2(\cdot), \ldots, y_n(\cdot)\}$ is a system of generators for the vector space $V(t_0)$.

Let $a_1,\ a_2,\ \ldots,\ a_n \in K$ such that $a_1 y_1(\cdot) + a_2 y_2(\cdot) + \ldots + a_n y_n(\cdot) = 0$, i.e. $a_1 y_1(t) + a_2 y_2(t) + \ldots + a_n y_n(t) = 0$, $\forall t \geq t_0$. For $t = t_0$, we obtain $\sum_{j=1}^{n} a_j y_j(t_0) = 0$, i.e., $\sum_{j=1}^{n} a_j v_j = 0$. Consequently $a_j = 0$, $\forall j$. Hence $y_1(\cdot), y_2(\cdot), \ldots, y_n(\cdot)$ are linearly independent, i.e., $\{y_1(\cdot), y_2(\cdot), \ldots, y_n(\cdot)\}$ is a basis of the vector space $V(t_0)$; the dimension of this space is obviously n.

c) Let $\{e_1, e_2, \ldots, e_n\}$ be the canonical basis of the space K^n. Hence $I_n = \begin{pmatrix} e_1 & e_2 & \ldots & e_n \end{pmatrix}$. From the definition of the matrix $\chi(t, t_0)$, it follows that $z_j(\cdot)$ is the solution of the recurrence (8) which verifies $z_j(t_0) = e_j$, $\forall j$. According to step *b*), it follows that $\{z_1(\cdot), z_2(\cdot), \ldots, z_n(\cdot)\}$ is a basis of the vector space $V(t_0)$. □

3.1. **Case of non-degenerate matrices**

In this subsection, we consider the functions $A_\alpha \colon \mathbb{Z}^m \to \mathcal{M}_n(K)$, $\alpha \in \{1, 2, \ldots, m\}$, such that for any $\alpha \in \{1, 2, \ldots, m\}$ and any $t \in \mathbb{Z}^m$, the matrix $A_\alpha(t)$ is invertible and $\forall t \in \mathbb{Z}^m$, $\forall \alpha, \beta \in \{1, 2, \ldots, m\}$ the relations (3) hold.

According to Theorem 2.2, for any pair $(t_0, x_0) \in \mathbb{Z}^m \times K^n$, there exists a unique function $x \colon \mathbb{Z}^m \to K^n$, which verifies the recurrence

$$x(t + 1_\alpha) = A_\alpha(t) x(t), \quad \forall t \in \mathbb{Z}^m, \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{12}$$

and the condition $x(t_0) = x_0$.

In this case the fundamental matrix can be defined on the set $\mathbb{Z}^m \times \mathbb{Z}^m$. For each $t_0 \in \mathbb{Z}^m$, $\chi(\,\cdot\,, t_0) \colon \mathbb{Z}^m \to \mathcal{M}_n(K)$ is the unique matrix solution of the recurrence $X(t+1_\alpha) = A_\alpha(t) X(t)$, $\forall \alpha \in \{1, 2, \ldots, m\}$, which verifies $X(t_0) = I_n$. In this way we obtain the fundamental matrix associated to the recurrence (12), i.e., the function $\chi(\,\cdot\,, \cdot\,) \colon \mathbb{Z}^m \times \mathbb{Z}^m \to \mathcal{M}_n(K)$.

The statements in Proposition 3.2 are maintained with few changes.

The statement *a*) rewrites $\chi(t, s)\chi(s, r) = \chi(t, r)$, $\forall t, s, r \in \mathbb{Z}^m$. The proof is similar to those given in the proof of Proposition 3.2.

The point *i*) becomes $\chi(t, s) = \chi(t, t_0)\chi(s, t_0)^{-1}$, $\forall t, s, t_0 \in \mathbb{Z}^m$. For $t_0 = t$, we obtain $\chi(t, s) = \chi(s, t)^{-1}$, $\forall t, s \in \mathbb{Z}^m$.

One can easily show that the point *j*) can be completed in this way:
if $\forall \alpha \in \{1, 2, \ldots, m\}$, *the matrix functions* $A_\alpha(\cdot)$ *are constant, then*
$\chi(t, s) = A_1^{(t^1 - s^1)} A_2^{(t^2 - s^2)} \cdot \ldots \cdot A_m^{(t^m - s^m)}$, $\quad \forall t, s \in \mathbb{Z}^m$.

The analog of Proposition 3.3 is:
The solution of the recurrence (12) *which verifies* $x(t_0) = x_0$, *is*
$x \colon \mathbb{Z}^m \to K^n$, $\quad x(t) = \chi(t, t_0) x_0$, $\quad \forall t \in \mathbb{Z}^m$.
If $\forall \alpha \in \{1, 2, \ldots, m\}$, *the matrix functions* $A_\alpha(\cdot)$ *are constants, then*
$x(t) = A_1^{(t^1 - t_0^1)} A_2^{(t^2 - t_0^2)} \cdot \ldots \cdot A_m^{(t^m - t_0^m)} x_0$, $\quad \forall t \in \mathbb{Z}^m$.

