# BINOMIAL COEFFICIENTS AND BINARY PARTITIONS

Mircea Merca[1]

*Building on Nathan J. Fine's work, we know that binomial coefficients can be expressed as sums of products of binomial coefficients, with each product corresponding to an integer partition. Expanding this perspective, we unveil a refined and elegant decomposition of binomial coefficients through binary partitions. We built upon these concepts by exploring their q-analogues using Gaussian polynomials.*

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## 1. **Introduction.**

Mathematics provides various tools for counting and arranging objects. Among these, binomial coefficients and integer partitions are fundamental in combinatorial mathematics. They play an essential role in probability, algebra, and number theory.

The binomial coefficient, denoted as $\binom{n}{k}$, quantifies the number of ways to select $k$ elements from a set of $n$ elements, disregarding order. It plays a fundamental role in the binomial theorem, expressed as

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k. \tag{1}$$

The multinomial coefficient extends the concept of the binomial coefficient to partitioning $n$ objects into $k$ groups of specified sizes $n_1, n_2, \ldots, n_k$. It is given by

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2!\, \ldots\, n_k!}, \quad \text{where} \quad n_1 + n_2 + \cdots + n_k = n,$$

and arises in the multinomial theorem, which generalizes the binomial expansion:

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_1,n_2,\ldots,n_k \geq 0}} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}.$$

The right-hand side of this relation represents a summation over all compositions of $n$. As noted in [1], a composition of a positive integer $n$ is an ordered way of expressing $n$ as a sum of positive integers, namely,

$$n = n_1 + n_2 + \cdots + n_k. \tag{2}$$

[1]Department of Mathematical Methods and Models, Fundamental Sciences Applied in Engineering Research Center, National University of Science and Technology Politehnica Bucharest, RO-060042 Bucharest, Romania and Academy of Romanian Scientists, RO-050044 Bucharest, Romania, E-mail: `mircea.merca@upb.ro` (corresponding author)

If the order of the summands $n_i$ is ignored, the representation in (2) is referred to as an integer partition. This can be expressed as

$$n = t_1 + 2t_2 + \cdots + nt_n,$$

where each positive integer $i$ appears $t_i$ times in the partition. The total number of parts in this partition is given by

$$t_1 + t_2 + \cdots + t_n = k.$$

The multinomial coefficient can be represented in various forms, including as a product of binomial coefficients:

$$\binom{n_1 + n_2 + \cdots + n_k}{n_1, n_2, \ldots, n_k} = \prod_{i=1}^{k} \binom{n_1 + n_2 + \cdots + n_i}{n_i}.$$

An additional relationship between binomial and multinomial coefficients arises from a partitioned version of the binomial theorem, as stated in [4, Corollary 4]:

$$(x+y)^{n-1} = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1 + \cdots + t_n}{t_1, \ldots, t_n} x^{t_1+\cdots+t_n-1} y^{n-t_1-\cdots-t_n}. \qquad (3)$$

So from (1) and (3), we easily derive the following identity.

**Theorem 1.1.** *Let $n$ and $k$ be positive integers. Then,*

$$\binom{n-1}{k-1} = \sum_{\substack{t_1+2t_2+\cdots+nt_n=n \\ t_1+t_2+\cdots+t_n=k}} \prod_{i=1}^{n} \binom{t_1 + \cdots + t_i}{t_i}. \qquad (4)$$

We remark that this formula appears to have been first published in 1959 by N. J. Fine [2, Ex. 5, p. 87]. As a direct consequence of formula (4), the binomial coefficient

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

can be expressed in terms of multinomial coefficients as a summation over all partitions of $n$ into exactly $k$ parts.

For example, the partitions of 9 into exactly four parts are:

$$(6,1,1,1),\ (5,2,1,1),\ (4,3,1,1),\ (4,2,2,1),\ (3,3,2,1),\ (3,2,2,2).$$

According to Theorem 1.1, this yields:

$$\begin{aligned}\binom{9}{4} &= \frac{9}{4}\left(\binom{3}{3}\binom{4}{1} + \binom{2}{2}\binom{3}{1}\binom{4}{1} + \binom{2}{2}\binom{3}{1}\binom{4}{1}\right. \\ &\qquad \left. + \binom{1}{1}\binom{3}{2}\binom{4}{1} + \binom{1}{1}\binom{2}{1}\binom{4}{2} + \binom{3}{3}\binom{4}{1}\right) \\ &= \frac{9}{4}(4 + 12 + 12 + 12 + 12 + 4) = 126.\end{aligned}$$

Apart from this representation of $\binom{n}{k}$, there exists a decomposition based on partitions of $k$ into at most $n-k+1$ parts.

