TEMPERED INTERVAL-VALUED FUZZY HYPERGRAPHS

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In this article, we first present concepts and properties of interval-valued fuzzy hypergraphs. Then we introduce the notion of $A = [\mu^-, \mu^+]-$tempered interval-valued fuzzy hypergraphs and investigate some of their properties.

Keywords: interval-valued fuzzy hypergraphs, A-tempered interval-valued fuzzy hypergraph.
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1. Introduction

Zadeh [20] introduced the notion of interval-valued fuzzy sets as extensions of Zadeh's fuzzy set theory [19] for representing vagueness and uncertainty. Interval-valued fuzzy set theory reflects the uncertainty by the length of the interval membership degree $[\mu_1, \mu_2]$. In intuitionistic fuzzy set theory for every membership degree (μ_1, μ_2), the value $\pi = 1 - \mu_1 - \mu_2$ denotes a measure of non-determinacy (or undecidedness). Interval-valued fuzzy sets provide a more adequate description of vagueness than traditional fuzzy sets. It is therefore important to use interval-valued fuzzy sets in applications, such as fuzzy control. One of the computationally most intensive parts of fuzzy control is defuzzification [13]. Since interval-valued fuzzy sets are widely studied and used, we describe briefly the work of Gorzalczany on approximate reasoning [8, 9], Roy and Biswas on medical diagnosis [17], Turksen on multivalued logic [18] and Mendel on intelligent control [13].
Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert system. Kaufmann's initial definition of a fuzzy hypergraph [10] was based on Zadeh's fuzzy relations [19]. Lee-kwang *et al.* [11] generalized and redefined the concept of fuzzy hypergraphs whose basic idea was given by Kaufmann [10]. Further the concept of fuzzy hypergraphs was discussed in [7]. The concepts and applications of intuitionistic fuzzy hypergraphs are discussed in [3, 15]. Chen [5] introduced the concept of interval-valued fuzzy hypergraphs. In this article, we introduce the notion of $A = [\mu^-, \mu^+]-$tempered interval-valued fuzzy hypergraphs and investigate some of their properties.

2. Preliminaries

In this section, we review some elementary concepts whose understanding is necessary fully benefit from this paper.
A hypergraph is a pair $H^* = (V, E^*)$, where V is a finite set of nodes (vertices) and E^* is a set

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of edges (or hyperedges) which are arbitrary nonempty subsets of V such that $\bigcup_j E_j^* = V$. A hypergraph is a generalization of an ordinary undirected graph, such that an edge need not contain exactly two nodes, but can instead contain an arbitrary nonzero number of vertices. An ordinary undirected graph (without self-loops) is, of course, a hypergraph where every edge has exactly two nodes (vertices). A hypergraph is simple if there are no repeated edges and no edge properly contains another. Hypergraphs are often defined by an incidence matrix with columns indexed by the edge set and rows indexed by the vertex set. The rank $r(H)$ of a hypergraph is defined as the maximum number of nodes in one edge, $r(H) = \max_j |Ej|$, and the anti-rank $s(H)$ is defined likewise, i.e., $s(H) = \min_j |Ej|$. We say a hypergraph uniform if $r(H) = s(H)$. A uniform hypergraph of rank k is called k-uniform hypergraph. Hence a simple graph is a 2-uniform hypergraph, and thus all simple graphs are also hypergraphs. A hypergraph is vertex (resp. hyperedge) symmetric if for any two vertices (resp. hyperedges) v_i and v_j (resp. e_i and e_j), there is an automorphism of the hypergraph that maps v_i to v_j (resp. e_i to e_j). More concepts and applications of hypergraph theory can be found in [4].
Fuzzy set theory is an extension of ordinary set theory in which to each element a real number between 0 and 1, called the membership degree, is assigned. Unfortunately, it is not always possible to give an exact degree of membership. There can be uncertainty about the membership degree because of lack of knowledge, vague information, etc. A possible way to overcome this problem is to use interval-valued fuzzy sets, which assign to each element a closed interval which approximates the "real", but unknown, membership degree. The length of this interval is a measure for the uncertainty about the membership degree.
An *interval number* D is an interval $[a^-, a^+]$ with $0 \leq a^- \leq a^+ \leq 1$. The interval $[a, a]$ is identified with the number $a \in [0, 1]$. Let $D[0, 1]$ be the set of all closed subintervals of $[0, 1]$. Then, it is known that $(D[0, 1], \leq, \vee, \wedge)$ is a complete lattice with $[0, 0]$ as the least element and $[1, 1]$ as the greatest.