Let $t_0 \in \mathbb{Z}^m$. We denote $W(t_0) = \left\{ x \colon \mathbb{Z}^m \to K^n \;\middle|\; x \text{ is solution of the recurrence (12)} \right\}$.

With a proof similar to those for Theorem 3.1, we obtain:

a) *The set* $W(t_0)$ *is a* K *- vector space of dimension* n.

b) *Let* $\{v_1, v_2, \ldots, v_n\}$ *be a basis of* K^n. *For* $j \in \{1, 2, \ldots, n\}$, *we consider*

$y_j\colon \mathbb{Z}^m \to K^n$ as solution of the recurrence (12) *which verifies $y_j(t_0) = v_j$. Then the set $\{y_1(\cdot), y_2(\cdot), \dots, y_n(\cdot)\}$ is a basis of the space $W(t_0)$.*
c) If $z_1(t)$, $z_2(t)$, ..., $z_n(t)$ are the columns of the matrix fundamental $\chi(t, t_0)$, then $\{z_1(\cdot), z_2(\cdot), \dots, z_n(\cdot)\}$ is a basis of the space $W(t_0)$.

4. Solving the linear discrete multitime multiple recurrence with constant coefficients

Let $A_1, A_2, \dots, A_m \in \mathcal{M}_n(K)$ be constant matrices such that $A_\alpha A_\beta = A_\beta A_\alpha$, $\forall \alpha, \beta \in \{1, 2, \dots, m\}$. Let $t_0 \in \mathbb{Z}^m$ and $x_0 \in K^n$. According to Proposition 3.3, the function $x\colon \{t \in \mathbb{Z}^m \,|\, t \geq t_0\} \to K^n$, given by the formula (11), is the solution of the recurrence

$$x(t + 1_\alpha) = A_\alpha x(t), \quad \forall \alpha \in \{1, 2, \dots, m\}, \tag{13}$$

which verifies $x(t_0) = x_0$. If $\forall \alpha$, A_α is invertible, then according to results in Subsection 3.1, the recurrence (13) has a unique solution $x\colon \mathbb{Z}^m \to K^n$ which verifies $x(t_0) = x_0$. This is defined by the same formula (11), but for any $t \in \mathbb{Z}^m$.

We shall use the following result.

Theorem 4.1. *Let K be a field and let $\mathcal{F} \neq \emptyset$, $\mathcal{F} \subseteq \mathcal{M}_n(K)$, such that any two matrices from $\mathcal{F}$ commute. If any matrix in $\mathcal{F}$ is diagonalizable (over K), then there exists an invertible matrix $T \in \mathcal{M}_n(K)$, such that $\forall A \in \mathcal{F}$, $\exists D(A) \in \mathcal{M}_n(K)$, $D(A)$ diagonal matrix, for which $A = TD(A)T^{-1}$.*

We shall denote by $diag(d_1; d_2; \dots; d_n) \in \mathcal{M}_n(K)$, the diagonal matrix, which has on the principal diagonal the elements $d_1, d_2, \dots, d_n$, in this order.

If the matrix $A \in \mathcal{M}_n(K)$ has the columns $q_1, q_2, \dots, q_n$, we shall denote $A = col(q_1; q_2; \dots; q_n)$.

Theorem 4.2. *Let $A_1, A_2, \dots, A_m \in \mathcal{M}_n(K)$ be diagonalizable matrices (over K), such that $A_\alpha A_\beta = A_\beta A_\alpha$, $\forall \alpha, \beta \in \{1, 2, \dots, m\}$.*

Let $T = col(v_1; v_2; \dots; v_n) \in \mathcal{M}_n(K)$ be an invertible matrix such that
$$A_\alpha = T \cdot diag(\lambda_{1,\alpha}; \lambda_{2,\alpha}; \dots; \lambda_{n,\alpha}) \cdot T^{-1}, \quad \forall \alpha \in \{1, 2, \dots, m\},$$
where $\lambda_{1,\alpha}, \lambda_{2,\alpha}, \dots, \lambda_{n,\alpha} \in K$ (such T exists, according to Theorem 4.1).

Then, $\forall (t_0, x_0) \in \mathbb{Z}^m \times K^n$, the solution of the recurrence (13), *which verifies $x(t_0) = x_0$, is $x\colon \{t \in \mathbb{Z}^m \,|\, t \geq t_0\} \to K^n$,*

$$x(t) = \sum_{j=1}^{n} c_j \Big(\prod_{\alpha=1}^{m} \lambda_{j,\alpha}^{t^\alpha - t_0^\alpha} \Big) v_j, \quad \forall t \geq t_0, \tag{14}$$

where $(c_1, c_2, \dots, c_n)^\top = T^{-1} x_0$.

