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A DESCRIPTIVE APPROACH OF INTERSECTING GEOMETRICAL LOCI – THE CYLINDER CASE

G.OPREA, G. RUSE*

În ingineria mecanică, în stadiul de proiectare, apare uneori necesitatea folosirii locurilor geometrice în scopul pozitionării relative a unor componente mobile din ansambluri, supuse unor constrângeri geometrice si dimensionale.

Lucrarea trarează metoda intersectării locurilor geometrice, prin care un loc geometric necunoscut se determină prin intersectarea a două locuri goemetrice cunoscute. Abordarea problemei se face prin mijloacele geometriei descriptive, obtinându-se soluțiile grafice necesare proiectantului.

In the frame of mechanical engineering, during designing, it is sometimes necessary to use the geometrical loci for the relative positioning of some mobile parts, which belong to some assemblies, subject to geometrical and dimensional constraints.

The present paper deals with the method of intersecting geometrical loci, method which determines an unknown geometrical locus intersecting other two known ones. The problem is approached by the means of descriptive geometry, thus obtaining the graphical solutions necessary to the designer.

Key words: geometrical locus, cylinder of revolution, sphere, plane.

Introduction

One of the geometry specific character is to alternate the abstract mathematical thinking with the concrete one, associating a figure with a metric characteristic feature.

The aim of the descriptive geometry is to develop the spatial seeing in order to use it for the plane representation of three-dimensional objects. Therefore studying the geometrical loci is very useful for being able to visualize geometrical features common to some points subject to the same geometrical and dimensional constraints. The use of the geometrical loci as a method to solve some metric and geometrical construction problems could be sometimes important in the field of mechanical designing.

* Reader, Prof., Dept. of Descriptive Geometry and Engineering Graphics, University "Politehnica" of Bucharest, ROMANIA

The geometrical locus is defined as the point set *{L}* having a common characteristic feature. The syntagm « geometrical locus » is to be found for the first time in Euclid's work « The Elements », but it has been historically prefigured by Pitagora and Aristotel.

A geometrical locus problem systematically approached is characterized by the following :

- a set *{E}* of spatial fixed elements ;
- a set *{K}* of some scalar and vectorial constants ;
- a feature *F* related to the sets *{E}* and *{K}* which should be proved by the points of the geometrical locus *{L}*.

The descriptive geometry studies the geometrical surfaces from several points of view : of representation, cutting, intersecting and developing. Analysing some usual surfaces it is easy to establish their feature of being geometrical loci. The features *F* defining some of them as geometrical loci can be stated as the following cases :

- **1.a.** the plane *[P]* is the geometrical locus of the points which distance *d* to another given plane *[Q]* is a constant ; in this case :*{E}={[Q]}* and *{K}={d}*.
- **1.b.** the plane is the geometrical locus of the points equidistant to two fixed points *A* and *B* at the distance *d* (the median perpendicular plane) ; in this case :*{E}={A,B}* and *{K}={d}*.
- **1.c.** the plane is the geometrical locus of the points equidistant to the sides of a dihedron of angle *a*, defined by the fixed planes *[P]* and *[Q]* (bisecting plane) ; in this case : *{E}={[P], [Q]}* and *{K}={a}*.
- **2.a.** the cylinder of revolution is the geometrical locus of the points which distance *r* to a given line *D* is constant (the axis of the cylinder is *D* and its radius is *r*) ; in this case : *{E}={D}* and *{K}={r}*.
- **2.b.** the cylinder of revolution is the geometrical locus of the points which distance *r* to the surface of another coaxial cylinder of revolution *(C)* of radius *R* is constant ; in this case : *{E}={(C)}* and *{K}={r}*.
- **3.** the cone of revolution is the geometrical locus of the lines passing through the fixed point *V* (the vertex of the cone) belonging to a fixed line *D* and building a constant angle *a* with the line *D* (the axis of the cone); in this case : *{E}={D,V}* and *{K}={a}*.
- **4.a.** the sphere is the geometrical locus of the points which distance *r* to a fixed point *O* (the center of the sphere) is constant ; in this case : *{E}={O}* and *{K}={r}*.
- **4.b.** the sphere is the geometrical locus of the points which distance *r* to another given sphere (*S*) of radius *R* is constant ; in this case : *{E}={(S)}* and *{K}={r}*.

