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SZEGO'S THEOREM STARTING FROM JENSEN'S THEOREM

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Mai întâi vom introduce Teorema lui Jensen şi unele consecinţe ale sale pentru determinarea numărului zerourilor unei funcţii analitice în planul complex în interiorul discului $D(0;r)$. *Apoi vom prezenta Teorema lui Szego şi vom determina noi evaluări asupra numărului rădăcinilor reale ale unui polinom cu coeficienţi complecşi.*

Firstly, we will introduce Jensen's theorem and some useful consequences for giving the numbers of the zeros to the analytical complex functions inside the open disc $D(0;r)$. *Then, we will present Szego's Theorem and we will get new evaluation about the number of the real roots of a complex polynomial.*

Keywords: The number of real roots, Jensen's equality, Szego's theorem.

1. Introduction

There are many theorems about the numbers of the real roots to the complex polynomials. Some of them use, specially the Cauchy's theorem in complex plane and the others use, specially Jensen theorem as we can see in chapter 3.

If $P(x) = a_n x^n + ... + a_1 x + a_0 \in C[x],\ n \geq 1,\ a_0 \cdot a_n \neq 0$, the length of P is denoted by $L(P) = \sum_{i=0}^{n} |a_i|$ and we denoted with „*t*" the number of real roots of *P*, repeated according to their multiplicity, in the first class we see for example.

Theorem 3.3 : $t \cdot (t+1) < 4(n+1) \cdot \ln[\frac{L(P)}{\sqrt{|a_0 a_n|}}].$ (1)

Our theorem, use Jensen's theorem and follow an **I.Schur** method, see [1] to References and our result are: **Theorem 3.2**:

$$t \leq 2\sqrt{n} \cdot \left[\sqrt{\ln\left(\frac{L(P)}{|a_n|}\right)} + \sqrt{\ln\left(\frac{L(P)}{|a_0|}\right)} \right]. \quad (2)$$

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2. Jensen's Equality and its Applications

Theorem 2.1 Jensen's equality:
Be it $P(x)$ an analytic function in a region which contains the closed disk $\overline{D}(0;r)$; $r>0$, in the complex plane, if $n \geq 1$, $x_1, x_2, ..., x_n \in C$, $|x_i| < r$, $(\forall) i = \overline{1,n}$, are the zeros of P in the interior of $D(0;r)$ repeated according to their multiplicity and if $P(0) \neq 0$, then:

$$\ln|P(0)| = -\sum_{j=1}^{n} \ln(\frac{R}{|x_j|}) + \frac{1}{2\pi}\int_0^{2\pi} \ln|P(R \cdot e^{i\theta})| d\theta \quad \text{or:} \qquad (3)$$

$$n \ln R = \frac{1}{2\pi}\int_0^{2\pi} \ln|P(R \cdot e^{i\theta})| d\theta - \ln|P(0)| + \ln(\prod_{j=1}^{n} |x_j|). \qquad (4)$$

Proof: see [2] or [3] from References.

Remark 2.1

This formula establishes a connection between the absolute values of the zeros of the function P inside the disk $|z| < R$ and the values of $|P(z)|$ on the circle $|z| = R$, and can be seen as a generalization of the mean value property of harmonic functions.

Corrolary 2.1 Be it $P(x)$ an analytic function in a region which contains the closed disk $\overline{D}(0;1)$ in the complex plane, $n \geq 1$ is the number of all zeros of P and if, for $1 \leq s \leq n$, $x_1, x_2, ..., x_s$, $|x_i| < 1, (\forall) i = \overline{1,s}$, are the zeros of P in the interior of $D(0;1)$ repeated according to multiplicity, then:

a) $$\min_{|z=1|} |P(z)| < |P(0)| \qquad (5)$$

b) $$\min_{|z|=1} \{|F(z)|\} \leq \frac{1}{2\pi}\int_0^{2\pi} \ln|F(e^{i\theta})| d\theta \leq \max_{|z|=1} \{|F(z)|\}. \qquad (6)$$