If $\forall \alpha$, the matrix A_α is invertible, then the recurrence (13) *has a unique solution $x\colon \mathbb{Z}^m \to K^n$ which verifies $x(t_0) = x_0$. This solution is defined also by the formula* (14), *but for any $t \in \mathbb{Z}^m$.*

Proof. According to formula (11), for any $t \geq t_0$, we have

$$\begin{aligned} x(t) &= \Big(\prod_{\alpha=1}^{m} A_\alpha^{t^\alpha - t_0^\alpha} \Big) x_0 = \Big(\prod_{\alpha=1}^{m} \big(T \cdot diag(\lambda_{1,\alpha}; \lambda_{2,\alpha}; \dots; \lambda_{n,\alpha}) \cdot T^{-1} \big)^{t^\alpha - t_0^\alpha} \Big) x_0 \\ &= T \cdot \Big(\prod_{\alpha=1}^{m} diag\big(\lambda_{1,\alpha}^{t^\alpha - t_0^\alpha}; \lambda_{2,\alpha}^{t^\alpha - t_0^\alpha}; \dots; \lambda_{n,\alpha}^{t^\alpha - t_0^\alpha} \big) \Big) \cdot T^{-1} x_0 \\ &= col(v_1; v_2; \dots; v_n) \cdot diag\Big(\prod_{\alpha=1}^{m} \lambda_{1,\alpha}^{t^\alpha - t_0^\alpha}; \prod_{\alpha=1}^{m} \lambda_{2,\alpha}^{t^\alpha - t_0^\alpha}; \dots; \prod_{\alpha=1}^{m} \lambda_{n,\alpha}^{t^\alpha - t_0^\alpha} \Big) \cdot T^{-1} x_0 \end{aligned}$$

$$= col\left(\Big(\prod_{\alpha=1}^{m}\lambda_{1,\alpha}^{t^\alpha-t_0^\alpha}\Big)v_1;\Big(\prod_{\alpha=1}^{m}\lambda_{2,\alpha}^{t^\alpha-t_0^\alpha}\Big)v_2;\ldots;\Big(\prod_{\alpha=1}^{m}\lambda_{n,\alpha}^{t^\alpha-t_0^\alpha}\Big)v_n\right)\cdot(c_1,c_2,\ldots,c_n)^\top$$

$$=\sum_{j=1}^{n}c_j\Big(\prod_{\alpha=1}^{m}\lambda_{j,\alpha}^{t^\alpha-t_0^\alpha}\Big)v_j.$$

If all the matrices A_α are invertible, then we saw that the formula (11) is true for any $t\in\mathbb{Z}^m$. But A_α is invertible iff $\lambda_{j,\alpha}\neq 0$, $\forall j=\overline{1,n}$. We notice easily that in this case all equalities above are true for any $t\in\mathbb{Z}^m$. □

Remark 4.1. *If $T=col(v_1;v_2;\ldots;v_n)\in\mathcal{M}_n(K)$ is the invertible matrix which appears in Theorem 4.2, then $\{v_1,v_2,\ldots,v_n\}$ is a basis of K^n, and each v_j is an eigenvector for all the matrices A_α.*

Remark 4.2. *In the conditions of Theorem 4.2, the fundamental matrix is*

$$\chi(t,t_0)=T\cdot\Big(\prod_{\alpha=1}^{m}diag\big(\lambda_{1,\alpha}^{t^\alpha-t_0^\alpha};\lambda_{2,\alpha}^{t^\alpha-t_0^\alpha};\ldots;\lambda_{n,\alpha}^{t^\alpha-t_0^\alpha}\big)\Big)\cdot T^{-1},\quad \forall t\geq t_0.$$

If all the matrices A_α are invertible, then the foregoing formula is true for any $(t,t_0)\in\mathbb{Z}^m\times\mathbb{Z}^m$.

For $A\in\mathcal{M}_n(K)$ and $k\in\mathbb{N}$, we denote $S(k;A)=\begin{cases}I_n+A+\ldots+A^{k-1} & \text{if}\quad k\geq 1\\ O_n & \text{if}\quad k=0.\end{cases}$

Theorem 4.3. *For $\alpha\in\{1,2,\ldots,m\}$, we consider the matrices $A_\alpha\in\mathcal{M}_n(K)$, $b_\alpha\in K^n=\mathcal{M}_{n,1}(K)$ such that*

$$A_\alpha A_\beta=A_\beta A_\alpha,\quad\forall\alpha,\beta\in\{1,2,\ldots,m\}\tag{15}$$

$$(I_n-A_\alpha)b_\beta=(I_n-A_\beta)b_\alpha,\quad\forall\alpha,\beta\in\{1,2,\ldots,m\}.\tag{16}$$