Definition 2.1. *An interval-valued fuzzy relation R in a universe $X \times Y$ is a mapping $R : X \times Y \to D[0, 1]$ such that $R(x, y) = [R^-(x, y), R^+(x, y)] \in D[0, 1]$ for all pairs $(x, y) \in X \times Y$.*

Interval-valued fuzzy relations reflect the idea that membership grades are often not precise and the intervals represent such uncertainty.

Definition 2.2. *The interval-valued fuzzy set A in V is defined by*

$$A = \{(x, [\mu_A^-(x), \mu_A^+(x)]) : x \in V\},$$

where $\mu_A^-(x)$ and $\mu_A^+(x)$ are fuzzy subsets of V such that $\mu_A^-(x) \leq \mu_A^+(x)$ for all $x \in V$. If $G^* = (V, E)$ is a graph, then by an interval-valued fuzzy relation B on a set E we mean an interval-valued fuzzy set such that

$$\mu_B^-(xy) \leq \min(\mu_A^-(x), \mu_A^-(y)), \mu_B^+(xy) \leq \min(\mu_A^+(x), \mu_A^+(y))$$

for all $xy \in E$. In the Clustering, the interval-valued fuzzy set A, is called an interval-valued fuzzy class. We define the support of A by supp $(A) = \{x \in V \mid [\mu_A^-(x), \mu_A^+(x)] \neq [0, 0]\}$ and say A is nontrivial if supp(A) is nonempty.

Definition 2.3. *The height of an interval-valued fuzzy set $A = [\mu_A^-(x), \mu_A^+(x)]$ is defined as*

$$h(A) = \sup_{x \in V}(A)(x) = [\sup_{x \in V} \mu_A^-(x), \sup_{x \in V} \mu_A^+(x)].$$

We shall say that interval-valued fuzzy set A is normal if $A = [\mu_A^-(x), \mu_A^+(x)]$ =[1, 1] for all $x \in V$.

Definition 2.4. [5] *Let V be a finite set and let $E = \{E_1, E_2, \cdots, E_m\}$ be a finite family of nontrivial interval-valued fuzzy subsets of V such that*

$$V = \bigcup_j \text{supp}[\mu_j^-, \mu_j^+], \; j = 1, 2, \ldots, m,$$

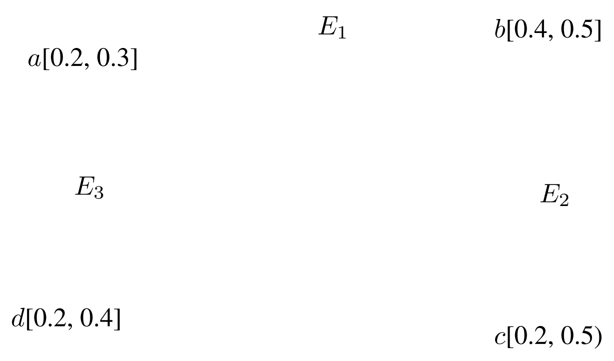
where $A = [\mu_j^-, \mu_j^+]$ is an interval-valued fuzzy set defined on $E_j \in E$. Then the pair $H = (V, E)$ is an interval-valued fuzzy hypergraph on V, E is the family of interval-valued fuzzy edges of H and V is the (crisp) vertex set of H. The order of H (number of vertices) is denoted by $|V|$ and the number of edges is denoted by $|E|$.

3. Tempered interval-valued fuzzy hypergraphs

Definition 3.1. *Let $A = [\mu_A^-, \mu_A^+]$ be an interval-valued fuzzy subset of V and let E be a collection of interval-valued fuzzy subsets of V such that for each $B = [\mu_B^-, \mu_B^+] \in E$ and $x \in V$, $\mu_B^-(x) \leq \mu_A^-(x)$, $\mu_B^+(x) \leq \mu_A^+(x)$. Then the pair (A, B) is an interval-valued fuzzy hypergraph on the interval-valued fuzzy set A. The interval-valued fuzzy hypergraph (A, B) is also an interval-valued fuzzy hypergraph on V = supp(A), the interval-valued fuzzy set A defines a condition for interval-valued in the edge set E. This condition can be stated separately, so without loss of generality we restrict attention to interval-valued fuzzy hypergraphs on crisp vertex sets.*