The present paper continues the author's concern regarding the geometrical loci problems [1] introducing and using the method of intersecting geometrical loci for solving the problems under discussion. In the literature this method is mentioned in many works such as [2], [3], [4].The descriptive geometry methods and notations used in the paper are in accord with the works [5], [6], [7].

1. The intersection of a cylinder of revolution and a plane

Let's consider the following problem of a geometrical locus : given two coaxial cylinders of revolution *(C_1)* and *(C_2)* having the bases in the horizontal plane of projection, and the radii r_1 and r_2 and a fixed plane *[P]* (fig. 1) , let's find the geometrical locus of the points equidistant to the cylinders and plane.

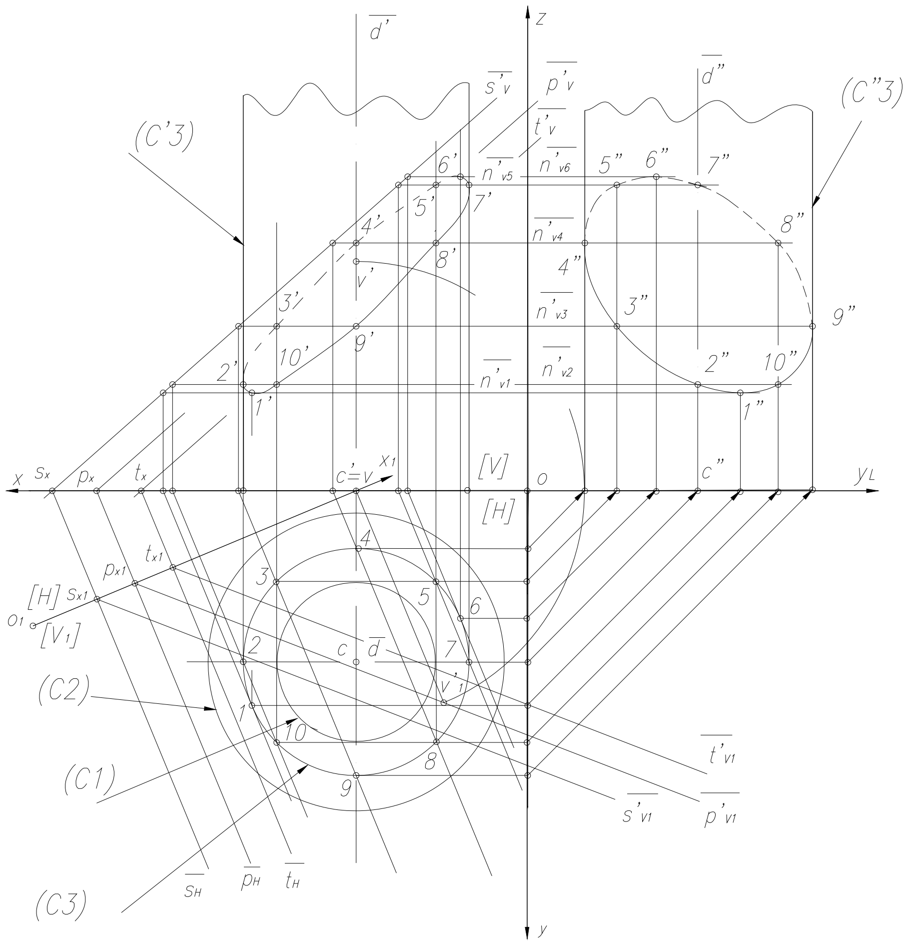

Fig. 1 The intersection of a cylinder of revolution and a plane

In this case the set of fixed elements $\{E\}$ consists of the two cylinders (C_1) and (C_2) and the plane *[P]*. The set of constants is $\{K\}=\{r_1, r_2\}$.

According to **2.b.** from **Introduction**, the geometrical locus of the points equidistant to the cylinders (C_1) and (C_2) is the cylinder of revolution (C_3), coaxial with (C_1) and (C_2) and having the radius: $r_3 = ½ (r_1 + r_2)$.

On the other hand, according to **1.a.** from **Introduction**, the geometrical locus of the points equidistant to the plane *[P]* consists of the system of the two parallel planes *[S]* and *[T]*, parallel and equidistant to the plane *[P]* (see fig.1).