Let $(t_0,x_0)\in\mathbb{Z}^m\times K^n$. The solution of the recurrence

$$x(t+1_\alpha)=A_\alpha x(t)+b_\alpha,\quad\forall\alpha\in\{1,2,\ldots,m\},\tag{17}$$

which verifies $x(t_0)=x_0$, is the function $x\colon\{t\in\mathbb{Z}^m\,|\,t\geq t_0\}\to K^n$, defined for any $t\geq t_0$ by

$$x(t)=\Big(\prod_{\alpha=1}^{m}A_\alpha^{t^\alpha-t_0^\alpha}\Big)x_0+S(t^1-t_0^1;A_1)b_1+\sum_{\beta=2}^{m}\Big(\prod_{\alpha=1}^{\beta-1}A_\alpha^{t^\alpha-t_0^\alpha}\Big)S(t^\beta-t_0^\beta;A_\beta)b_\beta,$$

if $m\geq 2$, respectively $x(t)=A_1^{t^1-t_0^1}x_0+S(t^1-t_0^1;A_1)b_1$, if $m=1$.

Proof. According to Theorem 2.1 and Remark 2.1 it follows that the recurrence (17) has a unique solution which verifies $x(t_0)=x_0$.

We prove the statement by induction on m, the number of components of t. For $m=1$, one verifies immediately, by direct computations, that for any $t\geq t_0$, the function $x(t)$ verifies the recurrence (17) and the condition $x(t_0)=x_0$.

Let $m\geq 2$. Suppose the statement is true for $m-1$ and we shall prove it for m. We denote $\tilde t=(t^2,\ldots,t^m)$; $\tilde t_0=(t_0^2,\ldots,t_0^m)$.

Let $\tilde x(\tilde t)=x(t_0^1,\tilde t)=x(t_0^1,t^2,\ldots,t^m)$. If $t^1>t_0^1$, then

$$\begin{aligned}x(t)&=x(t^1,\tilde t)=A_1x(t^1-1,\tilde t)+b_1=A_1^2x(t^1-2,\tilde t)+A_1b_1+b_1=\\&=\ldots=A_1^kx(t^1-k,\tilde t)+A_1^{k-1}b_1+\ldots+A_1b_1+b_1=\\&=\ldots=A_1^{t^1-t_0^1}x(t_0^1,\tilde t)+A_1^{t^1-t_0^1-1}b_1+\ldots+A_1b_1+b_1\\&=A_1^{t^1-t_0^1}\tilde x(\tilde t)+S(t^1-t_0^1;A_1)b_1.\end{aligned}$$

We have proved that if $t^1>t_0^1$, then $x(t)=A_1^{t^1-t_0^1}\tilde x(\tilde t)+S(t^1-t_0^1;A_1)b_1$; relation which is verified immediately for $t^1=t_0^1$.

For $\alpha \in \{2, \dots, m\}$, we denote $\tilde{1}_\alpha = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^{m-1}$; hence $1_\alpha = (0, \tilde{1}_\alpha)$. For $\alpha \geq 2$ and $t^1 = t_0^1$, the relations (17) become: $x((t_0^1, \tilde{t}) + (0, \tilde{1}_\alpha)) = A_\alpha x(t_0^1, \tilde{t}) + b_\alpha$, i.e., $\tilde{x}(\tilde{t} + \tilde{1}_\alpha) = A_\alpha \tilde{x}(\tilde{t}) + b_\alpha$, $\forall\, \tilde{t} \geq \tilde{t}_0$, $\forall \alpha \in \{2, \dots, m\}$.

Obviously $\tilde{x}(\tilde{t}_0) = x(t_0^1, \tilde{t}_0) = x(t_0) = x_0$. Since $\tilde{t}$ has $m-1$ components, from the induction hypothesis follows that $\forall \tilde{t} \geq \tilde{t}_0$ we have

$$\tilde{x}(\tilde{t}) = \Big(\prod_{\alpha=2}^{m} A_\alpha^{t^\alpha - t_0^\alpha}\Big) x_0 + S(t^2 - t_0^2; A_2) b_2 + \sum_{\beta=3}^{m} \Big(\prod_{\alpha=2}^{\beta-1} A_\alpha^{t^\alpha - t_0^\alpha}\Big) S(t^\beta - t_0^\beta; A_\beta) b_\beta,$$

if $m \geq 3$, respectively $\tilde{x}(\tilde{t}) = A_2^{t^2 - t_0^2} x_0 + S(t^2 - t_0^2; A_2) b_2$, if $m = 2$.

