POSITIVE SOLUTIONS FOR SINGULAR FDES

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In this article, we establish the existence of at least three positive solutions to a boundary-value problem of the nonlinear singular fractional differential equation. Our analysis rely on the well known fixed point theorem in the cone.

Keywords: Unbounded positive solution; fractional differential equation; fixed-point theorem.

1. Introduction

Fractional differential equations have many applications in modeling of physical and chemical processes and in engineering. In its turn, mathematical aspects of studies on fractional differential equations were discussed by many authors, see the text books [4,7,9], the survey paper [2], the papers [1,5,8,10,13] and the references therein.

The use of cone theoretic techniques in the study of the existence of solutions to boundary value problems has a rich and diverse history. Recently, E. R. Kaufmann and E. Mboumi in [11] studied the following boundary value problem for the fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha} u(t) + a(t) f(u(t)) = 0, \quad 0 < t < 1, 1 < \alpha < 2, \\ u(0) = 0, u'(1) = 0\,, \end{cases} \tag{1}$$

by using the Leggett-Williams fixed point theorem and the Krasnoselskii fixed point theorem under the assumptions:

(A1) $f : [0, +\infty) \to [0, \infty)$ is continuous;

(A2) $a \in L^{\infty}[0, 1]$;

(A3) there exists a constant $m > 0$ such that $a(t) \geq m$ a.e. $t \in [0, 1]$.

The authors in [11] proved that BVP(1) has at least one or three positive solutions. We note that the Green's function (or the Kernel see [11]) of the following homogeneous BVP

$$\begin{cases} D_{0^+}^{\alpha} u(t) = 0, \quad 0 < t < 1, 1 < \alpha < 2, \\ u(0) = 0, u'(1) = 0\,, \end{cases} \tag{2}$$

is as follows:

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, t \geq s, \\ t^{\alpha-1}(1-s)^{\alpha-2}, t \leq s. \end{cases}$$

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Let $\beta \in (0,1)$. G satisfies the conditions that

$$\beta s G(s,s) \le G(t,s) \text{ for all } t \in [\beta,1], s \in [0,1], \quad G(t,s) \le G(s,s) \text{ for all } t,s \in [0,1]. \tag{3}$$

One sees that (3) plays an important role in the proof of the theorems in references [10,11].

In this paper, we discuss the existence of three unbounded positive solutions to the boundary value problem of the nonlinear fractional differential equation of the form

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t,u(t)) = 0, \quad t \in (0,\infty), 1 < \alpha < 2, \\ \lim_{t\to 0} t^{2-\alpha} u(t) = 0, \\ D_0^{\alpha-1} u(1) = 0\,, \end{cases} \tag{4}$$

where D_{0+}^{α} (D^{α} for short) is the Riemann-Liouville fractional derivative of order α, and $f : (0,1] \times [0,\infty) \to [0,\infty)$ is a Caratheodory function, i.e., $f(\cdot,x) \in L^1(0,1]$ for each $x \in [0,\infty)$ and $f(t,\cdot)$ is continuous for almost all $t \in (0,1]$, furthermore, f satisfies that for each $r > 0$ there exists $\phi_r \in L^1[0,1]$ such that $|f(t,t^{\alpha-2}x)| \le \phi_r(t)$ holds for all $t \in (0,1]$ and $x \in [-r,r]$. We obtain the existence results for two and three unbounded positive solutions of BVP(4) by using the fixed point theorems in a cones. An example is presented to illustrate the main result. This example can not be covered by the theorems in references [10,11].

A difference to [11] is that f may be singular at zero and the positive solutions of BVP(4) may be unbounded ones since $\lim_{t\to 0} t^{2-\alpha} x(t) = 0$ for solution x of BVP(4).

2. Preliminary results

For the convenience of the reader, we present here the necessary definitions from fixed point theory and fractional calculus theory. These definitions and properties can be found in the literatures [3,4,6,7,9].

Definition 2.1. Let X be a real Banach space. The nonempty convex closed subset P of X is called a cone in X if $ax \in P$ for all $x \in P$ and $a \ge 0$, $x \in X$ and $-x \in X$ imply $x = 0$.

Definition 2.2. A map $\psi : P \to [0,+\infty)$ is a nonnegative continuous concave or convex functional map provided ψ is nonnegative, continuous and satisfies

$$\psi(tx + (1-t)y) \ge t\psi(x) + (1-t)\psi(y),$$

or

$$\psi(tx + (1-t)y) \le t\psi(x) + (1-t)\psi(y),$$

for all $x, y \in P$ and $t \in [0,1]$.

