U.P.B. Sci. Bull., Series A, Vol. 73, Iss. 2, 2011 ISSN 1223-7027

REMARKABLE STAR FAMILIES

Mihai REBENCIUC[1], Ioana PISICĂ[2]

În această lucrare generalizăm noțiunea de filtru pe o mulțime X prin noțiunea de pseudo-filtru pe X (ca familii stelate particulare); în acest demers folosim segmentele determinate pe X şi în mod esențial segmentele nedeterminate (inclusiv segmentul final) pe X care formează latici algebrice (ca sisteme de închidere algebrice – în contextul partiționării lui $\mathcal{P}(\mathcal{P}(X))$ în familii stelate şi familii reziduale). În final laticile algebrice ale pseudo-filtrelor pe X, respectiv ale filtrelor pe X (ca pseudo-filtre particulare) sunt asociate unor operatori parțiali de închidere pe $\mathcal{P}(X)$ - şi sunt menționate unele dezvoltări ale familiilor stelate.

In this paper we generalize the notion of filter on a set X using the notion of pseudo-filter on X (as particular starred families); in this conquest we use the determined segments on X and essentially the undetermined segments (including the final segment) on X which form algebraic lattices (as algebraic closure systems - in the context of partitioning $\mathcal{P}(\mathcal{P}(X))$ in starred families and residual families). Finally the algebraic lattices of the pseudo-filters on X, respectively of the filters on X (as particular pseudo-filters) are associated to partial closure operators on $\mathcal{P}(X)$ –and some developments of the starred families are mentioned.

Keywords: star family, final segment on a set, pseudo-filter on a set.
MSC2000: primary 54A99; secondary 03E99.

1. Introduction

In some mathematical disciplines (like sets theory or topology – see [5] - [8], respectively [1] - [4]) and in some fields of computer science and other technical sciences (for example, power engineering) the void family or families containing the void set are eliminated from families over a space set X (which is contextually non-ordinary, see consideration 2.1.i and where a space set is a set which has urelements as members, see [6]); in the following, these constitute the set of residual (non-star) families over X, denoted as nST(X) (see consideration 2.1.i again). It appears naturally that the set of star families ST(X) comes to complete a binary partition of $\mathcal{P}(\mathcal{P}(X))$ (see consideration 2.1.i again and again).

[1] Lecturer, Dept. of Mathematics III, University POLITEHNICA of Bucharest, Romania, e-mail: m_rebenciuc@yahoo.com
[2] Asist., Department of Electric Power Systems, University POLITEHNICA of Bucharest, Romania

The first examples of star families over X are: the family of supersets of an element $x \in X$, $\mathcal{P}_x(X)$ or generally of an subset(part) G of X, $\mathcal{P}_G(X)$ (generalizations of the corresponding neighborhoods' families) and the pre-filter over X (in a rephrased definition, see considerations 2.1.ii and 2.1.iii, respectively [1] - [4]). The determined segments over X (implicitly related to the relation $\subseteq$ in $\mathcal{P}(X)$, see consideration 2.1.iv) $[G,X]=\mathcal{P}_G(X)$, $[\emptyset,B]$ which represent a particularization of segments concerning more general relational structures are examples of star families and residual, respectively, frequently used within the paper; moreover, example 2.1 concerns the pre-filters as determined segments.

Undetermined segments over X (a generalization of determined segments over X, see Section 3) – the final segment, respectively the initial one over X $[\bullet,X],[\emptyset,\bullet]$ are star families (with the exception of $\emptyset$ and $\mathcal{P}(X)$), respectively residual; as a consequence, the pseudo-filter over X (as star final segment over X) and the filter over X (redefined as pseudo-filter over X that is also pre-filter over X, see Section 4) are star families. The sets of the above mentioned families – FS(X), IS(X), respectively P-FIL(X), FIL(X) are closure systems isomorphically associated to closure operators respectively defined as the closure of the argument family; in fact, FS(X), IS(X), P-FIL(X), FIL(X) are algebraic closure systems, therefore algebraic lattices with representation properties. More specifically, FS(X) and IS(X) are closure systems over $\mathcal{P}(X)$ – isomorphically associated to closure operators over $\mathcal{P}(X)$ and P-FIL(X), FIL(X) are closure systems over $\mathcal{P}^*(X)$ isomorphically associated to closure operators over $\mathcal{P}^*(X)$ (as partial closure operators over $\mathcal{P}(X)$). Moreover, FS(X), IS(X), P-FIL(X), FIL(X) are co-domains of closure operators isomorphically associated with each other – which have "min"-type expressions.

Consideration 2.2.i presents an adaptation to $\mathcal{P}(X)$ of the isomorphic association of the closure operator – closure system with the deduction of the co-domain property of the closure system and of the "min"-type expression of the closure operator (see eq. (1'), (2')). The representation properties are given in consideration 2.2.ii. Finally consideration 2.2.iii justifies (essentially) the maintaining of the isomorphism and of the above representation property in the case of the partiality (cases P-FIL(X) and FIL(X) from Section 4).

Other contributions:

- The algebraic lattices FS(X), IS(X), P-FIL(X), FIL(X) are tackled in comparison (see observations 3.2.ii, 4.2.i, 4.3.iii);
- A proof is given for the equivalence between the definition of a filter as particular pseudo-filter and the usual definitions (see observations 4.2.ii and theorem 4.2);
- Example 3.1 proves that determined segments are compact undetermined segments and gives representations for $\mathcal{P}^*(X) \in FS(X)$, $\mathcal{S}. \cup \{\emptyset\} \in IS(X)$;

in example 4.1 segments to the right prove to be compact filters with $\mathcal{P}^*(X) \in$ P-FIL(X) \ FIL(X).