Proof: **a)** In **Theorem 2.1.** Be it $R=1$. Then:

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \ln|P(e^{i\theta})| d\theta - \ln|P(0)| + \ln(\prod_{j=1}^{s} |x_i|),$$

But $0 \leq \prod_{i=1}^{s} |x_i| < 1 \Rightarrow \ln(\prod_{i=1}^{s} |x_i|) < 0$. So $\frac{1}{2\pi}\int_0^{2\pi} \ln|P(e^{i\theta})| d\theta < \ln|P(0)|$,

$$\frac{1}{2\pi}\cdot 2\pi\cdot\ln\left(\min_{|z=1|}|P(z)|\right)<\ln|P(0)|,$$

$$\min_{|z=1|}|P(z)|<|P(0)|. \qquad (7)$$

b) We can try by simillarity or see [4] to References.

Corrolary 2.2 If $P(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$, $a_i\in C;\ i=\overline{1,n}$, $n\geq 1$ and $R=\dfrac{L(P)}{|a_n|}$, $R>1$, where $x_1,x_2,...,x_n$ are the roots repeated according to multiplicity of $P(x)$ and $a_0\neq 0$, $a_n\neq 0$, then:

$$n=\frac{1}{2\pi i}\int_{|z|=\frac{L(F)}{|a_n|}}\frac{P'(z)}{P(z)}\,dz=\frac{\frac{1}{2\pi}\int_0^{2\pi}\ln|P[\frac{L(P)}{|a_n|}\cdot e^{i\theta}]|d\theta-\ln|a_n|}{\ln[L(P)]-\ln|a_n|} \qquad (8)$$

For proving see [4] to References.

Corollary 2.3 Be it $P(x)$ an analytic function in a region which contains the <u>closed disk</u> $\overline{D}(0;r)$, r>1 in the complex plane, $s\geq 1$ is the number of all zeros of P with, $x_1,x_2,...,x_s$, $|x_i|\leq 1$, $(\forall)i=\overline{1,s}$, are the zeros of P in the interior of $D(0;1)$ repeated according to multiplicity, then:

$$s<\frac{\ln\left[\max_{|z|=r}\left\{\frac{|P(z)|}{|P(0)|}\right\}\right]}{\ln r}. \qquad (9)$$

Proof: If $n\geq s\geq 1$, $x_1,x_2,..x_s,x_{s+1}...,x_n\in C$, $|x_i|<r$, $(\forall)i=\overline{1,n}$, are the zeros of P in the interior of $D(0;r)$ repeated according to their multiplicity. Then from Jensen Theorem we have:

$$n\ln r=\frac{1}{2\pi}\int_0^{2\pi}\ln|P(r\cdot e^{i\theta})|d\theta-\ln|P(0)|+\ln\left(\prod_{j=1}^{n}|x_j|\right).$$

Because for $i=\overline{1,s}$, $|x_i|\leq 1$, we have

$$0\leq\prod_{i=1}^{s}|x_i|\leq 1\Rightarrow\ln\left(\prod_{i=1}^{n}|x_i|\right)\leq\ln\left(\prod_{i=s+1}^{n}|x_i|\right)<\ln r^{n-s}=(n-s)\ln r.$$

Now we can write: $n\ln r-(n-s)\ln r<\frac{1}{2\pi}\int_0^{2\pi}\ln|P(r\cdot e^{i\theta})|d\theta-\ln|P(0)|$.