Example 3.1. *Consider an interval-valued fuzzy hypergraph $H = (V, E)$ such that $V = \{a, b, c, d\}$ and $E = \{E_1, E_2, E_3\}$, where*

$$E_1 = \{\frac{a}{[0.2, 0.3]}, \frac{b}{[0.4, 0.5]}\}, \; E_2 = \{\frac{b}{[0.4, 0.5]}, \frac{c}{[0.2, 0.5]}\}, \; E_3 = \{\frac{a}{[0.2, 0.3]}, \frac{d}{[0.2, 0.4]}\}.$$



Definition 3.2. *An interval-valued fuzzy set $A = [\mu_A^-, \mu_A^+] : V \to D[0, 1]$ is an elementary interval-valued fuzzy set if A is single valued on supp(A). An elementary interval-valued fuzzy hypergraph $H = (V, E)$ is an interval-valued fuzzy hypergraph whose edges are elementary.*

We explore the sense in which an interval-valued fuzzy graph is an interval-valued fuzzy hypergraph.

Table 1. The corresponding incidence matrix is given below:

M_H	E_1	E_2	E_3
a	$[0.2, 0.3]$	$[0, 0]$	$[0.2, 0.3]$
b	$[0.4, 0.5]$	$[0.4, 0.5]$	$[0, 0]$
c	$[0, 0]$	$[0.2, 0.5]$	$[0, 0]$
d	$[0, 0]$	$[0, 0]$	$[0.2, 0.4]$

Proposition 3.1. *Interval-valued fuzzy graphs and interval-valued fuzzy digraphs are special cases of the interval-valued fuzzy hypergraphs.*

An interval-valued fuzzy multigraph is a multivalued symmetric mapping $D = [\mu_D^-, \mu_D^+] : V \times V \to D[0, 1]$. An interval-valued fuzzy multigraph can be considered to be the "disjoint union" or "disjoint sum" of a collection of simple interval-valued fuzzy graphs, as is done with crisp multigraphs. The same holds for multidigraphs. Therefore, these structures can be considered as "disjoint unions" or "disjoint sums" of interval-valued fuzzy hypergraphs.

Definition 3.3. *An interval-valued fuzzy hypergraph $H = (V, E)$ is simple if $A = [\mu_A^-, \mu_A^+]$, $B = [\mu_B^-, \mu_B^+] \in E$ and $\mu_A^- \leq \mu_B^-$, $\mu_A^+ \leq \mu_B^+$ imply that $\mu_A^- = \mu_B^-$, $\mu_A^+ = \mu_B^+$. In particular, a (crisp) hypergraph $H^* = (V, E^*)$ is simple if X, $Y \in E^*$ and $X \subseteq Y$ imply that $X = Y$. An interval-valued fuzzy hypergraph $H = (V, E)$ is support simple if $A = [\mu_A^-, \mu_A^+]$, $B = [\mu_B^-, \mu_B^+]$ $\in E$, supp(A) = supp(B), and $\mu_A^- \leq \mu_B^-$, $\mu_A^+ \leq \mu_B^+$ imply that $\mu_A^- = \mu_B^-$, $\mu_A^+ = \mu_B^+$. An interval-valued fuzzy hypergraph $H = (V, E)$ is strongly support simple if $A = [\mu_A^-, \mu_A^+]$, $B = [\mu_B^-, \mu_B^+] \in E$ and supp(A) = supp(B) imply that $A = B$.*

Remark 3.1. *The definition 3.3 reduces to familiar definitions in the special case where H is a crisp hypergraph. The interval-valued fuzzy definition of simple is identical to the crisp definition of simple. A crisp hypergraph is support simple and strongly support simple if and only if it has no multiple edges. For interval-valued fuzzy hypergraphs all three concepts imply no multiple edges. Simple interval-valued fuzzy hypergraphs are support simple and strongly support simple interval-valued fuzzy hypergraphs are support simple. Simple and strongly support simple are independent concepts.*

Definition 3.4. *Let $H = (V, E)$ be an interval-valued fuzzy hypergraph. Suppose that $\alpha, \beta \in$ [0, 1]. Let*

- $E_{[\alpha,\beta]} = \{A_{[\alpha,\beta]} \,|\, A \in E\}$, $\quad A_{[\alpha,\beta]} = \{x \,|\, \mu_A^-(x) \leq \alpha \text{ or } \mu_A^+(x) \leq \beta\}$, *and*
- $V_{[\alpha,\beta]} = \bigcup_{A \in E} A_{[\alpha,\beta]}$.