2. Further considerations

Consideration 2.1.i (star families) Relative to a space set X={x} (contextually non-trivial, i.e. at least not void – or stronger $|X|\geq\omega_0$ in Card) consider the complete Boole algebras $\mathcal{P}(X)=\{A\}$,$\mathcal{P}(\mathcal{P}(X))=\{\mathcal{A}=\{A\}\}$ (the set of families of sets over X) – related to $\subseteq,\cup,\cap,\subset$ over $\mathcal{P}(X)$, respectively over $\mathcal{P}(\mathcal{P}(X))$ and bounded by $\emptyset$, X, respectively by $\emptyset$, $\mathcal{P}(X)$. Denote a star family over X a family of sets $\mathcal{S}$ over X with $\mathcal{S}\neq\emptyset$, $\emptyset\notin\mathcal{S}$; in fact, $ST(X)=\mathcal{P}^*(\mathcal{P}^*(X))=\{\mathcal{S}|\mathcal{S}$ *star family over X*} is a complete upper semi-lattice ($\cup$-semi-lattice) with $\mathcal{P}^*(X)$ as upper bound. The set of residual (non-star) families over X is $nST(X)=\mathcal{P}(\mathcal{P}(X)) \setminus \mathcal{P}^*(\mathcal{P}^*(X))=\{\mathcal{R} \mid \mathcal{R}=\emptyset$ or $\emptyset\in \mathcal{R}\}$ – with { ST(X), nST(X)} a partition of $\mathcal{P}(\mathcal{P}(X))$.

ii(star families of oversets and associated functions) The supersets family of preset $x\in X$, $\mathcal{P}_x(X)=\{S_{[x]}\in\mathcal{P}^*(X) \mid x\in S\}$ is a star family over X; to the family $\mathcal{A}\in\mathcal{P}^*(\mathcal{P}(X))$ we can associate the family $\mathcal{A}(x)=\mathcal{A}\cap\mathcal{P}_x(X)$ – star if $\mathcal{A}(x)\neq\emptyset$. More generally, the supersets families of $G\in\mathcal{P}^*(X)\setminus\{X\}$, $\mathcal{P}_G(X)=\{S_{[G]}\in\mathcal{P}^*(X) \mid G\subseteq S\}$is a star family over X; to the family $\mathcal{A}\in\mathcal{P}^*(\mathcal{P}(X))$ we can associate the family $\mathcal{A}(G)=\mathcal{A}\cap\mathcal{P}_G(X)$ – star if $\mathcal{A}(G)\neq\emptyset$. The function associated to the set of star families $\{\mathcal{S}(x) \mid x\in X\}$ is $\Phi_{[\mathcal{S}]}:X\to ST(X)$, $x\mapsto\mathcal{S}(x)$; the set $\Phi S(X)=\{\Phi_{[\mathcal{S}]} \mid \Phi_{[\mathcal{S}]}(x)=\mathcal{S}(x)\}$ is partially ordered by the relation $\leq$ induced by the relation $\subseteq$ in ST(X), i.e. $\Phi_{[\mathcal{S}']}\leq\Phi_{\mathcal{S}}$ iff for every $x\in X$, $\mathcal{S}'(x)\subseteq\mathcal{S}(x)$. Analogously for set $\{\mathcal{S}(G) \mid G\in\mathcal{P}^*(X)\setminus\{X\}\}\subset ST(X)$.

iii(pre-filters) A pre-filter $p\mathcal{F}$ over X is a star family over X which is, in addition, $\subseteq$ - lower (or to the left) filtered - $p\mathcal{F}\in ST(X)$ and any $\{A,B\}\subseteq p\mathcal{F}$ is $\subseteq$ - lower bounded in $p\mathcal{F}$, i.e. $\{A,B\}^-_{p\mathcal{F}}\neq\emptyset$ (reformulation of the classical definition from [1] – [4]); we denote $pFIL(X)=\{p\mathcal{F} \mid p\mathcal{F}$ *pre-filter over* X}.

Pre-filters over X are characterized by the following property:

P(pF) A family $\mathcal{A}\subseteq\mathcal{P}(X)$ is centered (with the property of finite intersection – for any $\mathcal{A}_f\subseteq\mathcal{A}$ finite, $\cap\mathcal{A}_f\neq\emptyset$) iff there is $p\mathcal{F}\in pFIL(X)$ so that $\mathcal{A}\subseteq p\mathcal{F}$ (reformulation of property 1.3.14 from [1]).

iv(determined segments) Consider the determined segments over X (implicitly related to the inclusion relation $\subseteq$ of $\mathcal{P}(X)$) $[A,X]_{[\subseteq]}=\mathcal{P}_A(X)$ (the segment to the right over X in $A\in\mathcal{P}^*(X)\setminus\{X\}$), $[\emptyset,B]_{[\subseteq]}=\{C\in\mathcal{P}(X) \mid C\subseteq B\}$ (the segment to the left over X in $B\in\mathcal{P}^*(X)\setminus\{X\}$), respectively $[A,B]_{[\subseteq]}= [A,X]\cap[\emptyset,B]$ (the segment bounded over X in A,B) – undeniably void if $B\subset A$; relatively to $\mathcal{P}(X)\setminus\{X\}$, respectively $\mathcal{P}^*(X)$ (P(X) without max, respectively min element) the corresponding determined segments over X are $[A,\rightarrow)_{[\subseteq]}=\mathcal{P}_A(X) \setminus \{X\}$, respectively $(\leftarrow,B]_{[\subseteq]}=\{C\in\mathcal{P}^*(X) | C\subseteq B\}$. Analogously, the open determined segments over X are defined (relatively to the strict inclusion relation $\subset$) – and the determined segments over $\mathcal{P}(X)$ (relatively to $\subseteq$ - implicitly or to $\subset$ - see [9] for more details, including relative to more general relational structures).

Example 2.1 (pre-filters as determined segments) For $\emptyset\neq A\subset B$ we have $[A,X]$, $(A,X]$, $[A,B]$, $(A,B]\in pFIL(X)$ – and $(\emptyset,B]$, $(\emptyset,B)\in ST(X)$ (and analogously relative to $\mathcal{P}(X) \setminus \{X\}$, respectively $\mathcal{P}^*(X)$).

Consideration 2.2.i(isomorphism) Consider the sets
$CO(\mathcal{P}(X))=\{\overline{C}:\mathcal{P}(\mathcal{P}(X))\rightarrow\mathcal{P}(\mathcal{P}(X)) | \overline{C}$ *closure operator over* $\mathcal{P}(X)\}$, respectively
$CS(\mathcal{P}(X))=\{cS\subseteq\mathcal{P}(\mathcal{P}(X)) \mid cS$ *closure system over* $\mathcal{P}(X)\}$. $\overline{C}$ has the following properties:

- Extensiveness, i.e. $\mathcal{G}\subseteq \overline{C}(\mathcal{G})$;
- Monotony, i.e. $\mathcal{E}\subseteq\mathcal{G}$ implies $\overline{C}(\mathcal{E})\subseteq \overline{C}(\mathcal{G})$;
- Idempotency, i.e. $(\overline{C}\circ\overline{C})(\mathcal{G}) = \overline{C}(\mathcal{G})$ (sufficiently $(\overline{C}\circ\overline{C})(\mathcal{G}) \subseteq \overline{C}(\mathcal{G})$ as given by the properties of extensiveness and monotony);

cS is $\cap$-complete semi-lattice bounded by $\emptyset$ and $\mathcal{P}(X)$ – in fact complete lattice (in completion cS is $\vee$ – complete semi-lattice – for cS'$\subseteq$ cS with $\vee$cS' = *sup* cS' = $\overline{C}(\cup cS')) \supseteq cS'$, i.e. cS is complete sub-lattice of $\mathcal{P}(\mathcal{P}(X))$ in case of equality).