**Theorem 1.2.** *Let $n$ and $k$ be positive integers. Then,*

$$\binom{n}{k} = \sum_{t_1+2t_2+\cdots+kt_k=k} \binom{n-k+1}{t_1+\cdots+t_k} \prod_{i=1}^{k} \binom{t_1+\cdots+t_i}{t_i}.$$

For instance, the partitions of 4 are:

$$(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1).$$

According to Theorem 1.2, we obtain:

$$\binom{9}{4} = \binom{6}{1}\binom{1}{1} + \binom{6}{2}\binom{1}{1}\binom{2}{1} + \binom{6}{2}\binom{2}{2} + \binom{6}{3}\binom{2}{2}\binom{3}{1} + \binom{6}{4}\binom{4}{4}$$
$$= 6 + 30 + 15 + 60 + 15 = 126.$$

A *binary partition* is a partition where each part is a non-negative power of 2. In this paper, motivated by Theorems 1.1 and 1.2, we introduce a new decomposition of the binomial coefficient $\binom{n}{k}$, which is expressed as a sum over binary partitions of $k$, where each part appears with multiplicity at most $n-k+1$.

**Theorem 1.3.** *Let $n$ and $k$ be positive integers. Then*

$$\binom{n}{k} = \sum_{2^0t_0+\cdots+2^kt_k=k} \prod_{i=0}^{k} \binom{n-k+1}{t_i}.$$

For example, the binary partitions of 4 are:

$$(4),\ (2,2),\ (2,1,1),\ (1,1,1,1).$$

Applying Theorem 1.3, we obtain:

$$\binom{9}{4} = \binom{6}{1} + \binom{6}{2} + \binom{6}{2}\binom{6}{1} + \binom{6}{4}$$
$$= 6 + 15 + 90 + 15 = 126.$$

We note the following consequence of Theorem 1.3, which concerns the decomposition of central binomial coefficients.

**Corollary 1.1.** *Let $n$ be positive integer. Then*

$$\binom{2n}{n} = \sum_{2^0t_0+\cdots+2^nt_n=n} \prod_{i=0}^{n} \binom{n+1}{t_i}.$$

Our theorems provide distinct decompositions of binomial coefficients, derived from partitions of $n$ into exactly $k$ parts, partitions of $k$ into at most $n-k+1$ parts, and binary partitions of $k$ where each part appears with multiplicity at most $n-k+1$. Notably, these three distinct combinatorial frameworks converge to the same result.

The remainder of the paper is structured as follows. In Section 2, we provide proofs of Theorems 1.2 and 1.3. Section 3 introduces Theorem 3.1, which reveals a further connection between binomial coefficients and binary partitions. In Section 4, we develop $q$-analogues of Theorems 1.3 and 3.1, and explore several of their ensuing implications.

## 2. **Proofs of Theorems 1.2 and 1.3.**

We establish these identities by transforming the generating function

$$\sum_{k=0}^{\infty}\binom{n+k}{k}z^k=\frac{1}{(1-z)^{n+1}}$$

into two equivalent forms:

$$\begin{aligned}\frac{1}{(1-z)^{n+1}}&=\left(1+z+z^2+z^3+\cdots\right)^{n+1}\\&=\sum_{t_0+t_1+t_2+\cdots=n+1}\binom{n+1}{t_0,t_1,t_2,\ldots}z^{0\cdot t_0+1\cdot t_1+2\cdot t_2+\cdots}\\&=\sum_{k=0}^{\infty}z^k\sum_{\substack{t_1+2t_2+\cdots+kt_k=k\\t_1+t_2+\cdots+t_k\le n+1}}\binom{n+1}{t_1,\ldots,t_k,n+1-t_1-\cdots-t_k}\\&=\sum_{k=0}^{\infty}z^k\sum_{\substack{t_1+2t_2+\cdots+kt_k=k\\t_1+t_2+\cdots+t_k\le n+1}}\binom{n+1}{t_1+\cdots+t_k}\prod_{i=1}^{n}\binom{t_1+\cdots+t_i}{t_i}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{(1-z)^{n+1}}&=\prod_{j=1}^{n+1}\frac{1}{1-z}=\prod_{j=1}^{n+1}\prod_{i=0}^{\infty}(1+z^{2^i})=\prod_{i=0}^{\infty}(1+z^{2^i})^{n+1}\\&=\prod_{i=0}^{\infty}\sum_{j=0}^{n+1}\binom{n+1}{j}t^{2^i\cdot j}=\sum_{k=0}^{\infty}z^k\sum_{2^0t_0+\cdots+2^kt_k=k}\prod_{j=0}^{k}\binom{n+1}{t_j}.\end{aligned}$$

The identities are derived by comparing the coefficients of $z^k$ and replacing $n$ with $n-k$.