If $E_{[\alpha,\beta]} \neq \emptyset$, then the crisp hypergraph $H_{[\alpha,\beta]} = (V_{[\alpha,\beta]}, E_{[\alpha,\beta]})$ is the $[\alpha, \beta]-$ level hypergraph of H.

Clearly, it is possible that $A_{[\alpha,\beta]} = B_{[\alpha,\beta]}$ for $A \neq B$, by using distinct markers to identity the various members of E a distinction between $A_{[\alpha,\beta]}$ and $B_{[\alpha,\beta]}$ to represent multiple edges in $H_{[\alpha,\beta]}$. However, we do not take this approach unless otherwise stated, we will always regard $H_{[\alpha,\beta]}$ as having no repeated edges.
The families of crisp sets (hypergraphs) produced by the $[\alpha, \beta]$-cuts of an interval-valued fuzzy hypergraph share an important relationship with each other, as expressed below: suppose $\mathbb{X}$ and $\mathbb{Y}$ are two families of sets such that for each set X belonging to $\mathbb{X}$ there is at least one set Y belonging to $\mathbb{Y}$ which contains X. In this case we say that $\mathbb{Y}$ *absorbs* $\mathbb{X}$ and

symbolically write $\mathbb{X} \sqsubseteq \mathbb{Y}$ to express this relationship between $\mathbb{X}$ and $\mathbb{Y}$. Since it is possible for $\mathbb{X} \sqsubseteq \mathbb{Y}$ while $\mathbb{X} \cap \mathbb{Y} = \emptyset$, we have that $\mathbb{X} \subseteq \mathbb{Y} \Rightarrow \mathbb{X} \sqsubseteq \mathbb{Y}$, whereas the converse is generally false. If $\mathbb{X} \sqsubseteq \mathbb{Y}$ and $\mathbb{X} \neq \mathbb{Y}$, then we write $\mathbb{X} \sqsubset \mathbb{Y}$.

Definition 3.5. *Let $H = (V, E)$ be an interval-valued fuzzy hypergraph, and for $[0, 0] < [s, t] \leq h(H)$. Let $H_{[s,t]}$ be the $[s, t]-$ level hypergraph of H. The sequence of real numbers*

$$\{[s_1, r_1], [s_2, r_2], \ldots, [s_n, r_n]\}, \quad [0, 0] < [s_1, r_1] < [s_2, r_2] < \cdots < [s_n, r_n] = h(H),$$

which satisfies the properties:

- *if $[s_{i+1}, r_{i+1}] < [u, v] \leq [s_i, r_i]$, then $E_{[u,v]} = E_{[s_i,r_i]}$, and*
- *$E_{[s_i,r_i]} \sqsubset E_{[s_{i+1},r_{i+1}]}$,*

is called the fundamental sequence of H, and is denoted by $F(H)$ and the set of $[s_i, r_i]$-level hypergraphs $\{H_{[s_1,r_1]}, H_{[s_2,r_2]}, \ldots, H_{[s_n,r_n]}\}$ is called the set of core hypergraphs of H or, simply, the core set of H, and is denoted by $C(H)$.

Definition 3.6. *Suppose $H = (V, E)$ is an interval-valued fuzzy hypergraph with*

$$F(H) = \{[s_1, r_1], [s_2, r_2], \ldots, [s_n, r_n]\},$$

and $s_{n+1} = 0$, $r_{n+1} = 0$, Then H is called sectionally elementary if for each edge $A = (\mu_A^-, \mu_A^+) \in E$, each $i = \{1, 2, \ldots, n\}$, and $[s_i, r_i] \in F(H)$, $A_{[s,t]} = A_{[s_i,r_i]}$ for all $[s, t] \in ([s_{i+1}, r_{i+1}], [s_i, r_i])$.

Clearly H is sectionally elementary if and only if $A(x) = (\mu_A^-(x), \mu_A^+(x)) \in F(H)$ for each $A \in E$ and each $x \in X$.