Function Ω: $CO(\mathcal{P}(X))\rightarrow CS(\mathcal{P}(X))$, $\overline{C} \mapsto cS=FP(\overline{C})$ (1)

(the set of fixed points of $\overline{C}$) is correctly defined (see theorem 1.5.7 from [10]); as consequence, the following equality stands:

$cS=codom(\overline{C})$ (1')

which results from (1) – keeping in mind the idempotency property of $\overline{C}$.

Reciprocally, the function

$\Sigma:CS(\mathcal{P}(X))\rightarrow CO(\mathcal{P}(X))$, $cS\mapsto \overline{C}$, $\overline{C}(\mathcal{G})=\cap cS[\mathcal{G}]$, $cS[\mathcal{G}]=\{\overline{\mathcal{S}}\in cS \mid \mathcal{G}\subseteq\overline{\mathcal{S}}\}$ (2)

is correctly defined (see theorem 1.5.6 from [10]); in fact, we have the expression:

$\overline{C}(\mathcal{G})=min\ cS[\mathcal{G}]$ (2')

because $\overline{C}(\mathcal{G})=inf\ cS[\mathcal{G}]$ and $\overline{C}(\mathcal{G})\in cS$ with $\mathcal{G}\subseteq \overline{C}(\mathcal{G})$, i.e. $\overline{C}(\mathcal{G})\in cS[\mathcal{G}]$.

Moreover, the Ω, Σ are bijective functions – $\Sigma=\Omega^{-1}$, $\Omega=\Sigma^{-1}$ and therefore we have the isomorphism $CO(\mathcal{P}(X))\simeq CS(\mathcal{P}(X))$ (see theorem 1.5.8 from [10]).

ii(representable families) By definition, $\overline{C}\in CO(\mathcal{P}(X))$ is algebraic if it holds the property:

P(ACO) $\forall S\in\mathcal{P}(X)$, $S\in \overline{C}(G)$ implies $\exists\mathcal{G}_f\subseteq\mathcal{G}$ finite family over X, $S\in \overline{C}(\mathcal{G}_f)$.

By definition, $cS \in CS(\mathcal{P}(X))$ is algebraic if associated $\overline{C} \in CO(\mathcal{P}(X))$ is algebraic; $cS \in CS(\mathcal{P}(X))$ is algebraic iff it satisfies the condition:

C(ACS) $\forall$ cS' $\in \mathcal{P}^*(cS)$ cS' $\subseteq$– upper (or to the right) filtered, $\cup$cS' $\in$ cS (see theorem 1.5.13 from [10]).

By definition, a lattice ALP $\subseteq \mathcal{P}(\mathcal{P}(X))$ is algebraic if:

- ALP is a complete lattice;
- Any family $\mathcal{L} \in$ ALP is representable through
  $\mathcal{L} = \vee$ ALP^c,
  where ALP^c is a set of compact families $\mathcal{L}^c$ over X – any $\vee$- covering with families from ALP of $\mathcal{L}^c$ contains a $\vee$- finite under-covering. In particular, in the case of association (2) with cS algebraic closure system over $\mathcal{P}(X)$, so algebraic lattice $\mathcal{L}^c$ is a compact family over X iff it holds the following condition:
  C(CF) $\exists \mathcal{G}_f \subseteq \mathcal{P}(X)$ finite family, $\mathcal{L}^c = \overline{C}(\mathcal{G}_f)$
  (see theorems 1.5.18 and 1.5.17 from [10]).

iii(partialness) The isomorphism $CO(\mathcal{P}(X)) \simeq CS(\mathcal{P}(X))$ is persistent in case of partialness, i.e. for $\overline{C} : \mathcal{P}(\mathcal{P}(X)) - - \rightarrow \mathcal{P}(\mathcal{P}(X))$ partial closure operator over $\mathcal{P}(X)$ with *dom*($\overline{C}$)=DO and *codom*($\overline{C}$)=cS $\subset$ DO $\subset \mathcal{P}(\mathcal{P}(X))$ – isomorphic with restriction $\overline{C}_D = \overline{C}|_{DO}$:DO$\rightarrow$DO because *dom*($\overline{C}_D$)=*dom*($\overline{C}$) $\cap$ DO=DO, *codom*($\overline{C}_D$)=$\overline{C}$(DO)=cS (in proofs of theorems 1.5.6, 1.5.7, 1.5.8 from [10], the definition expression (1) from point i or the property of idempotency of $\overline{C}$ are used, which depend on *dom*($\overline{C}$) - also see [11]; evidently restriction $\overline{C}_D$ verifies the properties of a closure operator – particularly over $\mathcal{P}^*(X)$ for DO $\subseteq$ ST(X) (see consideration 2.1.ii).

In addition, the following still stand:

- condition C(ACS) of algebraic closure system;
- algebraic lattice as algebraic closure system;
- condition C(CF) of compact family

(because the definition expression (1) from point i or the property of idempotency of $\overline{C}$ which depend on *dom*(C) are used in the profs of theorems 1.5.13, 1.5.17, 1.5.18 from [10].

3. Undetermined segments over sets

Definition 3.1(undetermined segments) A final segment over X (implicitly relative to the relation of inclusion $\subseteq$ in $\mathcal{P}$ (X)) is a family $[\bullet, X]_{[\subseteq]}$ with the property:

P(FS) $\forall F \in [\bullet, X]$, $[F,X] \subseteq [\bullet, X]$.

Analogously – but dually, an initial segment $[\emptyset, \bullet]_{[\subseteq]}$ over X is defined, i.e. with the property:

P(IS) $\forall I \in [\emptyset, \bullet]$, $[\emptyset, I] \subseteq [\emptyset, \bullet]$.