As the distance of a point belonging to the cylinder (C_3) to the cylinders (C_1) and (C_2) is: $d = ½ (r_2 – r_1)$, it results that the distance of the plane *[P]* to the planes *[S]* and *[T]* will be: $d=1/2(r_2–r_1)$ too.

The geometrical locus to be found will consist of the intersection curves of the cylinder (C_3) and the planes *[S]* and *[T]*, i.e. the two section ellipses cut in the cylinder (C_3) by the planes *[S]* and *[T]*. Fig.1 represents in three orthographic projections the ellipse cut by the plane *[S]* only in the cylinder (C_3). The points defining the ellipse were noted in the horizontal projection: *(1,2,3,4,5,6,7,8,9,10)*.

To determine the planes *[S]* and *[T]* we made a replacement of the vertical plane of projection for the plane *[P]*, which in the new projection system *[H]*, $[V_1]$ becomes perpendicular to the new vertical projecting plane $[V_1]$. The vertical traces s'_{V1} and t'_{V1}. of the planes *[S]* and *[T]* were determined in the new vertical projection, taking into account that the distances between the traces s'_{V1} and p'_{V1} on one hand, the traces t'_{V1} and p'_{V1} on the other hand,have both to be equal to $d = ½ (r_2 – r_1)$. Counter-revolving these traces we get s_H, s'_V, t_H, t'_V.

Sectioning the cylinder (C_3) by the plane *[S]* (s_H, s'_V) we obtain the ellipse of projections *(1 2....10 1, 1' 2' ....10' 1', 1" 2"....10" 1")*, the first solution of the problem. The second solution will be the ellipse cut in the cylinder (C_3) by the plane *[T]*.

2. The intersection of two cylinders of revolution

Let's consider the following problem of a geometrical locus: given the coaxial cylinders (C_1) and (C_2) with the axis Δ_1, similar to the ones in **chapter 1**, and a fixed line Δ_2, let's find the geometrical locus of the points equidistant to the two cylinders and the line Δ_2 (fig.2).

In this case the set of fixed elements is $\{E\} = \{(C_1), (C_2), \Delta_2\}$ and the set of constants is $\{K\} = \{r_1, r_2\}$.

According to **2.b.** from the **Introduction** the geometrical locus of the points equidistant to the cylinders of revolution (C_1) and (C_2) is the cylinder of revolution (C_3) coaxial with (C_1) and (C_2) and having the radius $r_3 = ½ (r_1 + r_2)$.