Definition 3.7. *A sequence of crisp hypergraphs $H_i = (V_i, E_i^*)$, $[1, 1] \leq i \leq [n, n]$, is said to be ordered if $H_1 \subset H_2 \subset \ldots \subset H_n$. The sequence $\{H_i \,|\, [1, 1] \leq i \leq [n, n]\}$ is simply ordered if it is ordered and if whenever $E^* \in E^*_{i+[1,1]} - E_i^*$, then $E^* \nsubseteq V_i$.*

Definition 3.8. *An interval-valued fuzzy hypergraph H is ordered if the H induced fundamental sequence of hypergraphs is ordered. The interval-valued fuzzy hypergraph H is simply ordered if the H induced fundamental sequence of hypergraphs is simply ordered.*

Example 3.2. *Consider the interval-valued fuzzy hypergraph $H = (V, E)$, where $V = \{a, b, c, d\}$ and $E = \{E_1, E_2, E_3, E_4, E_5\}$ which is represented by the following incidence matrix:*

Table 2. Incidence matrix of H

H	E_1	E_2	E_3	E_4	E_5
a	$[0.2, 0.7]$	$[0.0, 0.9]$	$[0, 0]$	$[0, 0]$	$[0.3, 0.4]$
b	$[0.2, 0.7]$	$[0.0, 0.9]$	$[0.0, 0.9]$	$[0.2, 0.7]$	$[0, 0]$
c	$[0, 0]$	$[0, 0]$	$[0.0, 0.9]$	$[0.2, 0.7]$	$[0.3, 0.4]$
d	$[0, 0]$	$[0.3, 0.4]$	$[0, 0]$	$[0.3, 0.4]$	$[0.3, 0.4]$

Clearly, $h(H) = [0.3, 0.9]$.
Now

$$E_{[0.1,0.9]} = \{\{a, b\}, \{b, c\}\}$$
$$E_{[0.2,0.7)} = \{\{a, b\}, \{b, c\}\}$$
$$E_{[0.3,0.4]} = \{\{a, b\}, \{a, b, d\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}\}.$$

Thus for $[0.3, 0.4] < [s, t] \leq [0.1, 0.9]$*,* $E_{[s,t]} = \{\{a, b\}, \{b, c\}\}$*, and for* $[0, 0] < [s, t] \leq [0.3, 0.4]$*,*

$$E_{[s,t]} = \{\{a, b\}, \{a, b, d\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}\}.$$

We note that $E_{[0.1,0.9]} \subseteq E_{[0.3,0.4]}$*. The fundamental sequence is* $F(H)$=$\{[s_1, r_1] = [0.1, 0.9], [s_2, r_2] = [0.3, 0.4]\}$ *and the set of core hypergraph is* $C(H) = \{H_1 = (V_1, E_1) = H_{[0.1,0.9]}, H_2 = (V_2, E_2) = H_{[0.3,0.4]}\}$*, where*

$$V_1 = \{a, b, c\},\ E_1 = \{\{a, b\}, \{b, c\}\}$$

$$V_2 = \{a, b, c, d\},\ E_2 = \{\{a, b\}, \{a, b, d\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}\}.$$

H *is support simple, but not simple.* H *is not sectionally elementary since* $E_{1[s,t]} \neq E_{1[0.1,0.9]}$ *for* $s = 0.2$*,* $t = 0.7$*. Clearly, interval-valued fuzzy hypergraph* H *is simply ordered.*

Proposition 3.2. *Let* $H = (V, E)$ *be an elementary interval-valued fuzzy hypergraph. Then* H *is support simple if and only if* H *is strongly support simple.*

Proof. Suppose that H is elementary, support simple and that $\text{supp}(A) = \text{supp}(B)$. We assume without loss of generality that $h(A) \leq h(B)$. Since H is elementary, it follows that $\mu^-_A \leq \mu^-_B$, $\mu^+_A \leq \mu^+_B$ and since H is support simple that $\mu^-_A = \mu^-_B$, $\mu^+_A = \mu^+_B$. Therefore H is strongly support simple. The proof of converse part is obvious. □

The complexity of an interval-valued fuzzy hypergraph depends in part on how many edges it has. The natural question arises: is there an upper bound on the number of edges of an interval-valued fuzzy hypergraph of order n?