We denote $FS(X)_{[\subseteq]} = \{[\bullet, X]_{[\subseteq]} | [\bullet, X]$ *final segment over* X$\}$, respectively $IS(X)_{[\subseteq]} = \{[\emptyset, \bullet]_{[\subseteq]} | [\emptyset, \bullet]$ *initial segment over* X$\}$.

Observation 3.1.i (alternatives) Relative to $\mathcal{P}(X) \setminus \{X\}$, respectively $\mathcal{P}^*(X)$ ($\mathcal{P}$(X) without max, respectively min) the corresponding undetermined segments over X are $[\bullet, \rightarrow)_{[\subseteq]} \subseteq \mathcal{P}(X) \setminus \{X\}$ respectively $(\leftarrow, \bullet]_{[\subseteq]} \subseteq \mathcal{P}^*(X)$; analogously for open undetermined segments over X $(\bullet, X]$, $[\emptyset, \bullet)$ respectively $(\bullet, \rightarrow)$, $(\leftarrow, \bullet)$.

ii) (intersections) Analogously to the case of bounded segment over X as intersection of determined segments over X, we can define the undetermined segment over X as intersection between a final and an initial segment over X – corresponding to definition 3.1 and cases from point i (for more details, including more general relational structures, see [12]).

Definition 3.2 (closures of families) The final closure over $\mathcal{P}$(X) of the family $\mathcal{G} \in \mathcal{P}(\mathcal{P}(X))$ (implicitly relative to the inclusion relation $\subseteq$ in $\mathcal{P}$(X)) is the family:

$\vec{C}(\mathcal{G})_{[\subseteq]} = \{F \in \mathcal{P}(X) \mid \exists G \in \mathcal{G}, F \in [G,X]_{[\subseteq]}\}$.

Analogously – but dually, the initial closure over $\mathcal{P}$(X) - relative to a family $\mathcal{G} \in \mathcal{P}(\mathcal{P}(X))$ is defined by

"$\overleftarrow{C}(\mathcal{G})_{[\subseteq]} = \{I \in \mathcal{P}(X) \mid \exists G \in \mathcal{G}, I \in [\emptyset, G]_{[\subseteq]}\}$".

Theorem 3.1.i (expressions of undetermined segments) For the undetermined segments $[\bullet, X] \in FS(X)$ and $[\emptyset, \bullet] \in IS(X)$ the following expressions hold, respectively:

$[\bullet, X] = \cup \{[F,X] \mid F \in [\bullet, X]\} = \{F \in \mathcal{P}(X) \mid \exists F' \in [\bullet, X], F \in [F', X]\}$,

$[\emptyset, \bullet] = \cup \{[\emptyset, I] \mid I \in [\emptyset, \bullet] = \{I \in \mathcal{P}(X) | \exists I' \in [\emptyset, \bullet], I \in [\emptyset, I']\}$.

ii (closure operators over $\mathcal{P}$(X)) $\vec{C}, \overleftarrow{C}$: $\mathcal{P}(\mathcal{P}(X)) \rightarrow \mathcal{P}(\mathcal{P}(X))$, $\vec{C}(\mathcal{G}) = \{F \in \mathcal{P}(X) | \exists G \in \mathcal{G}, F \in [G,X]\}$, $\overleftarrow{C}(\mathcal{G}) = \{I \in \mathcal{P}(X) | \exists G \in \mathcal{G}, I \in [\emptyset, G]\}$ are closure operators over $\mathcal{P}$(X) – associated respectively to FS(X), IS(X) as closure systems over $\mathcal{P}$(X).

iii (expressions of closures) The following expressions are standing for families closures:

$\vec{C}(\mathcal{G})$ = min FS(X)[$\mathcal{G}$], FS(X)[$\mathcal{G}$] = {[•, X] ∈FS(X) | $\mathcal{G}$⊆[•, X]},

$\overleftarrow{C}(\mathcal{G})$= min IS(X)[$\mathcal{G}$], IS(X)[$\mathcal{G}$] = {[∅, •] ∈IS (X) | $\mathcal{G}$⊆[∅, •]}.

iv (codomains, partition) We have *codom*($\vec{C}$)= FS(X) – with partition {$\vec{C}$(nST(X))={∅,$\mathcal{P}$(X)}, $\vec{C}$(ST(X))= FS(X)\{∅, $\mathcal{P}$(X)}} and *codom* ($\overleftarrow{C}$)= IS(X)⊆nST(X).

Proof. i. We have (by definition) [•,X]= ∪ { F | F∈ [•,X]} ⊆∪ { [F, X] | F∈ [•, X]}⊆[•, X] – and analogously for [∅,•].

ii. Among the definition properties of a closure operator (see consideration 2.2.i), the extensiveness property relative to $\vec{C}$ - $\mathcal{G}$⊆$\vec{C}(\mathcal{G})$ is obvious (by definition), and the monotony property results from the sequence

$\mathcal{E}$⊆$\mathcal{G}$, F∈$\vec{C}(\mathcal{E})$ iff ∃E∈$\mathcal{E}$, F∈[E,X], therefore ∃ E ∈$\mathcal{G}$, F ∈[E, X], i.e. F∈$\vec{C}(\mathcal{G})$.

Finally, the idempotency property ($\vec{C}\circ\vec{C}$)($\mathcal{G}$)⊆$\vec{C}(\mathcal{G})$ results from the sequence: F∈($\vec{C}\circ\vec{C}$)($\mathcal{G}$)=$\vec{C}(\vec{C}(\mathcal{G}))$ iff ∃F'∈$\vec{C}(\mathcal{G})$, F∈[F',X], therefore ∃G∈$\mathcal{G}$, F'∈ [G,X] and consequently G⊆F'⊆F, i.e. G⊆F, F∈$\vec{C}(\mathcal{G})$.

FS(X) is the closure system over $\mathcal{P}$(X) isomorphically associated to $\vec{C}$ as closure operator over $\mathcal{P}$(X) because we have the isomorphism CO($\mathcal{P}$(X))≃CS($\mathcal{P}$(X)) and the equality FS(X)=FP($\vec{C}$) can be verified (see consideration 2.2.i - (1)); indeed, if [•,X]∈FS(X), then we have $\vec{C}$([•,X])=[•,X] resulting from the sequence

F∈$\vec{C}$([•,X]) iff ∃S∈[•,X], F∈[S,X] iff F∈[•,X]

and reciprocal, $\mathcal{G}$∈FP($\vec{C}$) has the expression

$\mathcal{G}$={F∈$\mathcal{P}$(X) | ∃G∈$\mathcal{G}$, F∈[G,X]},

thus $\mathcal{G}$∈FS(X) (see point i).