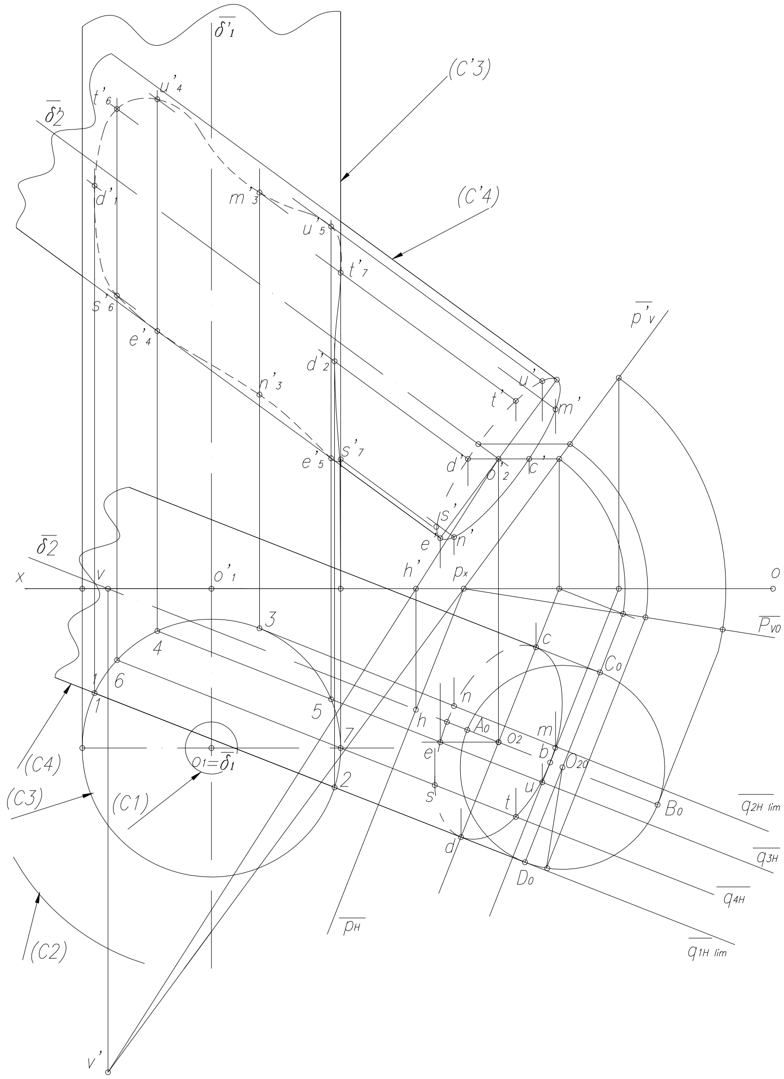

Fig. 2 The intersection of two cylinders of revolution

According to **2.a.** from the **Introduction**, the geometrical locus of the points at the distance r_4 to a given fixed line Δ_2 is the cylinder of revolution *(C4)* having the axis Δ_2 and the radius r_4.

As the distance of a point belonging to the cylinder (C_3) to the cylinders (C_1) and (C_2) is: $d = ½ (r_2 - r_1)$, it results that the radius of the cylinder *(C4)* will be: $r_4 = d = ½ (r_2 - r_1)$.

The geometrical locus to be found will be the line of intersection of the cylinders of revolution *(C₃)* of radius $r_3 = ½ (r_1 + r_2)$, having the axis Δ_1 and (C_4) of radius $r_4 = ½ (r_2 – r_1)$, having the axis Δ_2.

Fig.2 represents in two orthographic projections the line of intersection of the cylinders (C_3) and (C_4).

The plane *[P]* of the basis of the cylinder (C_4) was arbitrarily chosen, observing the condition of perpendicularity of *[P]* and Δ_2. The radius of the basic circle of the cylinder (C_4) was taken: $r_4 = ½ (r_2 – r_1)$ in the revolved position of the basic plane (the revolved vertical trace of the plane was noted by: p_{V0} and the center of the basic circle by: O_{20}).

The line of intersection $(D_1S_6E_4N_3E_5S_7D_2T_7U_5M_3U_4T_6D_1)$ of the cylinders (C_3) and (C_4) was determined using auxiliary vertical planes having the horizontal traces: q_{1Hlim}, q_{2Hlim}, q_{3H} and q_{4H}. These planes cut simultaneously both bases of the cylinders (C_3) and (C_4). The intersection of these planes and the two cylinders consists of a number of concurrent generatrices defining the points of the intersection line.

On the representation in fig.2 only the points defining the vertical projection of the line of intersection were noted.

As we can see, the horizontal projection of this line matches the arc of the basic circle of the cylinder *(1643572)* and in the vertical projection only the branch $(s'_7 d'_2 t'_7)$ of the line of intersection is visible.

This line of intersection having the horizontal projection *(1643572)* and the vertical projection $(d'_1 s'_6 e'_4 ... t'_6 d'_1)$ is the solution of the problem.

3. The intersection of a cylinder of revolution and a sphere

Let's consider the following problem of a geometrical locus: given the coaxial cylinders (C_1) and (C_2) of axis *D (d, d')* and of radii r_1, r_2, similar to the ones in **chapter 1**, and a sphere *(S)* of radius *R*, let's find the geometrical locus of the points equidistant to the two cylinders and the sphere (see fig.3).

In this case the set of fixed elements is $\{E\} = \{(C_1), (C_2), (S)\}$ and the set of constants is $\{K\} = \{r_1, r_2, R\}$.

According to **2.b.** from the **Introduction** the geometrical locus of the points equidistant to the cylinders of revolution (C_1) and (C_2) is the cylinder of revolution (C_3) coaxial with (C_1) and (C_2) and having the radius: $r_3 = ½ (r_1 + r_2)$.

According to **4.b.** from the **Introduction**, the geometrical locus of the points at the distance *a* to a given sphere *(S)* consists of two spheres (S_4) and (S_5) concentric with the sphere *(S)* and having the radii: $r_4 = R + a$ and $r_5 = R – a$ respectively.