Proposition 3.3. *Let* $H = (V, E)$ *be a simple interval-valued fuzzy hypergraph of order* n*. Then there is no upper bound on* $|E|$*.*

Proof. Let $V = \{x, y\}$, and define E_N= $\{A_i = [\mu^-_{A_i}, \mu^+_{A_i}] \mid i = 1, 2, \ldots, N\}$, where

$$\mu^-_{A_i}(x) = \frac{1}{i+1},\ \mu^+_{A_i}(x) = 1 - \frac{1}{i+1},$$

$$\mu^-_{A_i}(y) = \frac{1}{i+1},\ \mu^+_{A_i}(y) = \frac{i}{i+1}.$$

Then $H_N = (V, E_N)$ is a simple interval-valued fuzzy hypergraph with N edges. This ends the proof. □

Proposition 3.4. *Let* $H = (V, E)$ *be a support simple interval-valued fuzzy hypergraph of order* n*. Then there is no upper bound on* $|E|$*.*

Proposition 3.5. *Let* $H = (V, E)$ *be a strongly support simple interval-valued fuzzy hypergraph of order* n*. Then there is no upper bound on* $|E| \leq 2^n - 1$ *if and only if* $\{supp(A) \mid A \in E\} = P(V) - \emptyset$*.*

Proposition 3.6. *Let* $H = (V, E)$ *be an elementary simple interval-valued fuzzy hypergraph of order* n*. Then there is no upper bound on* $|E| \leq 2^n - 1$ *if and only if* $\{supp(A) \mid A \in E\} = P(V) - \emptyset$*.*

Proof. Since H is elementary and simple, each nontrivial $W \subseteq V$ can be the support of at most one $A = (\mu^-_A, \mu^+_A) \in E$. Therefore, $|E| \leq 2^n - 1$. To show there exists an elementary, simple H with $|E| = 2^n - 1$, let $E = \{A = (\mu^-_A, \mu^+_A) \mid W \subseteq V\}$ be the set of functions defined by

$$\mu^-_A(x) = \frac{1}{|W|},\ \text{if } x \in W,\quad \mu^-_A(x) = 0,\ \text{if } x \notin W,$$

$$\mu_A^+(x) = 1 - \frac{1}{|W|}, \text{ if } x \in W, \quad \mu_A^+(x) = 1, \text{if } x \notin W.$$

Then each one element has height $[0,1]$, each two elements has height $[0.5, 0.5]$ and so on. Hence H is an elementary and simple, and $|E| = 2^n - 1$. □

Proposition 3.7.
(*a*) *If $H = (V, E)$ is an elementary interval-valued fuzzy hypergraph, then H is ordered.*
(*b*) *If H is an ordered interval-valued fuzzy hypergraph with simple support hypergraph, then H is elementary.*

Consider the situation where the vertex of a crisp hypergraph is fuzzified. Suppose that each edge is given a uniform degree of membership consistent with the weakest vertex of the edge. Some constructions describe the following subclass of interval-valued fuzzy hypergraphs.

Definition 3.9. *An interval-valued fuzzy hypergraph $H = (V, E)$ is called a $A = [\mu_A^-, \mu_A^+]$-tempered interval-valued fuzzy hypergraph of $H = (V, E)$ if there is a crisp hypergraph $H^* = (V, E^*)$ and an interval-valued fuzzy set $A = [\mu_A^-, \mu_A^+] : V \to D(0,1]$ such that $E = \{B_F = [\mu_{B_F}^-, \mu_{B_F}^+] \mid F \in E^*\}$, where*

$$\mu_{B_F}^-(x) = \begin{cases} \min(\mu_A^-(y) \mid y \in F) & \text{if } x \in F, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_{B_F}^+(x) = \begin{cases} \min(\mu_A^+(y) \mid y \in F) & \text{if } x \in F, \\ 1 & \text{otherwise.} \end{cases}$$

Let $A \otimes H$ denote the A-tempered interval-valued fuzzy hypergraph of H determined by the crisp hypergraph $H = (V, E^*)$ and the interval-valued fuzzy set $A : V \to D(0,1]$.