An analogous procedure can be performed in order to verify the affirmation "$\overleftarrow{C}$ - closure operator over $\mathcal{P}$(X)" and the isomorphic association $\overleftarrow{C}$≃IS(X).

iii. The expressions of closures are particularizations of (2) from consideration 2.2.i for FS(X), IS(X)∈CS($\mathcal{P}$(X)).

iv. The equalities relative to codomains are particularizations of equality (1') from consideration 2.2.i for $\vec{C}$, $\overleftarrow{C}$∈CO($\mathcal{P}$(X)), and the inclusion IS(X) ⊆nST(X) is obvious (by definition). Related to the partition, we have $\vec{C}(\mathcal{G})$= ∅ iff $\mathcal{G}$= ∅, i.e. ∅∈FP($\vec{C}$) and $\mathcal{R}$∈nST(X) \ {∅} iff $\vec{C}(\mathcal{R})$={$\mathcal{P}$(X)} because ∅∈$\mathcal{R}$⊆$\vec{C}(\mathcal{R})$ (see point ii); moreover, {ST(X), nST(X)} is a partition of $\mathcal{P}$($\mathcal{P}$(X)) (see consideration 2.1.i and in addition the property of maintaining the intersection when a section of a univocal relation to the right is taken – for details in case of binary categorical and (pre-) univocal relations, see [9]).

Observation 3.2.i (using the inverse). The proof of point ii of the above theorem can use the inverse of the bijective function Ω: CO($\mathcal{P}$(X))→CS($\mathcal{P}$(X)), i.e. Σ:CS($\mathcal{P}$(X)) →CO($\mathcal{P}$(X)), FS(X)↦ $\vec{C}$ with validations FS(X)$\in$CS($\mathcal{P}$(X)) and $\vec{C}(\mathcal{G})$= $\cap$ FS(X)[$\mathcal{G}$], obtaining the equality FS(X)=FP($\vec{C}$) (see consideration 2.2.i).

ii (the algebraic lattices FS(X), IS(X)) As closure system over $\mathcal{P}$(X), FS(X) is a complete lattice, bounded by $\varnothing$ and $\mathcal{P}$(X) – but as complete sub-lattice of $\mathcal{P}(\mathcal{P}$(X)) (i.e. the operations $\vee$ and $\cup$ coincide, see consideration 2.2.i); in fact FS(X) \ {$\varnothing$} is lower bounded by {X}$\in$FS(X) and FS(X) \ {$\mathcal{P}$(X)} is upper bounded by $\mathcal{P}$*(X) $\in$FS(X).

Particularly, FS(X) is an algebraic closure system over $\mathcal{P}$(X) (satisfies C(ACS) from consideration 2.2.ii), so FS(X) is algebraic lattice and as consequence, any [$\bullet$,X] $\in$FS(X) is represented by:

[$\bullet$,X]= $\cup$ fS(X)^c,

where fS(X)$^c\subseteq$FS(X) is a set of final compact segments [$\bullet$,X]^c – that satisfy any of the two conditions of compact family – finite sub-cover, respectively

C(CF) [$\bullet$,X]^c=$\vec{C}(\mathcal{G}_f)$, $\mathcal{G}_f\subseteq$P(X) finite family (see consideration 2.2.ii).

IS(X) is the dual analogous of FS(X):

- IS(X) is a complete lattice, bounded by $\varnothing$ and P(X) – but as complete sub-lattice of $\mathcal{P}(\mathcal{P}$(X)), where IS(X) \ {$\varnothing$} is lower bounded by {$\varnothing$}$\in$IS(X);
- Particularly, IS(X) is an algebraic closure system over $\mathcal{P}$(X), so IS(X) is algebraic lattice with the representation of any initial segment through [$\varnothing$,$\bullet$]= $\cup$ iS(X)^c,
  where IS(X)$^c\subseteq$IS(X) is a set of initial compact segments – that satisfy any of the two conditions of compact family.

iii (about the alternatives of initial star segments) A "star" alternative to IS(X)$\subseteq$nST(X) is the set (IS(X)={(←,$\bullet$]$_{[\subseteq]}$|(←,$\bullet$] *initial segment over* X}$\subseteq$ST(X) – but which is not a closure system over $\mathcal{P}$*(X), and „associated" ($\overleftarrow{C}$: $\mathcal{P}(\mathcal{P}$*(X))→$\mathcal{P}(\mathcal{P}$*(X)) is not a closure operator over $\mathcal{P}$*(X) (generally, it does not hold the extensiveness property – see observation 3.1.i and theorem 3.1, points i, ii); the same "de-structuring" is also characteristic for the other version of initial segments like (←,$\bullet$) (see observation 3.1.i).

Example 3.1 (undetermined segments as determined segments) Let be A, B$\in\mathcal{P}$*(X), a$\in$A, b$\in$B, $\mathcal{S}_\bullet\subset$P*(X) the family of singletons of X; we have $\mathcal{P}$(X) =

$[\varnothing,X] \in FS(X) \cap IS(X)$ and $\mathcal{P}_A(X)=[A,X]=\vec{C}(\{A\})$ – in particular $\mathcal{P}_a(X) = [\{a\},X] = \vec{C}(\{\{a\}\})$ are final compact segments and $\mathcal{P}^*(X)=(\varnothing,X]= \vec{C}(\mathcal{S}.)= \vec{C}(\mathcal{P}^*(X))= \cup\{\mathcal{P}_x(X) \mid x\in X\}$ is a final segment that is not compact – but it is represented by the respective union of final compact segments (see considerations 2.1.ii, 2.1.iv, the proof from point ii of theorem 3.1 and observation 3.2.ii).

Dually, $[\varnothing,B] = \overleftarrow{C}(\{B\})$ – in particular $[\varnothing,\{b\}]=\overleftarrow{C}(\{\{b\}\})$ are initial compact segments and $\mathcal{S}.\cup\{\varnothing\}=\overleftarrow{C}(\mathcal{S}.)= \overleftarrow{C}(\mathcal{S}.\cup\{\varnothing\})=\cup\{[\varnothing,\{x\}] \mid \{x\}\in\mathcal{S}.\}$ is an initial segment that is not compact – but it is represented by the respective union of initial compact segments.