Definition 2.3. An operator $T : X \to X$ is completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Suppose that ψ is a nonnegative functional on a cone P of the real Banach space X. We define the following sets by

$$\begin{aligned} &P_r = \{y \in P : \ \ ||y|| < r\}, \\ &P(\psi; a, b) = \{y \in P : \ \ a \le \psi(y),\ ||y|| < b\}, \\ &P(\psi, d) := \{x \in P : \psi(x) < d\}. \end{aligned}$$

Lemma 2.1[6]. Suppose that $T: \overline{P}_c \to \overline{P}_c$ is a completely continuous operator and ψ a nonnegative continuous concave functional on P such that $\psi(y) \leq ||y||$ for all $y \in \overline{P}_c$. Furthermore, suppose that there exist $0 < a < b < d \leq c$ such that

(E1) $\{y \in P(\psi; b, d) | \psi(y) > b\} \neq \emptyset$ and $\psi(Ty) > b$ for $y \in P(\psi; b, d)$;

(E2) $||Ty|| < a$ for $||y|| \leq a$;

(E3) $\psi(Ty) > b$ for $y \in P(\psi; b, c)$ with $||Ty|| > d$.

Then T has at least three fixed points y_1, y_2 and y_3 such that $||y_1|| < a$, $b < \psi(y_2)$ and $||y_3|| > a$ with $\psi(y_3) < b$.

Lemma 2.2[3]. Suppose that P is a cone in the real Banach space X, $\phi, \gamma : P \to [0, \infty)$ are two continuous increasing functionals, $\theta : P \to [0, \infty)$ is a continuous functional and there exist positive numbers M and c such that

(i) $T : \overline{P}(\gamma, c) \to P$ is completely continuous;

(ii) $\theta(0) = 0$ and $\gamma(x) \leq \theta(x) \leq \phi(x)$, $||x|| \leq M\gamma(x)$ for all $x \in \overline{P}(\gamma, c)$;

(iii) there exist constants $0 < a < b < c$ such that $\theta(\lambda x) \leq \lambda\theta(x)$ for all $\lambda \in [0, 1]$ and $x \in \partial P(\theta, b)$;

(iv) $\gamma(Tx) > c$ for all $x \in \partial P(\gamma, c)$; $\theta(Tx) < b$ for all $x \in \partial P(\theta, b)$; $P(\phi, a) \neq \emptyset$ and $\phi(Tx) > a$ for all $x \in \partial P(\phi, a)$;

then T has two fixed points x_1, x_2 in $P(\gamma, c)$ such that

$$\phi(x_1) > a,\ \theta(x_1) < b < \theta(x_2),\ \ \gamma(x_2 < c.$$

Lemma 2.3[3]. Suppose that P is a cone in a real Banach space X, $\phi, \gamma : P \to [0, \infty)$ are two continuous increasing functionals, $\theta : P \to [0, \infty)$ is a continuous functional and there exist positive numbers $M, c > 0$ such that (i), (ii) and (iii) in Lemma 2.4 hold and

(iv) $\gamma(Tx) < c$ for all $x \in \partial P(\gamma, c)$; $\theta(Tx) > b$ for all $x \in \partial P(\theta, b)$; $P(\phi, a) \neq \emptyset$ and $\phi(Tx) < a$ for all $x \in \partial P(\phi, a)$;

then T has two fixed points x_1, x_2 in $P(\gamma, c)$ such that

$$\phi(x_1) > a,\ \theta(x_1) < b < \theta(x_2),\ \ \gamma(x_2 < c.$$

Definition 2.4[1]. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to R$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right-hand side exists.

Definition 2.5[1]. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to R$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^{n+1}}{dt^{n+1}} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n - 1 < \alpha \leq n$, provided that the right-hand side is point-wise defined on $(0, \infty)$.

Lemma 2.4[1]. Let $n - 1 < \alpha \leq n$, $u \in C^0(0, 1) \bigcap L^1(0, 1)$. Then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n},$$

where $C_i \in R$, $i = 1, 2, \ldots n$.

Lemma 2.5[1]. The relations $I_{0+}^{\alpha}I_{0+}^{\beta}\varphi = I_{0+}^{\alpha+\beta}\varphi,\ D_{0+}^{\alpha}I_{0+}^{\alpha} = \varphi$ are valid in following case: $Re\beta > 0,\ Re(\alpha+\beta) > 0,\ \varphi \in L_1(0,1)$.