Hence, if $m \geq 3$, for any $t \geq t_0$, we have $x(t) = A_1^{t^1 - t_0^1} \tilde{x}(\tilde{t}) + S(t^1 - t_0^1; A_1) b_1$

$$= A_1^{t^1 - t_0^1} \Big(\prod_{\alpha=2}^{m} A_\alpha^{t^\alpha - t_0^\alpha}\Big) x_0 + A_1^{t^1 - t_0^1} S(t^2 - t_0^2; A_2) b_2$$

$$+ \sum_{\beta=3}^{m} A_1^{t^1 - t_0^1} \Big(\prod_{\alpha=2}^{\beta-1} A_\alpha^{t^\alpha - t_0^\alpha}\Big) S(t^\beta - t_0^\beta; A_\beta) b_\beta + S(t^1 - t_0^1; A_1) b_1$$

$$= \Big(\prod_{\alpha=1}^{m} A_\alpha^{t^\alpha - t_0^\alpha}\Big) x_0 + S(t^1 - t_0^1; A_1) b_1 + \sum_{\beta=2}^{m} \Big(\prod_{\alpha=1}^{\beta-1} A_\alpha^{t^\alpha - t_0^\alpha}\Big) S(t^\beta - t_0^\beta; A_\beta) b_\beta.$$

If $m = 2$, for any $t \geq t_0$, we have $x(t) = A_1^{t^1 - t_0^1} \tilde{x}(\tilde{t}) + S(t^1 - t_0^1; A_1) b_1 =$

$= A_1^{t^1 - t_0^1} A_2^{t^2 - t_0^2} x_0 + A_1^{t^1 - t_0^1} S(t^2 - t_0^2; A_2) b_2 + S(t^1 - t_0^1; A_1) b_1.$ □

Theorem 4.4. *Consider the matrices* $A_\alpha \in \mathcal{M}_n(K)$, $b_\alpha \in K^n = \mathcal{M}_{n,1}(K)$, *which for any* $\alpha, \beta \in \{1, 2, \dots, m\}$ *satisfy the conditions* (15) *and* (16).

a) Suppose there exists an index $\alpha_0 \in \{1, 2, \dots, m\}$, *for which the matrix* $I_n - A_{\alpha_0}$ *is invertible. Let* $v \in K^n$, *such that* $(I_n - A_{\alpha_0}) v = b_{\alpha_0}$, *i.e.* $v = (I_n - A_{\alpha_0})^{-1} b_{\alpha_0}$. *Then*

$$(I_n - A_\alpha) v = b_\alpha, \quad \forall \alpha \in \{1, 2, \dots, m\}. \tag{18}$$

b) Suppose there exists $v \in K^n$, *such that, for any* $\alpha \in \{1, 2, \dots, m\}$, *the relations* (18) *are true.*

Let $(t_0, x_0) \in \mathbb{Z}^m \times K^n$. *Then, the solution of the recurrence* (17), *which verifies* $x(t_0) = x_0$, *is the function* $x\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\} \to K^n$, *defined for any* $t \geq t_0$ *by*

$$x(t) = \Big(\prod_{\alpha=1}^{m} A_\alpha^{t^\alpha - t_0^\alpha}\Big) \cdot (x_0 - v) + v. \tag{19}$$

If furthermore, $\forall \alpha$, *the matrix* A_α *is invertible, then the recurrence* (17) *has a unique solution* $x\colon \mathbb{Z}^m \to K^n$, *which verifies* $x(t_0) = x_0$. *This is defined also by the formula* (19), *but for any* $t \in \mathbb{Z}^m$.

Proof. a) From the relation $(I_n - A_\alpha) b_{\alpha_0} = (I_n - A_{\alpha_0}) b_\alpha$ it follows the equality: $(I_n - A_\alpha)(I_n - A_{\alpha_0}) v = (I_n - A_{\alpha_0}) b_\alpha \iff (I_n - A_{\alpha_0})(I_n - A_\alpha) v = (I_n - A_{\alpha_0}) b_\alpha$. Since $I_n - A_{\alpha_0}$ is invertible, we obtain $(I_n - A_\alpha) v = b_\alpha$.

b) Let $x(\cdot)$ be the solution of the recurrence (17), which verifies $x(t_0) = x_0$. We denote $y(\cdot) = x(\cdot) - v$, i.e. $x(\cdot) = y(\cdot) + v$. We have

$$y(t + 1_\alpha) + v = A_\alpha \big(y(t) + v\big) + b_\alpha, \quad \forall \alpha \in \{1, 2, \dots, m\},$$

$$\iff y(t + 1_\alpha) = A_\alpha y(t) - (I_n - A_\alpha) v + b_\alpha, \quad \forall \alpha \in \{1, 2, \dots, m\}.$$

Since the relations (18) are true, it follows that $y(\cdot)$ is the solution of the recurrence (13) which verifies $y(t_0) = x_0 - v$. According to Proposition 3.3, $\forall t \geq t_0$ we have

$$y(t)=\Big(\prod_{\alpha=1}^{m}A_\alpha^{t^\alpha-t_0^\alpha}\Big)\cdot(x_0-v). \tag{20}$$

From the equality $y(t)=x(t)-v$, we obtain the relation (19).