4. Pseudo-filters over sets

Definition 4.1 (pseudo-filter) A pseudo-filter $\mathcal{F}$ on X is a starred family on X and also a final segment on X; we denote P-FIL(X)=$\{\mathcal{F} \mid \mathcal{F}$ pseudo-filter on X$\}$ = ST(X)$\cap$FS(X).

Observation 4.1 (the set P-FIL(X)) The set of final residual (non-star) segments on X is actually the set of improper final segments on X $\{\varnothing,\mathcal{P}(X)\}\subset FS(X)$; consequently P-FIL(X) = FS(X) \ $\{\varnothing,\mathcal{P}(X)\}= \vec{C}$ (ST(X)) (see theorem 3.1.iv).

Theorem 4.1.i (closure operator over $\mathcal{P}^*(X)$) $\vec{C}_{P\text{-}F}:ST(X)\to ST(X)$, $\vec{C}_{P\text{-}F}= \vec{C}|_{ST(X)}$ is a closure operator on $\mathcal{P}^*(X)$ – associated with P-FIL(X) as closure system over $\mathcal{P}^*(X)$.

ii (expression) We have the following expression:

$$\vec{C}_{P\text{-}F}(\mathcal{S})=\min P\text{-}FIL(X)[\mathcal{S}],\ P\text{-}FIL(X)[\mathcal{S}]=\{\mathcal{F}\in P\text{-}FIL(X) \mid \mathcal{S}\subseteq\mathcal{F}\}.$$

iii (codomain) We have the following equality:

$$\text{codom}\,(\vec{C}_{P\text{-}F})=P\text{-}FIL(X).$$

Proof i. $\vec{C}_{P\text{-}F}$ is closure operator over $\mathcal{P}^*(X)$ – as the restriction on ST(X) of $\vec{C}$ as closure operator over $\mathcal{P}(X)$ and P-FIL(X) is the closure system over $\mathcal{P}^*(X)$ associated to $\vec{C}_{P\text{-}F}$ in accordance to isomorphism $CO(\mathcal{P}^*(X))\simeq CS(\mathcal{P}^*(X))$ - obtained by „the partialness" of the isomorphism $CO(\mathcal{P}(X))\simeq CS(\mathcal{P}(X))$ at ST(X) and the equality P-FIL(X)=FP($\vec{C}_{P\text{-}F}$) (see considerations 2.2.i, 2.2.iii); the above equality results from the equalities:

$$FS(X) = FP(\vec{C}),\ \{\varnothing,\mathcal{P}(X)\} = FP(\vec{C}|_{nST(X)}),\ FP(\vec{C}) = FP(\vec{C}_{P\text{-}F})\cup FP(\vec{C}|_{nST(X)}),$$

where the last equality with disjunctive union is obtained from the equality (with disjunctive union)

$$\vec{C} = \vec{C}_{P\text{-}F}\cup \vec{C}|_{nST(X)})$$

(see theorem 3.1 – proof on points ii and iv, respectively [12]).

ii. The closing expression is a particularization of expression (2) from consideration 2.2.i for P-FIL(X) $\in$CS($\mathcal{P}$*(X)); this way we have $\vec{C}_{P\text{-}F}(\mathcal{S})$=min FS(X)[$\mathcal{S}$] = min P-FIL(X)[$\mathcal{S}$] (see theorem 3.1.iii and observation 4.1)

iii. The equality results from the equalities:

codom ($\vec{C}_{P\text{-}F}$) = $\vec{C}$(ST(X)) = P-FIL(X)

(see consideration 2.2.iii and observation 4.1).

Observation 4.2.i (the algebraic lattice P-FIL(X)) As closure system over P*(X)

P-FIL(X) = FS(X) \ {$\varnothing$,$\mathcal{P}$(X)} = $\vec{C}$(ST(X)) = codom($\vec{C}_{P\text{-}F}$)

(according to observation 4.1 and theorem 4.1.iii) is a complete lattice which is bounded by {X}=$\cap$P-FIL(X), $\mathcal{P}$*(X)=$\cup$P-FIL(X) – but as a complete sub-lattice of FS(X) with the same properties (see consideration 2.2.ii and observation 3.2.ii):

- $\bigvee$ p-FIL(X) = sup p-FIL(X) = $\bar{C}_{P\text{-}F}$($\cup$p-FIL(X)) = $\cup$p-FIL(X), p-FIL(X) $\subseteq$ P-FIL(X);
- In algebraic lattice P-FIL(X) (as an algebraic closure system on $\mathcal{P}^*$(X)) any pseudo-filter is represented by

$\mathcal{F}$ = $\bigvee$ p -FIL(X)^c = $\cup$p -FIL(X)^c,

where p-FIL(X)^c$\subseteq$P-FIL(X) is a set of compact pseudo-filters on X (as closings of finite starred families on X).

ii (filter) A filter $\mathcal{F}$ on X can be defined as a pseudo-filter on X which is also a pre-filter on X - sufficiently $\mathcal{F}$ is an $\subseteq$ - lower (or to the left) filtering family(see observation 4.1 and consideration 2.1.iii); we notate FIL(X)={$\mathcal{F}$ | $\mathcal{F}$ filter on X}= P-FIL(X)$\cap$pFIL(X).

The equivalence with the usual definitions (where some of them are adapted for $\mathcal{P}$(X)) is settled in the following theorem.

Theorem 4.2 (equivalences) The following equivalences stand:

i. $\mathcal{F}$ is a filter on X iff $\mathcal{F}$ is a pseudo-filter on X and an $\subseteq$ - lower (or to the left) filtering family – observation 4.2. ii;

ii. $\mathcal{F}$ is a filter on X iff $\mathcal{F}$ is a pseudo-filter on X and $\cap$-semi-lattice (adaptation of definition 1.3.12 from [1], respectively the adaptation for $\mathcal{P}$(X) of conditions (10), (11), page 77 from [10];

iii. $\mathcal{F}$ is a filter on X iff $\mathcal{F}$ is a pre-filter on X and final segment on X (adaptation of definition 1.3.12.b from [1]);

iv. $\mathcal{F}$ is a filter on X iff $\mathcal{F}$ is starred family on X and verifies

„Eq. $\forall$ A,B$\in\mathcal{P}$(X), A,B$\in\mathcal{F}$ iff A$\cap$B$\in\mathcal{F}$.''