## 3. **Alternating sum over binary partitions**

Theorems 1.1–1.3 can be understood as recurrence relations for binomial coefficients. The following result establishes an additional connection between binomial coefficients and binary partitions.

**Theorem 3.1.** *Let $n$ and $k$ be non-negative integers. Then*

$$\binom{n}{k}=\sum_{2^0t_0+\cdots+2^kt_k=k}(-1)^{t_1+\cdots+t_k}\prod_{i=0}^{k}\binom{n-1+t_i}{t_i}.$$

*Proof.* According to binomial theorem (1), we can write

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}z^k &= (1+z)^n = \frac{1}{(1-z)^n}\prod_{i=1}^{\infty}\frac{1}{(1+z^{2^i})^n}\\ &= \left(\sum_{k=0}^{\infty}\binom{n-1+k}{k}z^k\right)\prod_{i=1}^{\infty}\sum_{k=0}^{\infty}\binom{n-1+k}{k}(-z^{2^i})^k\\ &= \sum_{k=0}^{\infty}z^k\sum_{2^0t_0+\cdots+2^kt_k=k}(-1)^{k+t_0+\cdots+t_k}\prod_{i=0}^{k}\binom{n-1+t_i}{t_i}.\end{aligned}$$

The proof follows by extracting the coefficients of $z^k$. □

For example, the binary partitions of 4 are:

$$(4),\ (2,2),\ (2,1,1),\ (1,1,1,1).$$

Applying Theorem 3.1, we obtain:

$$\begin{aligned}\binom{9}{4} &= -\binom{9}{1}+\binom{10}{2}-\binom{9}{1}\binom{10}{2}+\binom{12}{4}\\ &= -9+45-405+495=126.\end{aligned}$$

We remark the following consequences of Theorem 3.1:

**Corollary 3.1.** *Let $n$ be a non-negative integer. Then*

a) $$\sum_{2^0t_0+\cdots+2^nt_n=n}(-1)^{t_1+\cdots+t_n}\prod_{i=0}^{n}\binom{n-1+t_i}{t_i}=1;$$

b) $$\sum_{2^0t_0+\cdots+2^nt_n=n}(-1)^{t_1+\cdots+t_n}\prod_{i=0}^{n}\binom{n+t_i}{t_i}=n+1.$$

## 4. **Gaussian polynomials**

Gaussian polynomials, also known as $q$-binomial coefficients, constitute a fundamental class of $q$-analogues in mathematics. They extend the classical binomial coefficients by introducing a parameter $q$, yielding expressions that interpolate between discrete combinatorial counts and richer algebraic structures. The $q$-binomial coefficient is defined as:

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\begin{cases}\frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, & \text{if } k\in\{0,1,\ldots,n\},\\ 0, & \text{otherwise,}\end{cases}$$

where:

$$(a;q)_n=\begin{cases}(1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{for } n>0,\\ 1, & \text{for } n=0\end{cases}$$

is the $q$-shifted factorial. These expressions reduce to their classical counterparts in the limit as $q\to 1$:

$$\lim_{q\to 1}\begin{bmatrix}n\\k\end{bmatrix}_q=\binom{n}{k}.$$

Notably, $\begin{bmatrix}n\\k\end{bmatrix}_q$ is a polynomial in $q$ with non-negative integer coefficients.

The Gaussian binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ admits a standard combinatorial interpretation as the generating function for integer partitions fitting inside a $k \times (n-k)$ rectangle, where each partition contributes $q^{|\lambda|}$, with $|\lambda|$ denoting its size. For instance, $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = q^4 + q^3 + 2q^2 + q + 1$ enumerates the six partitions that fit in a $2 \times 2$ box: $(2,2)$, $(2,1)$, $(2)$, $(1,1)$, $(1)$, and $()$, corresponding to sizes $4,3,2,2,1$, and $0$, respectively.