If $\forall\alpha$, the matrix A_α is invertible, then according to the remarks in Subsection 3.1, it follows that the relation (20) is true for any $t\in\mathbb{Z}^m$; consequently, also the formula (19) is valid for any $t\in\mathbb{Z}^m$. □

Remark 4.3. *The statement: "there exists $v\in K^n$, such that, for any α, the relations* (18) *are true" is equivalent to the fact that the recurrence* (17) *has a constant solution $x(\cdot)=v$.*

5. **Recurrences on a monoid**

Let M be a nonvoid set, let $(N,\cdot,E)$ be a monoid and let $\varphi\colon N\times M\to M$ be an action of the monoid N on the set M, i.e.

$$\varphi(AB,x)=\varphi\big(A,(B,x)\big),\ \ \varphi(e,x)=x,\quad \forall A,B\in N,\forall x\in M. \tag{21}$$

For any $A\in N$, $x\in M$, we denote $\varphi(A,x)=Ax$ (not to be confused with the operation of monoid N). The relations (21) become

$(AB)x=A(Bx),\ \ ex=x,\quad \forall A,B\in N,\forall x\in M.$

We denote by $\mathcal{Z}$ one of the sets $\mathbb{Z}^m$ or $\{t\in\mathbb{Z}^m \,|\, t\ge t_1\}$ (with $t_1\in\mathbb{Z}^m$).

For each $\alpha\in\{1,2,\dots,m\}$, we consider the functions $A_\alpha\colon\mathcal{Z}\to N$, which define the recurrence

$$x(t+1_\alpha)=A_\alpha(t)x(t),\quad \forall\alpha\in\{1,2,\dots,m\}, \tag{22}$$

with the unknown function $x\colon\{t\in\mathcal{Z}\,|\,t\ge t_0\}\to M$, $t_0\in\mathcal{Z}$.

Remark 5.1. *For $(N,\cdot,E)=(\mathcal{M}_n(K),\cdot,I_n)$, $M=K^n=\mathcal{M}_{n,1}(K)$ and the action*

$$\varphi\colon\mathcal{M}_n(K)\times K^n\to K^n,\quad \varphi(A,x)=Ax,\quad \forall A\in\mathcal{M}_n(K),\ \forall x\in K^n,$$

the recurrence (22) *becomes the recurrence* (8).

With a similar proof with those in Theorem 2.1, it follows

Theorem 5.1. *a) If, for any $(t_0,x_0)\in\mathcal{Z}\times M$, there exists at least one function $x\colon\{t\in\mathcal{Z}\,|\,t\ge t_0\}\to M$, which, for any $t\ge t_0$, verifies the recurrence* (22) *and the condition $x(t_0)=x_0$, then*

$$A_\alpha(t+1_\beta)A_\beta(t)x=A_\beta(t+1_\alpha)A_\alpha(t)x, \tag{23}$$

$\forall t\in\mathcal{Z},\ \forall x\in M,\ \forall\alpha,\beta\in\{1,2,\dots,m\}$.

b) If the relations (23)*, are satisfied, then, for any $(t_0,x_0)\in\mathcal{Z}\times M$, there exists a unique function $x\colon\{t\in\mathcal{Z}\,|\,t\ge t_0\}\to M$, which, for any $t\ge t_0$ verifies the recurrence* (22) *and the condition $x(t_0)=x_0$.*

Now we consider the action of the monoid N on himself, $\xi\colon N\times N\to N$, defined by $\xi(A,X)=A\cdot X$, $\forall A,X\in N$ ("$\cdot$" is the operation considered on N).

In this case, being given the functions $A_\alpha\colon\mathcal{Z}\to N$, $\alpha\in\{1,2,\dots,m\}$, the analogue of the recurrence (22) is

$$X(t+1_\alpha)=A_\alpha(t)X(t),\quad \forall\alpha\in\{1,2,\dots,m\}, \tag{24}$$

with the unknown function $X\colon\{t\in\mathcal{Z}\,|\,t\ge t_0\}\to N$, $t_0\in\mathcal{Z}$.

By doing like in the proof of Theorem 2.1, it is shown that

Theorem 5.2. *a) If, for any $t_0\in\mathcal{Z}$, there exists at least one function $X\colon\{t\in\mathcal{Z}\,|\,t\ge t_0\}\to N$, which, for any $t\ge t_0$, verifies the recurrence* (24) *and the condition $X(t_0)=E$, then*

$$A_\alpha(t+1_\beta)A_\beta(t)=A_\beta(t+1_\alpha)A_\alpha(t), \tag{25}$$

$\forall t\in\mathcal{Z},\ \forall\alpha,\beta\in\{1,2,\dots,m\}$.

b) If the relations (25)*, are satisfied, then, for any $(t_0,X_0)\in\mathcal{Z}\times N$, there exists a unique function $x\colon\{t\in\mathcal{Z}\,|\,t\ge t_0\}\to N$, which, for any $t\ge t_0$ verifies the recurrence* (24) *and the condition $X(t_0)=X_0$.*

Definition 5.1. *Suppose that the functions* $A_\alpha\colon \mathcal{Z} \to N$, $\alpha \in \{1, 2, \ldots, m\}$, *verify the relations* (25).