(adaptation for $\mathcal{P}$(X) of condition (12), page 77 from [10]).

Proof. The equivalence i $\Leftrightarrow$ iii is immediate (by definition). In verifying the following equivalences we use the expression

„E. $A \cap B = \inf_{\subseteq} \{A,B\} \{\text{in } \mathcal{F}\}$''.

The equivalence ii $\Leftrightarrow$ iv – actually the equivalence (10), (11) $\Leftrightarrow$ (12) is verified at page 77 from [10].

The equivalence ii $\Leftrightarrow$ iii is only mentioned in [1]. The implication ii $\Rightarrow$ iii is immediate (by definition and according to expression E); the mutual implication results by definition and from the dual completeness of $\mathcal{F} \in$ P-FIL(X) – if $\{A,B\}^{-}_{\mathcal{F}} \neq \emptyset$, then $A \cap B \in \mathcal{F}$ because we have successively

$C \in \{A,B\}^{-}_{\mathcal{F}}$, $[C,X] \subseteq \mathcal{F}$ with $A, B \in [C,X]$, $A \cap B \in [C,A] \cap [C,B] \subseteq [C,X]$

(for details – including more general relational structures see[9]).

Theorem 4.3.i (restriction at pFIL(X)) $\vec{C}_F$:pFIL(X)→pFIL(X) which is defined by $\vec{C}_F = \vec{C}_{P\text{-}F}|_{pFIL(X)} = \vec{C}|_{pFIL(X)}$ is the closure operator on $\mathcal{P}^*(X)$ – associated with FIL(X) as closure system on $\mathcal{P}^*(X)$.

ii (expression) We have the following expression

$\vec{C}_F(p\mathcal{F})$=min FIL(X)[$p\mathcal{F}$], FIL(X)[$p\mathcal{F}$]=$\{\mathcal{F} \in$ FIL(X) | $p\mathcal{F} \subseteq \mathcal{F}\}$.

iii (codomain) We have the equality

codom ($\vec{C}_F$)=FIL(X).

Proof i. $\vec{C}_F$ is correctly defined because

$\vec{C}_{P\text{-}F}|_{pFIL(X)} = (\vec{C}|_{ST(X)})|_{pFIL(X)} = \vec{C}|_{ST(X) \cap pFIL(X)} = \vec{C}|_{pFIL(X)}$

(see consideration 2.1.iii and [12]); also $\vec{C}_F$ is closure operator on $\mathcal{P}^*(X)$ – as a closure operator restriction (the restriction at pFIL(X) of $\vec{C}_{P\text{-}F}$ on $\mathcal{P}^*(X)$ or equivalently of $\vec{C}$ on $\mathcal{P}(X)$) and FIL(X) is the closure system on $\mathcal{P}^*(X)$ associated to $\vec{C}_F$ according to the isomorphism CO($\mathcal{P}^*(X)$)≃CS($\mathcal{P}^*(X)$) (obtained by „the partialness" of the isomorphism CO($\mathcal{P}(X)$) ≃CS($\mathcal{P}(X)$) at pFIL(X)⊂ST(X)) and the equality FIL(X)=FP($\vec{C}_F$) (see considerations 2.2.i, 2.2.iii); the equality above results from the equality sequence

P-FIL(X)=FP($\vec{C}_{P\text{-}F}$), FIL(X)=P-FIL(X)∩pFIL(X)=FP($\vec{C}_{P\text{-}F}$)∩pFIL(X)=
FP($\vec{C}_{P\text{-}F}|_{pFIL(X)}$) = FP($\vec{C}_F$)

(see observation 4.2 ii and the proof at point i of theorem 4.1).

ii. The closing expression is a particularization of expression (2) from consideration 2.2.i for FIL(X)∈CS($\mathcal{P}^*$(X)).

iii. We have the equalities (according to consideration 2.2.iii):

codom($\vec{C}_F$)= $\vec{C}$(pFIL(X))=FIL(X).

Observation 4.3.i (other proofs) The results from points ii, iii of theorem 4.3. are known – but in other context and with other proofs (see [1]).

ii (other expressions) The closing from the theorem 4.3.ii also has the following expressions:

$$\vec{C}_F(p\mathcal{F})=\min \text{P-FIL}(X)[p\mathcal{F}]=\min FS(X)[p\mathcal{F}]$$

because $\vec{C}_F(p\mathcal{F})=\vec{C}_{P\text{-}F}(p\mathcal{F})=\vec{C}(p\mathcal{F})$ (see theorems 4.1.i, 4.3.i, 3.1.ii and observations 4.1, 4.2.ii).

iii (the algebraic lattice FIL(X)) Generally the set of the filters of a lattice L is an algebraic closure system on L (more specific, a complete lattice – but not a sub-lattice of L), so algebraic lattice – with representation expressions of a filter and of the disjunction of two filters (see consideration 2.2.ii and [10]).

As closure system over $\mathcal{P}^*(X)$ FIL(X) is a $\cap$- complete semi-lattice bounded by $\{X\}=\cap$FIL(X) – as a $\cap$-complete sub-semi-lattice of P-FIL(X) and $\vee$-complete semi-lattice, i.e.

$\vee$ fIL(X)=sup fIL(X)=$\vec{C}_F(\cup$ fIL(X)$)\supset\cup$fIL(X), fIL(X)$\subseteq$FIL(X (see observation 4.2.i).

In the algebraic lattice FIL(X), any filter $\mathcal{F}$ is represented by

$\mathcal{F}=\vee$ fIL$(X)^c$,

where fIL$(X)^c\subseteq$FIL(X) is a set of compact filters on X (as closings of finite pre-filters on X).

Also we have $\mathcal{F}_1\vee\mathcal{F}_2=\{F_1\cap F_2 \mid F_1\in\mathcal{F}_1, F_2\in\mathcal{F}_2\}$, $\mathcal{F}_1, \mathcal{F}_2\in$FIL(X).

Example 4.1 (pseudo-filter, filters) Let be $a\in A\in\mathcal{P}^*(X)$; $\mathcal{P}_A(X)$ = [A,X] = $\vec{C}_{P\text{-}F}(\{A\})$ = $\vec{C}_F(\{A\})$ – in particular $\mathcal{P}_a(X)$=[{a},X]= $\vec{C}_{P\text{-}F}(\{\{a\}\})$= $\vec{C}_F(\{\{a\}\})$ are compact filters ({A}, {{a}}$\in$pFIL(X), but $\mathcal{P}^*(X)=(\varnothing,X]\in$P-FIL(X) \ FIL(X) - $\mathcal{P}^*(X)$ $\notin$ sections 1, 2pFIL(X) and is represented by the set $\{\mathcal{P}_x(X) \mid x\in X\}$ of compact pseudo-filters (see example 3.1 and observation 4.1.).