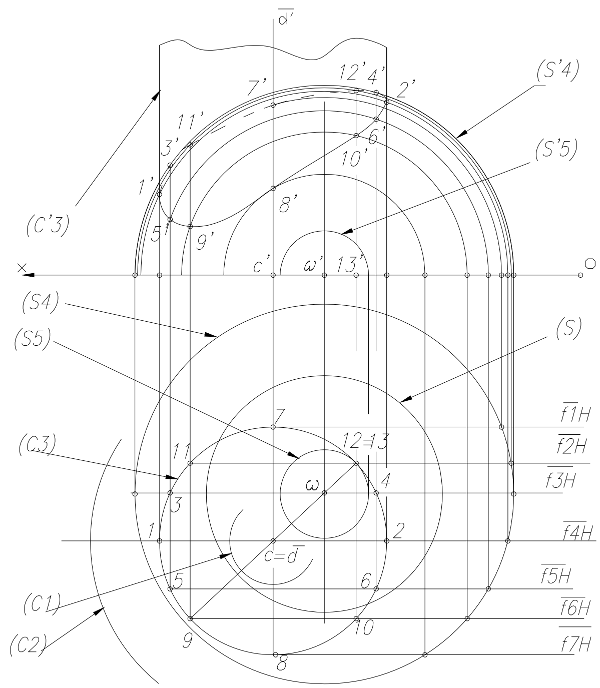

Fig. 3 The intersection of cylinder of revolution and a sphere

As the distance of a point belonging to the cylinder *(C_3)* to the cylinders *(C_1)* and *(C_2)* is: $d = ½ (r_2 – r_1)$, it results that $a = d$ and the radii of the spheres *(S_4)* and *(S_5)* will be: $r_4 = R + ½ (r_2 – r_1)$ and $r_5 = R - ½ (r_2 – r_1)$.

The geometrical locus to be found will be the line of intersection of the cylinder *(C_3)* and the spheres *(S_4)* and *(S_5)*.

Fig.3 represents in two orthographic projections the line of intersection of the cylinder *(C_3)* of axis *D (d, d')*, having the basic circle of center *C (c, c')* and the sphere *(S_4)* having the center *Ώ (ώ, ώ')* laying on the horizontal plane of projection.

The line of intersection *(1 3 11 7 12 4 2 6 10 8 9 5 1, 1' 3' 11' 7' 12' 4' 2' 6' 10' 8' 9' 5' 1')* of the cylinder *(C_3)* and the sphere *(S_4)* was determined using the seven auxiliary frontal planes having the horizontal traces f_{1H}, f_{2H},…f_{7H} simultaneously cutting the cylinder by generatrices and the sphere by circles. The points of intersection of the generatrices and circles belonging to the same frontal plane will define the line of intersection.

As we can see, the horizontal projection of this line matches the basic circle of the cylinder *(C_3)* and in the vertical projection only the branch *(2' 6' 10' 8' 9' 5' 1')* of the line of intersection is visible.

In fig.3 just one half of the sphere *(S_4)* was represented. Taking the whole sphere we obtain another line of intersection symmetrical with the first one with respect to the horizontal plane of projection.

The coordinates of the problem were chosen so that the sphere *(S_5)* is inner tangent to the cylinder *(C_3)*.

In this case the intersection of the cylinder *(C_3)* and the sphere *(S_5)* consists of one single point only. This point *(13, 13')* is the tangency point of the basic circle of the cylinder *(C_3)* and the great circle of the sphere *(S_5)* in the horizontal plane of projection.

It results that the solution of the problem consists of the lines of intersection of the cylinder *(C_3)* and the sphere *(S_4)* and the point *(13, 13')* of tangency of the cylinder *(C_3)* and the sphere *(S_5)*.

Conclusions

The paper points out the possibilities of descriptive geometry to solve spatial geometrical loci problems by graphic means, in a simple and elegant manner. The method of approaching the three examples analysed above was the method of intersecting the geometrical loci.

Solving this kind of problems by the means of analytical geometry it would lead to a difficult system of algebraic equations which solution will be another complicated equation. On the contrary, the descriptive answer to these problems will give clear and concrete graphic solutions, allowing an easy intuition and visualization of the solution.

The paper can be useful to the designers in mechanical engineering, particularly to the ones who design spatial mechanisms, in the cases when a moving element of an assembly has to observe some geometrical and dimensional constraints.

R E F E R E N C E S

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