Example 3.3. *Consider the interval-valued fuzzy hypergraph $H = (V, E)$, where $V = \{a, b, c, d\}$ and $E = \{E_1, E_2, E_3, E_4\}$ which is represented by the following incidence matrix:*

Table 3. Incidence matrix of H

H	E_1	E_2	E_3	E_4
a	$[0.2, 0.7]$	$[0, 0]$	$[0, 0]$	$[0.2, 0.7]$
b	$[0.2, 0.7]$	$[0.3, 0.4]$	$[0.0, 0.9]$	$[0, 0]$
c	$[0, 0]$	$[0, 0]$	$[0.0, 0.9]$	$[0.2, 0.7]$
d	$[0, 0]$	$[0.3, 0.4]$	$[0, 0]$	$[0, 0]$

Then

$$E_{[0.0,0.9]} = \{\{b, c\}\}$$
$$E_{[0.2,0.7)} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$$

and

$$E_{[0.3,0.4]} = \{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}\}.$$

Define $A = [\mu_A^-, \mu_A^+] : V \to D(0,1]$ by

$$\mu_A^-(a) = 0.2, \ \mu_A^-(b) = \mu_A^-(c) = 0.0, \ \mu_A^-(d) = 0.3,$$
$$\mu_A^+(a) = 0.7, \ \mu_A^+(b) = \mu_A^+(c) = 0.9, \ \mu_A^+(d) = 0.4.$$

Note that

$$\mu^-_{B_{\{a,b\}}}(a) = \min(\mu^-_A(a), \mu^-_A(b)) = 0.0,\ \mu^-_{B_{\{a,b\}}}(b) = \min(\mu^-_A(a), \mu^-_A(b)) = 0.0,$$
$$\mu^-_{B_{\{a,b\}}}(c) = 0.0,\ \mu^-_{B_{\{a,b\}}}(d) = 0.0,$$

$$\mu^+_{B_{\{a,b\}}}(a) = \min(\mu^+_A(a), \mu^+_A(b)) = 0.7,\ \mu^+_{B_{\{a,b\}}}(b) = \min(\mu^+_A(a), \mu^+_A(b)) = 0.7$$
$$\mu^+_{B_{\{a,b\}}}(c) = 1.0,\ \mu^+_{B_{\{a,b\}}}(d) = 1.0.$$

Thus

$$E_1 = [\mu^-_{B_{\{a,b\}}}, \mu^+_{B_{\{a,b\}}}],\ E_2 = [\mu^-_{B_{\{b,d\}}}, \mu^+_{B_{\{b,d\}}}],$$
$$E_3 = [\mu^-_{B_{\{b,c\}}}, \mu^+_{B_{\{b,c\}}}],\ E_4 = [\mu^-_{B_{\{a,c\}}}, \mu^+_{B_{\{a,c\}}}].$$

Hence H is A-tempered hypergraph.

Proposition 3.8. *An interval-valued fuzzy hypergraph H is a A-interval-valued fuzzy hypergraph of some crisp hypergraph H^* if and only if H is elementary, support simple and simply ordered.*

Proof. Suppose that $H = (V, E)$ is a A-tempered interval-valued fuzzy hypergraph of some crisp hypergraph H^*. Clearly, H is elementary and support simple. We show that H is simply ordered. Let

$$C(H) = \{(H^*_1)^{r_1} = (V_1, E^*_1),\ (H^*_2)^{r_2} = (V_2, E^*_2), \cdots,\ (H^*_n)^{r_n} = (V_n, E^*_n)\}.$$

Since H is elementary, it follows from Proposition 3.18 that H is ordered. To show that H is simply ordered, suppose that there exists $F \in E^*_{i+1} \setminus E^*_i$. Then there exists $x^* \in F$ such that $\mu^-_A(x^*) = r_{i+1}$, $\mu^+_A(x^*) = \acute{r}_{i+1}$. Since $\mu^-_A(x^*) = r_{i+1} < r_i$ and $\mu^+_A(x^*) = \acute{r}_{i+1} < \acute{r}_i$, it follows that $x^* \notin V_i$ and $F \nsubseteq V_i$, hence H is simply ordered.