5. Conclusions

The main purpose of this paper is the generalization - by the notion of pseudo-filter on a set X of the notion of filter $\mathcal{F}$ on X which is defined usually (see [1] – [4]) as a pre-filter on X with the additional property

„P . $\forall F\in\mathcal{F}$, $\forall S\in\mathcal{P}(X)$, $F\subseteq S$ implies $S\in\mathcal{F}$";

hereinafter in defining of a pre-filter $p\mathcal{F}$ on X the void family and the families which contain the void family are eliminated, i.e. the residual families on X and so $p\mathcal{F}$ is a particular starred family (see [1] - [4], respectively considerations 2.1.i, 2.1.iii). Also, concerning $(\mathcal{P}(X),\subseteq)$ as a partial order structure the segment to the right on X $[G,X]=\mathcal{P}_G(X)$ is a starred family on X (actually pre-filter on X) and the segment to the left on X [∅,B] is a residual family on X (see section 1 and consideration 2.1.iv for details and example 2.1).

Hereinafter the undetermined segments on X are defined (see definition 3.1) – the final segment [•,X] (by the property „P(FS)" in which the segment to the right intervenes and which is equivalent with property „P") and the initial segment [∅,B] (by duality) and the operational expressions are obtained (see theorem 3.1.i). The main results concerning FS(X), IS(X) (the sets of final segments on X, respectively of initial segments on X) consist in the attributes of closure systems and codomains, respectively algebraic lattices with representation properties (for details see section 1 - and theorem 3.1, points ii, iv, respectively observation 3.2.ii); the proof from points ii and iv of theorem 3.1, respectively the affirmations from point ii of observation 3.2 are simplified or justified according to considerations 2.2.i, 2.2.ii.

A pseudo-filter $\mathcal{F}$ on X is a starred family on X which is also a final segment on X – and we have P-FIL(X) = ST(X)$\cap$FS(X) = FS(X) \ {∅,$\mathcal{P}$(X)} (expressions of the pseudo-filter set on X, see definition 4.1 and observation 4.1); relatively to P-FIL(X) the closure system and codomain, respectively algebraic lattice with representation properties attributes are maintained (for details see sections 1, 2 - and theorem 4.1, points i, ii, respectively observation 4.2.i).

Finally a filter $\mathcal{F}$ on X can be defined as a pseudo-filter on X which is also a pre-filter on X (sufficient $\mathcal{F}$ is an $\subseteq$- lower filtering family) – definition which is equivalent to the usual definitions (see observation 4.2.ii and theorem 4.2); also FIL(X) (the set of filters on X) has attributes of P-FIL(X) (for details see sections 1, 2 - and theorem 4.3, points i, iii, respectively observation 4.3.iii).

We mention that the development of consideration 2.2.iii (see also section 1) may constitute the subject of a separate paper.

From the developments relative to starred families we mention that of comparability (with the relation $\subseteq$ and with other relations), respectively of utility in different approach of data structures(see [13] and [15], [16] for some considerations).Also pseudo-filters on a set intervene in the definition of pseudo-topological structures (see [14]).

R E F E R E N C E S

[1] *Gh. Sirețchi,* Analiză Matematică, vol. III, Fasc. 2, Topologie generală, Universitatea din Bucureşti, Bucureşti (Mathematical Analysis, vol. III, sheaf 2, General Topology, University of Bucharest, Bucharest), 1983.

[2] *R. Engelking,* General Topology, Monografie Matematyczue, tom. 60, Warszawa, 1977.

[3] *L. A. Steen, J. A. Seebach Jr.,* Counterexamples in Topology, Springer – Verlag, New York, 1978. Reprinted by Dover Publications, New York, 1995.

[4] *S. Willard,* General Topology, Dover Publications, New York, 2004.

[5] *A. Fraenkel, Y. Bar - Hillel, A .Levy,* Foundation of Set Theory. Studies in Logic and the Foundation of Mathematics, Amsterdam: North Holland, 1973.

[6] *J. R. Shoenfield,* Axioms of set theory. In Handbook of Mathematical Logic, ed. J. Barwise, Studies in Logic and the Foundations of Mathematics, 90, Amsterdam: North Holland, 1977, pp. 321–344.
[7] *Y. N. Moschovakis*, Set Theory Notes. Undergraduate Texts in Mathematics, Heidelberg: Springer – Verlag, 1994.
[8] *T. Jech,* Set Theory, 2nd ed. Springer – Verlag, New York, 1997.
[9] *M. Rebenciuc*, Relații în categorii, logici neclasice, Teză de doctorat, Universitatea din Bucureşti, Facultatea de Matematică (Relations in categories, non-classical logics, PhD Thesis, University of Bucharest, Faculty of Mathematics), 1998 – 1999.
[10] *I. Purdea*, *Gh. Pic*, Tratat de algebră modernă, **vol. I**, Ed. Academiei României, Bucureşti (Treatise of modern algebra, **vol. I**, Publishing house of Romanian Academy, Bucharest), 1977.
[11] *M. Rebenciuc,* Binary relations – addenda 2 (sections, composabilities). U.P.B. Sci. Bull. Series A, Vol. 71, No. 1, 2009, pp. 21-32.
[12] *M. Rebenciuc*, Binary relations – addenda 1 (kernel, restrictions and inducing, relational morphisms). U.P.B. Sci. Bull. Series A, Vol. 70, No. 3, 2008, pp. 11-22.
[13] *M. Rebenciuc, I. Pisică,* About comparability relative to star families – with applications (preprint), 2010.
[14] *M. Rebenciuc,* Pseudo-topological structures (preprint), 2010.
[15] *M. Rebenciuc,* About regularity, Symposium on Advanced topics in electrical engineering-ATEE 2000, Bucharest, Section II. Mathematics, 2000, pp. 81-84.
[16] *M. Rebenciuc,* (Pseudo-)Ordered groups and extensions, Symposium on Advanced topics in electrical engineering-ATEE 2002, Bucharest, Section Mathematics, 2002, pp. 158-160.