A profound interpretation of the $q$-binomial coefficient arises in finite geometry: when $q$ is a prime power, $\begin{bmatrix} n \\ k \end{bmatrix}_q$ counts the number of $k$-dimensional subspaces of an $n$-dimensional vector space over the finite field $\mathbb{F}_q$. This count corresponds to the number of points in the finite Grassmannian $\mathrm{Gr}(k, \mathbb{F}_q^n)$, highlighting the significance of $q$-binomial coefficients in enumerative combinatorics and algebraic geometry.

We remark the following $q$-analog of Theorems 1.3.

**Theorem 4.1.** *Let $n$ and $k$ be non-negative integers. Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{2^0 t_0 + \cdots + 2^k t_k = k} \prod_{i=0}^{k} q^{2^i \binom{t_i}{2}} \begin{bmatrix} n-k+1 \\ t_i \end{bmatrix}_{q^{2^i}}.$$

*Proof.* We begin with a well known identity which was proved by Cauchy [3, Theorem 26]:

$$\sum_{k=0}^{\infty} \begin{bmatrix} n-1+k \\ k \end{bmatrix}_q z^k = \prod_{j=0}^{n-1} \frac{1}{1-q^j z}.$$

Each term in the product on the right-hand side can be expanded using the identity:

$$\frac{1}{1-q} = \prod_{i=0}^{\infty} (1+q^{2^i}),$$

valid for $|q| < 1$. Applying this to each factor $1/(1-q^j z)$, we obtain:

$$\prod_{j=0}^{n-1} \frac{1}{1-q^j z} = \prod_{j=0}^{n-1} \prod_{i=0}^{\infty} (1+(q^j z)^{2^i}) = \prod_{i=0}^{\infty} \prod_{j=0}^{n-1} (1+q^{j \cdot 2^i} z^{2^i}).$$

Each inner product over $j$ can be expressed as a sum using Rothe's $q$-binomial theorem [3, Theorem 12]:

$$\prod_{j=0}^{n-1} (1+q^{j \cdot 2^i} z^{2^i}) = \sum_{j=0}^{n} q^{\binom{j}{2} \cdot 2^i} \begin{bmatrix} n \\ j \end{bmatrix}_{q^{2^i}} z^{j \cdot 2^i}.$$

Combining the results from the previous steps, we arrive at:

$$\sum_{k=0}^{\infty} \begin{bmatrix} n-1+k \\ k \end{bmatrix}_q z^k = \sum_{k=0}^{\infty} z^k \sum_{2^0 t_0 + \cdots + 2^k t_k = k} \prod_{j=0}^{k} q^{2^j \binom{t_j}{2}} \begin{bmatrix} n \\ t_j \end{bmatrix}_{q^{2^j}}.$$

The proof follows by extracting the coefficients of $z^k$ and replacing $n$ by $n+1-k$. $\square$

We now present the following $q$-analogue of Corollary 1.1:

**Corollary 4.1.** *Let $n$ be a non-negative integer. Then*

$$\begin{bmatrix}2n\\ n\end{bmatrix}_q = \sum_{2^0t_0+\cdots+2^nt_n=n} \prod_{i=0}^{n} q^{2^i\binom{t_i}{2}} \begin{bmatrix}n+1\\ t_i\end{bmatrix}_{q^{2^i}}.$$

Similarly, we obtain the following $q$-analogue of Theorem 3.1:

**Theorem 4.2.** *Let $n$ and $k$ be non-negative integers. Then*

$$q^{\binom{k}{2}}\begin{bmatrix}n\\ k\end{bmatrix}_q = \sum_{2^0t_0+\cdots+2^kt_k=k} (-1)^{t_1+\cdots+t_k} \prod_{i=0}^{k} \begin{bmatrix}n-1+t_i\\ t_i\end{bmatrix}_{q^{2^i}}.$$

*Proof.* The proof closely mirrors that of the preceding theorem; however, in this instance, we commence with Rothe's $q$-binomial theorem [3, Theorem 12] and subsequently invoke Cauchy's identity [3, Theorem 26], omitting the detailed steps for brevity:

$$\begin{aligned}
\sum_{j=0}^{n}(-1)^j\, q^{\binom{j}{2}}\begin{bmatrix}n\\ j\end{bmatrix}_q z^j &= \prod_{j=0}^{n-1}(1-q^jz) = \prod_{j=0}^{n-1}\prod_{i=0}^{\infty}\frac{1}{1+(q^jz)^{2^i}}\\
&= \prod_{i=0}^{\infty}\prod_{j=0}^{n-1}\frac{1}{1+q^{j\cdot 2^i}z^{2^i}}\\
&= \prod_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\begin{bmatrix}n-1+j\\ j\end{bmatrix}_{q^{2^i}} z^{j\cdot 2^i}\\
&= \sum_{k=0}^{\infty} z^k \sum_{2^0t_0+\cdots+2^kt_k=k}(-1)^{k+t_0+\cdots+t_k}\prod_{i=0}^{k}\begin{bmatrix}n-1+t_i\\ t_i\end{bmatrix}_{q^{2^i}}.
\end{aligned}$$

This concludes the proof. □

The following identities arise as natural analogues of Corollary 3.1.