Conversely, suppose $H = (V, E)$ is elementary, support simple and simply ordered. Let

$$C(H) = \{(H^*_1)^{r_1} = (V_1, E^*_1),\ (H^*_2)^{r_2} = (V_2, E^*_2), \cdots,\ (H^*_n)^{r_n} = (V_n, E^*_n)\}$$

where $D(H) = \{r_1, r_2, \cdots, r_n\}$ with $0 < r_n < \cdots < r_1$. Since $(H^*)^{r_n} = H^*_n = (V_n, E^*_n)$ and define $A = [\mu^-_A, \mu^+_A] : V_n \to D(0, 1]$ by

$$\mu^-_A(x) = \left\{ \begin{array}{ll} r_1 & if\ x \in V_1, \\ r_i & if\ x \in V_i \setminus V_{i-1}, i = 1, 2, \cdots, n \end{array} \right\}$$
$$\mu^+_A(x) = \begin{cases} s_1 & if\ x \in V_1, \\ s_i & if\ x \in V_i \setminus V_{i-1}, i = 1, 2, \cdots, n \end{cases}$$

We show that $E = \{B_F = [\mu^-_{B_F}, \mu^+_{B_F}] \mid F \in E^*\}$, where

$$\mu^-_{B_F}(x) = \begin{cases} \min(\mu^-_A(y) \mid y \in F) & \text{if } x \in F, \\ 0 & \text{otherwise,} \end{cases}$$
$$\mu^+_{B_F}(x) = \begin{cases} \min(\mu^+_A(y) \mid y \in F) & \text{if } x \in F, \\ 1 & \text{otherwise.} \end{cases}$$

Let $F \in E^*_n$. Since H is elementary and support simple, there is a unique interval-valued fuzzy edge $C_F = [\mu^-_{C_F}, \mu^+_{C_F}]$ in E having support E^*. Indeed, distinct edges in E must have distinct supports that lie in E^*_n. Thus, to show that $E = \{B_F = [\mu^-_{B_F}, \mu^+_{B_F}] \mid F \in E^*_n\}$, it suffices to show that for each $F \in E^*_n$, $\mu^-_{C_F} = \mu^-_{B_F}$ and $\mu^+_{C_F} = \mu^+_{B_F}$. As all edges are elementary and different edges have different supports, it follows from the definition of fundamental sequence that $h(C_F)$ is equal to some number r_i of $D(H)$. Consequently, $E^* \subseteq V_i$. Moreover, if $i > 1$, then $F \in E^* \setminus E^*_{i-1}$. Since $F \subseteq V_i$, it follows from the definition of $A = [\mu^-_A, \mu^+_A]$ that for each $x \in F$, $\mu^-_A(x) \geq r_i$ and $\mu^+_A(x) \geq s_i$. We claim that $\mu^-_A(x) = r_i$ and $\mu^+_A(x) = s_i$, for some $x \in F$. If not, then by definition of $A = [\mu^-_A, \mu^+_A]$, $\mu^-_A(x) \geq r_i$ and $\mu^+_A(x) \geq s_i$ for all $x \in F$ which implies that $F \subseteq V_{i-1}$ and so $F \in E^* \setminus E^*_{i-1}$ and since H is simply ordered $F \subsetneq V_{i-1}$, a contradiction. Thus it follows from the definition of B_F that $B_F = C_F$. This completes the proof. □

As a consequence of the above theorem we obtain.

Proposition 3.9. *Suppose that H is a simply ordered interval-valued fuzzy hypergraph and $F(H) = \{r_1, r_2, \cdots, r_n\}$. If H^{r_n} is a simple hypergraph, then there is a partial interval-valued fuzzy hypergraph $\acute{H}$ of H such that the following assertions hold:*

(1) *$\acute{H}$ is a A-interval-valued fuzzy hypergraph of H_n.*
(2) *$E \sqsubseteq \acute{E}$.*
(3) *$F(\acute{H}) = F(H)$ and $C(\acute{H}) = C(H)$.*

4. **Conclusions**

Hypergraphs are the generalization of graphs in case of set of multiarity relations. The hypergraph has been considered a useful tool for modeling system architectures or data structures and to represent a partition, covering and clustering in the area of circuit design. The concept of ordinary hypergraphs was extended to fuzzy theory. The interval-valued fuzzy sets constitute a generalization of Zadeh's fuzzy set theory. In this paper, we have discussed the concept of interval-valued fuzzy hypergraphs. The interval-valued fuzzy hypergraphs are more flexible and compatible than the previous works due to the fact that they allowed the degree of membership of a vertex to a hyperedge to be represented by subinterval values in [0, 1] rather than the crisp real values between 0 and 1.

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