Lemma 2.6. Suppose that $1 < \alpha < 2$. Given $h \in C(0,1]$. Then the unique solution of

$$\begin{cases} D^{\alpha}u(t) + h(t) = 0, 0 < t < 1, \\ \lim_{t\to 0} t^{2-\alpha}u(t) = 0, \\ D^{\alpha-1}u(1) = 0\,, \end{cases} \tag{5}$$

is

$$u(t) = \int_0^1 G(t,s)h(s)ds, \tag{6}$$

where

$$G(t,s) = \begin{cases} -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & s \le t, \\ \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t \le s. \end{cases} \tag{7}$$

Proof. We may apply Lemma 2.4 to reduce BVP(5) to an equivalent integral equation

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + c_1t^{\alpha-1} + c_2t^{\alpha-2}$$

for some $c_i \in R, i = 1,2$. We get

$$t^{2-\alpha}u(t) = -t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + c_1t + c_2$$

and

$$D^{\alpha-1}u(t) = -\int_0^t h(s)ds + c_1\Gamma(\alpha).$$

From the boundary conditions in (5), since $\lim_{s\to 0}\Gamma(s) = \infty$, we get

$$c_2 = 0,$$
$$-\int_0^1 h(s)ds + c_1\Gamma(\alpha) = 0.$$

It follows that

$$c_1 = \frac{1}{\Gamma(\alpha)}\int_0^1 h(s)ds,$$

and

$$c_2 = 0.$$

Therefore, the unique solution of BVP(5) is

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 h(s)ds = \int_0^1 G(t,s)h(s)ds.$$

Here G is defined by (7). Reciprocally, let u satisfy (6). Then

$$\lim_{t\to 0} t^{2-\alpha}u(t) = 0, D^{\alpha-1}u(1) = 0\,.$$

Furthermore, we have $D^{\alpha}u(t) = -h(t)$. The proof is complete.

Lemma 2.7. Suppose that $1 < \alpha < 2$ and $\beta \in (0,1)$. Then $G(t,s)$ satisfies the following properties:

(i) $G(t,s) \ge 0$ for all $t, s \in [0,1]$;

(ii) $G(t,s) \le G(s,s)$ for all $t,s \in [0,1]$;
(iii) $\min_{t\in[\beta,1]} G(t,s) \ge \beta G(s,s)$ for all $s \in [0,1]$.

Proof. One sees from (7) that $G(t,s) \ge 0$ for all $t,s \in [0,1]$.

It is easy to see that $G(t,s) \le G(s,s)$ for $t \le s$. When $t \ge s$, since

$$[t^{\alpha-1} - (t-s)^{\alpha-1}]' = (\alpha-1)t^{\alpha-2}\left[1 - \left(1 - \frac{s}{t}\right)^{\alpha-2}\right] \le 0$$

Then $G(t,s) \le G(s,s)$ for $t \ge s$. Hence $G(t,s) \le G(s,s)$ for all $t,s \in [0,1]$.

Let $F(s) = 1 - (1-s)^{\alpha-1} - \beta s^{\alpha-1}$. It is easy to see that $F(0) = 0$ and $F(1) = 1 - \beta > 0$. Since

$$F'(s) = (\alpha-1)s^{\alpha-2}\left[\left(\frac{1}{s} - 1\right)^{\alpha-2} - \beta\right]\begin{cases} \ge 0, s \in \left(0, \frac{1}{\beta^{\frac{1}{\alpha-2}}+1}\right], \\ \le 0, s \in \left[\frac{1}{\beta^{\frac{1}{\alpha-2}}+1}, 1\right], \end{cases}$$

we get that $1 - (1-s)^{\alpha-1} \ge \beta s^{\alpha-1}$.

For $1 \ge t \ge s$, we have

$$G(t,s) \ge G(1,s) = -\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \ge \beta\frac{s^{\alpha-1}}{\Gamma(\alpha)}$$

For $\beta \le t \le s$, we have

$$G(t,s) \ge G(\beta,s) = \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} \ge \beta\frac{s^{\alpha-1}}{\Gamma(\alpha)}$$

$\min_{t\in[\beta,1]} G(t,s) \ge \beta G(s,s)$ for all $s \in [0,1]$. The proof is completed.

For our construction, we let $X = C(0,1]$ and $\|u\| = \sup_{t\in(0,1]} t^{2-\alpha}|u(t)|$ which is a Banach space. We seek solutions of (4) that lie in the cone

$$P = \left\{u \in X : u(t) \ge 0,\ 0 < t \le 1,\ \min_{t\in[\eta,1]} u(t) \ge \beta^{\alpha}||u||\right\}.$$

Define the operator $T : P \to X$, by

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds\,.$$

One sees from Lemma 2.7 that

$$||Tu|| = \max_{t\in(0,1]} t^{2-\alpha}(Tu)(t) \le \int_0^1 G(s,s)f(s,u(s))ds$$

and

$$\min_{t\in[\eta,1]} t^{2-\alpha}(Tu)(t) = \min_{t\in[\eta,1]} \beta^{2-\beta}\int_0^1 G(t,s)f(s,u(s))ds \ge \beta^{\alpha}\int_0^1 G(s,s)f(s,u(s))ds.$$

Hence

$$\min_{t\in[\eta,1]} t^{2-\alpha}(Tu)(t) \ge \beta^{\alpha}||u||.$$

It follows that $Tu \in P$. Then $T : P \to P$ is well defined.