For each $t_0 \in \mathcal{Z}$, *we denote* $\chi(\,\cdot\,, t_0)\colon \{t \in \mathcal{Z} \,|\, t \geq t_0\} \to N$ *the unique solution of the recurrence* (24) *which verifies* $X(t_0) = E$.

The function $\chi(\,\cdot\,,\cdot\,)\colon \{(t, s) \in \mathcal{Z} \times \mathcal{Z} \,|\, t \geq s\} \to N$ *is called the fundamental (transition) function associated to the recurrence* (22).

This is the analog of the fundamental solution associated to the recurrence (8), introduced in Definition 3.1.

For $\alpha \in \{1, 2, \ldots, m\}$ and $k \in \mathbb{N}$, we consider the function $C_{\alpha,k}\colon \mathcal{Z} \to N$, defined formally by the relation (10), replacing I_n with E, but now $A_\alpha\colon \mathcal{Z} \to N$, hence $A_\alpha(\cdot)$ are not matrix functions.

Remark 5.2. *If the functions* $A_\alpha\colon \mathcal{Z} \to N$, $\alpha \in \{1, 2, \ldots, m\}$, *verify the relations* (25), *then Proposition* 3.2 *can be rewritten with the help of* $A_\alpha(\cdot)$, $C_{\alpha,k}(\cdot)$, $\chi(\,\cdot\,,\cdot\,)$ *discussed in this section (instead of matrices); the proof is identically to those in Proposition* 3.2.

Analogous to Proposition 3.3, we have

Proposition 5.1. *We consider the functions* $A_\alpha\colon \mathcal{Z} \to N$, $\alpha \in \{1, 2, \ldots, m\}$, *for which the relations* (25) *are satisfied. Let* $(t_0, x_0) \in \mathcal{Z} \times M$. *Then, the unique function* $x\colon \{t \in \mathcal{Z} \,|\, t \geq t_0\} \to M$, *which, for any* $t \geq t_0$, *verifies the recurrence* (22) *and the condition* $x(t_0) = x_0$, *is*

$$x(t) = \chi(t, t_0)x_0, \quad \forall t \geq t_0.$$

If $\forall\alpha \in \{1, 2, \ldots, m\}$, *the functions* $A_\alpha(\cdot)$ *are constant, then*

$$x(t) = A_1^{(t^1 - t_0^1)} A_2^{(t^2 - t_0^2)} \cdot \ldots \cdot A_m^{(t^m - t_0^m)} x_0, \quad \forall t \geq t_0.$$

We return to the recurrence (2), i.e. $x(t + 1_\alpha) = A_\alpha(t)x(t) + b_\alpha(t)$; where K is a field and $A_\alpha\colon \mathcal{Z} \to \mathcal{M}_n(K)$, $b_\alpha\colon \mathcal{Z} \to K^n = \mathcal{M}_{n,1}(K)$ verify the relations (3) and (4). Let $x\colon \{t \in \mathbb{Z}^m \,|\, t \geq t_0\} \to K^n$ the function which, for any $t \geq t_0$, verifies the recurrence (2) and the condition $x(t_0) = x_0$ ($t_0 \in \mathcal{Z}$, $x_0 \in K^n$).

We shall assume in addition that $\forall\alpha \in \{1, 2, \ldots, m\}$, $\forall t \in \mathcal{Z}$, the matrix $A_\alpha(t)$ is invertible. Let $\chi(\cdot, \cdot)$ be the transition (fundamental) matrix associated to the linear homogeneous recurrence (8). According to Proposition 3.2, $\forall t \geq s$, the matrix $\chi(t, s)$ is invertible. Since $\forall t \geq t_0$, $\chi(t + 1_\alpha, t_0) = A_\alpha(t)\chi(t, t_0)$ or $A_\alpha(t) = \chi(t + 1_\alpha, t_0)\chi(t, t_0)^{-1}$, it follows that the equality (2) is equivalent to

$$x(t + 1_\alpha) = \chi(t + 1_\alpha, t_0)\chi(t, t_0)^{-1}x(t) + b_\alpha(t)$$

$$\iff \chi(t + 1_\alpha, t_0)^{-1}x(t + 1_\alpha) = \chi(t, t_0)^{-1}x(t) + \chi(t + 1_\alpha, t_0)^{-1}b_\alpha(t).$$

Let $\widetilde{x}\colon \{t \in \mathbb{Z}^m \,|\, t \geq t_0\} \to K^n$, $\widetilde{x}(t) = \chi(t, t_0)^{-1}x(t)$, $\forall t \geq t_0$. For $\alpha \in \{1, 2, \ldots, m\}$, let $\widetilde{A}_\alpha\colon \{t \in \mathbb{Z}^m \,|\, t \geq t_0\} \to K^n$, $\widetilde{A}_\alpha(t) = \chi(t + 1_\alpha, t_0)^{-1}b_\alpha(t)$, $\forall t \geq t_0$.