And next $P(0)=a_0\neq 0$, and

$$\int_0^{2\pi}\ln|P(r\cdot e^{i\theta})|d\theta<\int_0^{2\pi}\ln[\max_{|z|=r}|P(z)|]d\theta=2\pi\cdot\ln[\max_{|z|=r}|P(z)|],$$

Therefore, from the previous relations we have:

$$s\ln(r)\leq\frac{1}{2\pi}\cdot 2\pi\cdot\ln[\max_{|z|=r}|P(z)|]-\ln|a_0|,$$

and now:

$$s<\frac{\ln\left[\max_{|z|=r}\left\{\frac{|P(z)|}{|P(0)|}\right\}\right]}{\ln r}.$$

3. Szego's Theorem

Proposition 3.1 Be it the polynomial $P(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0\in C[x]$, with $a_0\cdot a_n\neq 0$ and $Q_m(x)=n^{-m}\left(x\frac{d}{dx}\right)^m[P(x)]$, $m\in N$, meaning Q_m is defined by the relation

$$Q_m(x)=n^{-1}\cdot x\cdot\frac{dQ_{m-1}(x)}{dx}. \quad (10)$$

a) If we denote by α the number of the real roots, having absolute value bigger or equal with 1 for the polynomial Q_m and by b the number of the real roots bigger or equal with 1 then $b\leq m+\alpha$.

b) $$Q_m(x)=\sum_{i=0}^{n}\left(\frac{i}{n}\right)^m a_ix^i \quad (11)$$

Proof:

a) Using **Rolle's Theorem**

$$m=1\Rightarrow Q_1(x)=\frac{1}{n}\cdot x\cdot P'(x);\ Q_1(x)=0\Rightarrow\begin{cases}x=0\\ \text{or}\\ P'(x)=0\end{cases} \quad (12)$$

From **Rolle's Theorem** result that: $P'(x)$ has at least b - 1 real roots having absolute value bigger or equal with 1. (12')

From (12) and (12') result that $Q_1(x)$ has at least (b- 1) + m =(b- 1) + 1= b roots. It is known the fact that the degree of $Q_1(x) = n$ is the degree of $P(x)$ and we repeat the process. It results $b \leq m + \alpha$, with α the number of roots having absolute value bigger or equal with 1 for $Q(x) = Q_m(x)$.

b) Let see now the expression to $Q(x) = Q_m(x)$.

First we take m=1 then m=2. Obtain that:

$$Q_1(x) = n^{-1}xP'(x) = \frac{1}{n} \cdot x \cdot (na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + ... + a_1$$
$$Q_1(x) = \frac{n}{n}a^n x^n + \frac{n-1}{n}a_{n-1}x^{n-1} + ... + \frac{2}{n}a_2 x^2 + \frac{1}{n}a_1 x. \qquad (13)$$

$$Q_2(x) = n^{-1}x\left(\frac{n^2}{n}a_n x^{n-1} + \frac{(n-1)^2}{n}a_{n-1}x^{n-2} + ... + \frac{2^2}{n}a_2 x + \frac{1}{n}a_1\right),$$

$$Q_2(x) = \frac{n^2}{n^2}a_n x^n + \frac{(n-1)^2}{n^2}a_{n-1}x^{n-1} + ... + \frac{2^2}{n^2}a_2 x^2 + \frac{1}{n^2}a_1 x = \sum_{i=0}^{n}\left(\frac{i}{n}\right)^2 a_i x^i .$$

(14)

Now by induction about m we can obtain:

$$Q(x) = Q_m(x) = n^{-m}\left(x\frac{d}{dx}\right)^m \left[P(x)\right] = \sum_{i=0}^{n}\left(\frac{i}{n}\right)^m a_i x^i \qquad (15)$$

Theorem 3.1 For $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \in C[x]$, with $a_0 \cdot a_n \neq 0$; the length of P is denoted by $L(P) = \sum_{i=0}^{n} | a_i |$ and let b the number of the real roots bigger or equal with 1. Then

$$b \leq 2\sqrt{n \cdot \ln\left(\frac{L(P)}{|a|}\right)} . \qquad (16)$$

Demonstration:

From previous proposition for a natural m and for the polynomial

$Q(x)=Q_m(x)=\sum_{i=0}^{n}\left(\frac{i}{n}\right)^m a_i x^i$ we have $b\le m+\alpha$ where α is the number of the real roots, having absolute value bigger or equal with 1 for the polynomial Q.
We consider $revQ$ the reciprocal polynomials of $Q(x)$:

$$revQ(x)=x^n\cdot Q(x^{-1})=\sum_{i=0}^{n}\left(\frac{n-i}{n}\right)^m a_{n-i}\cdot x^i=\sum_{i=0}^{n}\left(1-\frac{i}{n}\right)^m\cdot a_{n-i}\cdot x^i. \tag{17}$$

If $Q(x)=0$ with $|x|\ge 1\Rightarrow revQ\left(\frac{1}{x}\right)=0;\ \left|\frac{1}{x}\right|\le 1$ and now we see that: α is the number of the real roots bigger or equal with 1 for $Q(x)$ if and only if α is the number of the real roots small or equal with 1 for $revQ(x)$.
Be it $r\in R;\ r>1$. From Jensen's inequality results that the total number of the roots (complex) of the polynomial $revQ$ from $|z|\le 1$ are delimitated by the quantity of the total number of the roots (complex) of the polynomial $revQ$ from $|z|\le r$ and taking in Jensen formula only the roots with moduly at most equally to one we obtain from **Corollary 2.3:**

$$\alpha<\frac{\ln\left[\max\limits_{|z|=r}\left\{\frac{|revQ(z)|}{|revQ(0)|}\right\}\right]}{\ln r}. \tag{18}$$

We pick

$$r=e^{m/n}\Rightarrow \max\left\{|revQ(z)|;\ |z|=r\right\}\le\sum_{j=0}^{n}\left|a_{n-j}\right|\cdot\left(1-\frac{j}{n}\right)^m e^{m(j/n)}. \tag{19}$$

And if we note $g(x)=(1-x)^m\cdot e^{mx}$ then for $0\le\frac{j}{n}=x\le 1,\ j\in\{0,1,...,n\}$ we will have:

$$(1-x)^m\cdot e^{mx}=\left(1-\frac{j}{n}\right)^m\cdot e^{m(j/n)}. \tag{20}$$

$$\begin{aligned}g'(x)&=m\cdot(1-x)^{m-1}\cdot(-1)\cdot e^{mx}+(1-x)^m\cdot e^{mx}\cdot m\\ g'(x)&=m\cdot(1-x)^{m-1}\cdot e^{mx}\cdot(-x)\end{aligned} \tag{21}$$

Then $g'(x)\le 0$ for $0\le x\le 1$ and because of that

$$g\left(\frac{j}{n}\right)\le g(0)\ (\forall)\ j\in\{0,1,...,n\}. \tag{22}$$

We obtain

$$\left(1-\frac{j}{n}\right)^m e^{m(j/n)}=1. \tag{23}$$

From (19) and (23) we have:

$$\max\left\{\left|revQ(z)\right|;\ |z|=r\right\}\le\sum_{j=0}^{n}\left|a_{n-j}\right|\left(1-\frac{j}{n}\right)^m e^{m(j/n)}\le\sum_{j=0}^{n}\left|a_{n-j}\right|=L(P) \tag{24}$$

Because $revQ(0)=a_n$ from (18) relation we obtain: $\alpha<\dfrac{\ln\dfrac{L(P)}{|a_n|}}{\dfrac{m}{n}}\Rightarrow\alpha<\dfrac{n}{m}\ln\dfrac{L(P)}{|a_n|}$.