**Corollary 4.2.** *Let $n$ be a non-negative integer. Then*

a) $$\sum_{2^0t_0+\cdots+2^nt_n=n}(-1)^{t_1+\cdots+t_n}\prod_{i=0}^{n}\begin{bmatrix}n-1+t_i\\ t_i\end{bmatrix}_{q^{2^i}} = q^{\binom{n}{2}};$$

b) $$\sum_{2^0t_0+\cdots+2^nt_n=n}(-1)^{t_1+\cdots+t_n}\prod_{i=0}^{n}\begin{bmatrix}n+t_i\\ t_i\end{bmatrix}_{q^{2^i}} = \frac{1-q^n}{1-q}\,q^{\binom{n}{2}}.$$

In [5, Theorem 1.1], it is established that the reciprocal of the finite product $(q;q)_k$ admits two distinct representations as sums over all integer partitions of $k$:

$$\frac{q^{\binom{k}{2}}}{(q;q)_k} = \sum_{t_1+2t_2+\cdots+kt_k=k}(-1)^{k+t_1+\cdots+t_k}\prod_{i=1}^{n}\binom{t_1+\cdots+t_i}{t_i}\frac{1}{(q;q)_i^{t_i}},$$

$$\frac{1}{(q;q)_k} = \sum_{t_1+2t_2+\cdots+kt_k=k}(-1)^{k+t_1+\cdots+t_k}\prod_{i=1}^{n}\binom{t_1+\cdots+t_i}{t_i}\frac{q^{t_i\binom{i}{2}}}{(q;q)_i^{t_i}}.$$

Taking the limit as $n\to\infty$ in Theorems 4.1 and 4.2, we derive analogous expressions for the reciprocal of $(q;q)_k$ as sums indexed by binary partitions of $k$:

**Corollary 4.3.** *Let $k$ be a non-negative integer. Then*

a) $$\frac{1}{(q;q)_k} = \sum_{2^0t_0+\cdots+2^kt_k=k} \prod_{i=0}^{k} \frac{q^{2^i\binom{t_i}{2}}}{(q^{2^i};q^{2^i})_{t_i}};$$

b) $$\frac{q^{\binom{k}{2}}}{(q;q)_k} = \sum_{2^0t_0+\cdots+2^kt_k=k} (-1)^{t_1+\cdots+t_k} \prod_{i=0}^{k} \frac{1}{(q^{2^i};q^{2^i})_{t_i}}.$$

## 5. **Concluding remarks**

We have explored multiple perspectives on the binomial coefficient $\binom{n}{k}$ by leveraging diverse combinatorial structures, including integer partitions, compositions, and binary partitions. Through Theorems 1.1–3.1, we derived elegant decompositions that express $\binom{n}{k}$ in terms of products of binomial or multinomial coefficients, each summation corresponding to a distinct combinatorial interpretation.

These results illuminate deep connections between classical combinatorial tools and richer algebraic frameworks. Notably, the introduction of binary partitions with bounded multiplicities and their role in summing over $\binom{n-k+1}{t_i}$ terms presents a new and intuitive representation of binomial coefficients. This complements the traditional partition-based identities and offers insights into recurrence relations and generating functions.

We further extended these ideas into the realm of $q$-analogues through the Gaussian polynomials, capturing both the algebraic generality and enumerative richness that $q$-binomial coefficients provide. These extensions emphasize that the structure and behavior of binomial coefficients are far more nuanced and versatile than their original definition might suggest.

Altogether, the identities and decompositions presented here not only unify multiple strands of combinatorics and algebra but also open avenues for further investigation in number theory, symbolic computation, and finite geometry.

An intriguing open question remains: does there exist $d > 2$ such that analogous results can be established for d-ary partitions? Exploring such generalizations may reveal deeper structural patterns and broaden the applicability of these combinatorial identities within higher arity frameworks.

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