We have $\widetilde{x}(t_0) = \chi(t_0, t_0)^{-1}x(t_0) = x(t_0)$.

From the above it follows that $x(\cdot)$ is a solution of the recurrence (2) which verifies $x(t_0) = x_0$, if and only if $\widetilde{x}(\cdot)$ is the solution of the recurrence

$$\widetilde{x}(t + 1_\alpha) = \widetilde{A}_\alpha(t) + \widetilde{x}(t), \quad \forall t \geq t_0, \quad \forall\alpha \in \{1, 2, \ldots, m\}, \tag{26}$$

which verifies $\widetilde{x}(t_0) = x_0$. We find that the recurrence (26) is of type (22), where: $(N, \cdot, E) = (K^n, +, 0)$, $M = N = K^n$ and the action is

$$\psi\colon K^n \times K^n \to K^n, \quad \psi(\widetilde{A}, \widetilde{x}) = \widetilde{A} + \widetilde{x}, \quad \forall\widetilde{A} \in K^n, \ \forall\widetilde{x} \in K^n.$$

The relation (25) corresponding to the recurrence (26) is: $\widetilde{A}_\alpha(t + 1_\beta) + \widetilde{A}_\beta(t) = \widetilde{A}_\beta(t + 1_\alpha) + \widetilde{A}_\alpha(t) \iff \chi(t + 1_\beta + 1_\alpha, t_0)^{-1}b_\alpha(t + 1_\beta) + \chi(t + 1_\beta, t_0)^{-1}b_\beta(t)$ $\iff \chi(t + 1_\beta + 1_\alpha, t_0)^{-1}b_\alpha(t + 1_\beta) + \chi(t + 1_\beta, t_0)^{-1}b_\beta(t) = \chi(t + 1_\alpha + 1_\beta, t_0)^{-1}$

$b_\beta(t+1_\alpha)+\chi(t+1_\alpha,t_0)^{-1}b_\alpha(t) \Longleftrightarrow b_\alpha(t+1_\beta)+\chi(t+1_\alpha+1_\beta,t_0)\chi(t+1_\beta,t_0)^{-1}b_\beta(t)$ $= b_\beta(t+1_\alpha)+\chi(t+1_\alpha+1_\beta,t_0)\chi(t+1_\alpha,t_0)^{-1}b_\alpha(t) \Longleftrightarrow A_\alpha(t+1_\beta)b_\beta(t)+b_\alpha(t+1_\beta) =$ $A_\beta(t+1_\alpha)b_\alpha(t)+b_\beta(t+1_\alpha)$; and this is the relation (4), which is satisfied.

Let $\widetilde{\chi}(\,\cdot\,,\cdot\,)\colon \{(t,s)\in\mathbb{Z}^m\times\mathbb{Z}^m \,|\, t\geq s\geq t_0\}\to K^n$ be the fundamental function associated to the recurrence (26). According to Proposition 5.1, we have $\widetilde{x}(t)=\widetilde{\chi}(t,t_0)+x_0$. Since $x(t)=\chi(t,t_0)\widetilde{x}(t)$, it follows that

$$x(t)=\chi(t,t_0)x_0+\chi(t,t_0)\widetilde{\chi}(t,t_0).$$

According to Remark 5.2, the matrix $\widetilde{\chi}(t,t_0)$ writes as a sum of matrices $\widetilde{C}_{\alpha,k}(\cdot)$ (analogue of the relation in the step f) of Proposition 3.2, but with the operation " + " instead of multiplication), and: $\widetilde{C}_{\alpha,k}(t)=\sum_{j=1}^{k}\widetilde{A}_\alpha(t+(k-j)\cdot 1_\alpha)$, if $k\geq 1$, and $\widetilde{C}_{\alpha,0}(t)=0$; i.e. the analogue of the formula (10).

Acknowledgments

The work has been funded by the Sectoral Operational Programme Human Resources Development 2007-2013 of the Ministry of European Funds through the Financial Agreement POSDRU/159/1.5/S/132395.

Partially supported by Academy of Romanian Scientists. Special thanks goes to Prof. Dr. Ionel Ţevy, who was willing to participate in our discussions about multivariate sequences and to suggest the name "multiple recurrences".

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