Then $b\le m+\alpha\Rightarrow b<m+\dfrac{n}{m}\cdot\ln\dfrac{L(P)}{|a_n|}$. Be it $m=\left[\left(n\cdot\ln\dfrac{L(P)}{|a_n|}\right)^{1/2}\right]+1.$

It results

$$b<\left[\left(n\cdot\ln\frac{L(P)}{|a_n|}\right)^{1/2}\right]+1+\frac{n}{\left[\left(n\cdot\ln\frac{L(P)}{|a_n|}\right)^{1/2}\right]+1}\cdot\ln\frac{L(P)}{|a_n|}.$$

$$b<\left[\left(n\cdot\ln\frac{L(P)}{|a_n|}\right)^{1/2}\right]+1+\frac{n\cdot\ln\frac{L(P)}{|a_n|}}{\left[\left(n\ \ln\frac{L(P)}{|a_n|}\right)^{1/2}\right]+1},$$

$$b<\left[\left(n\cdot\ln\frac{L(P)}{|a_n|}\right)^{1/2}\right]+1+\left(n\cdot\ln\frac{L(P)}{|a_n|}\right)^{1/2}. \tag{25}$$

And because b is natural from the last relation we obtain: $b\le 2\sqrt{n\cdot\ln\left(\dfrac{L(P)}{|a_n|}\right)}$

Theorem 3.2 For $P(x)=a_nx^n+a_{n\text{-}1}x^{n\text{-}1}+...+a_1x+a_0\in C[x]$, with $a_0\cdot a_n\ne 0$;

the length of P is denoted by $L(P)=\sum_{i=0}^{n}|a_i|$ and let be t the number of all the real roots of P. Then:

$$t \le 2\sqrt{n}\cdot\left[\sqrt{\ln\left(\frac{L(P)}{|a_n|}\right)}+\sqrt{\ln\left(\frac{L(P)}{|a_0|}\right)}\right]. \tag{26}$$

Demonstration:

a) Be it b the number of the real roots bigger or equal with 1.
Then $s=t-b$ is the number of the real roots smaller than 1.
We can observe that s is the number of real roots bigger or equal with 1 for the reciprocal polynomial

$$revP = x^n \cdot P\left(\frac{1}{x}\right) = a_0x^n + a_1x^{n-1}+...+a_n.$$

We have demonstrated in previous theorem that

$$b \le 2\cdot\sqrt{n\cdot\ln\left(\frac{L(P)}{|a_n|}\right)} \text{ for } P=a_nx^n+a_{n-1}x^{n-1}+...+a_0. \tag{27}$$

Using a similar method we have: $s \le 2\sqrt{n\ln\frac{L(revP)}{|a_0|}}$.

But $L(revP)=L(P)$, then $s \le 2\cdot\sqrt{n\ln\frac{L(P)}{|a_0|}}$

Then:

$$t=b+s\le 2\sqrt{n}\cdot\left[\sqrt{\ln\left(\frac{L(P)}{|a_n|}\right)}+\sqrt{\ln\left(\frac{L(P)}{|a_0|}\right)}\right] \tag{28}$$

and the relation was proved.

Corrolary 3.1 For $P(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0\in C[x]$, with $a_0\cdot a_n\ne 0$ with length of P, $L(P)=\sum_{i=0}^{n}|a_i|$ and t the number of all the real roots of P.
Then:

$$t^2 \le 8n\cdot\ln\left(\frac{L(P)}{\sqrt{|a_0a_n|}}\right). \tag{29}$$

Proof: From the last Theorem we have: $t \le 2\sqrt{n} \cdot \left[\sqrt{\ln\left(\frac{L(P)}{|a_n|}\right)} + \sqrt{\ln\left(\frac{L(P)}{|a_0|}\right)} \right]$.

We use the relation, $(\forall)\ i, j \in N$, we can write:

$$(i+j)^2 = i^2 + j^2 + 2ij \le i^2 + j^2 + 2 \cdot \frac{i^2 + j^2}{2} \le 2\left(i^2 + j^2\right).$$

Then $t^2 \le 2 \cdot n \left[\sqrt{\ln\left(\frac{L(P)}{|a_n|}\right)} + \sqrt{\ln\left(\frac{L(P)}{|a_0|}\right)} \right]^2 \le 2n \cdot 2 \left\{ \left[\sqrt{\ln\left(\frac{L(P)}{|a_n|}\right)} \right]^2 + \sqrt{\ln\left(\frac{L(P)}{|a_0|}\right)}^2 \right\}$,

$$t^2 \le 4n \cdot \ln\left(\frac{[L(P)]^2}{|a_0 \cdot a_n|}\right),\ t^2 \le 8n \cdot \ln\left(\frac{L(P)}{\sqrt{|a_0 \cdot a_n|}}\right). \quad (30)$$

Corrolary 3.2 For $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 \in C[x]$ with $a_0 \cdot a_n \ne 0$, $L(P) = \sum_{i=0}^{n} |a_i|$ a polynomial which have at least one real root bigger or equal to one and at least one root smaller than one, with length of P, and "t" the number of all the real roots of P. Then:

$$t \le 4n \cdot \ln\left(\frac{L(P)}{\sqrt{|a_0 a_n|}}\right) \quad (31)$$

Proof: Be it b the number of the real roots bigger or equal with 1 and s=t-b the number of the real roots smaller than 1, we have obtained: $b \le 2\sqrt{n \cdot \ln\left(\frac{L(P)}{|a_n|}\right)}$,

$s \le 2 \cdot \sqrt{n \ln \frac{L(P)}{|a_0|}}$.

From hypotesis $1 \le b, 1 < s$ then $b \le b^2$, $s < s^2$. Then we have:

$$t \le b + s \le b^2 + s^2 \le 2n \cdot \left[\ln\left(\frac{L(P)}{|a_n|}\right) + \ln\left(\frac{L(P)}{|a_0|}\right) \right] = 4n \cdot l\,n\left(\frac{L(P)}{\sqrt{|a_0 a_n|}}\right). \quad (32)$$

Theorem 3.3 (G. Szegö).

If $P(x) = a_n x^n + ... + a_1 x + a_0 \in C[x]$, $n \ge 1$, $a_0 \cdot a_n \ne 0$, then if we noted with „t" the number of real roots of P we have:

$$t\cdot(t+1)<4(n+1)\cdot\ln[\frac{L(P)}{\sqrt{|a_0 a_n|}}]. \quad (33)$$

For proving see [5].

Corrolary 3.3. If $P(x)=a_n x^n+...+a_1 x+a_0\in Z[x]$, $n\geq 1$, $a_0\cdot a_n\neq 0$, for „t”, the number of real roots of P, we have: $t^2<4\cdot(n+1)\cdot\ln[L(P)]$. (34)

Demonstration: $a_0\neq 0$, $a_n\neq 0$ and $a_0, a_n\in Z$.

Then $|a_0|\geq 1, |a_n|\geq 1$ and $L(P)>1, \sqrt{|a_0\cdot a_n|}\geq 1$, $\ln L(P)>0$.

Also it is easy to prove that: $\frac{L(P)}{\sqrt{|a_0 a_n|}}>1$ and $\ln[\frac{L(P)}{\sqrt{|a_0 a_n|}}]>0$.

Now from **Theorem 2.3. (G. Szegö)** we have:

$$t^2<t\cdot\ (t+1)<4(n+1)\cdot\ln[\frac{L(P)}{\sqrt{|a_0 a_n|}}]. \quad (35)$$

Then because: for $n\geq 1$, and $\sqrt{|a_0\cdot a_n|}\geq 1$ so $\frac{L(P)}{\sqrt{|a_0\cdot a_n|}}\geq L(P)$ and

$\ln[\frac{L(P)}{\sqrt{|a_0\cdot a_n|}}]\geq\ln[L(P)]$, the relation become: $t^2<4\cdot(n+1)\cdot\ln[L(P)]$.

Remark 3.1 Demonstrations for **G. Szegö's** theorems we can see in [1], [5] , [6]. The author follow a method from [1] and give the new relations as we can see in **Theorem 3.1** and **Theorem 3.2**.

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