Lemma 2.8. Suppose that $f(t,x)$ is continuous on $(0,1]\times R$ and satisfies that for each $r>0$ there exists $\phi_r\in L^1(0,1]$ such that $|f(t,t^{\alpha-2}x)|\le\phi_r(t)$ for all $t\in(0,1]$ and $|x|\le r$. Then T is completely continuous.

Proof. We divide the proof into three steps.

Step 1. T is continuous.

Let $\{y_n\}$ be a sequence such that $y_n\to y$ in X. Let

$$r=\max\left\{\sup_{t\in(0,1]}t^{2-\alpha}y_n(t),\ \sup_{t\in(0,1]}t^{2-\alpha}y(t)\right\}.$$

Then for $t\in(0,1]$, we have

$$\begin{aligned}&t^{2-\alpha}|(Ty_n)(t)-(Ty)(t)|\\ =\ &\left|\int_0^1 t^{2-\alpha}G(t,s)f(s,y_n(s))ds-\int_0^1 t^{2-\alpha}G(t,s)f(s,y(s))ds\right|\\ \le\ &\int_0^1 t^{2-\alpha}G(t,s)|f(s,y_n(s))-f(s,y(s))|ds\\ \le\ &\frac{1}{\Gamma(\alpha)}\int_0^1|f(s,s^{\alpha-2}2^{2-\alpha}y_n(s))-f(s,s^{\alpha-2}2^{2-\alpha}y(s))|ds\\ \le\ &2\frac{1}{\Gamma(\alpha)}\int_0^1\phi_r(s)ds.\end{aligned}$$

Since $f(t,s^{\alpha-2}x)$ is continuous in x, we have $||Ty_n-Ty||\to0$ as $n\to\infty$.

Step 2. T maps bounded sets into bounded sets in X.

It suffices to show that for each $l>0$, there exists a positive number $L>0$ such that for each $x\in M=\{y\in X:||y||\le l\}$, we have $||Ty||\le L$. By the definition of T, we get

$$\begin{aligned}t^{2-\alpha}|(Ty)(t)|&=\int_0^1 t^{2-\alpha}G(t,s)f(s,y(s))ds\\ &\le\frac{1}{\Gamma(\alpha)}\int_0^1 f(s,s^{\alpha-2}2^{2-\alpha}y(s))ds\\ &\le\frac{1}{\Gamma(\alpha)}\int_0^1\phi_l(s)ds.\end{aligned}$$

It follows that $||Ty||\le\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^1\phi_l(s)ds$ for each $y\in\{y\in X:||y||\le l\}$. So T maps bounded sets into bounded sets in X.

Step 3. T maps bounded sets into equicontinuous sets in X.

Let $y\in M=\{y\in X:||y||\le l\}$ be defined in Step 2.

Firstly, we prove that $\{Ty:y\in M\}$ is equicontinuous on each compact subinterval $[t_1,t_2]$ of $(0,1]$, where $t_1,t_2\in(0,1]$ with $t_1<t_2$. We have

$$\begin{aligned}&|t_1^{2-\alpha}(Ty)(t_1)-t_2^{2-\alpha}(Ty)(t_2)|\\ =\ &\left|\int_0^1 t_1^{2-\alpha}G(t_1,s)f(s,y(s))ds-\int_0^1 t_2^{2-\alpha}G(t_2,s)f(s,y(s))ds\right|\end{aligned}$$

$$\begin{aligned}
&\le \int_0^1 |t_1^{2-\alpha}G(t_1,s)-t_2^{2-\alpha}G(t_2,s)|f(s,s^{\alpha-2}s^{2-\alpha}y(s))ds\\
&\le \int_0^{t_1}\left[\left|\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}-t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\right|+\frac{|t_1-t_2|}{\Gamma(\alpha)}\right]f(s,s^{\alpha-2}s^{2-\alpha}y(s))ds\\
&\quad+\int_{t_1}^{t_2}|t_1^{2-\alpha}G(t_1,s)-t_2^{2-\alpha}G(t_2,s)|f(s,s^{\alpha-2}s^{2-\alpha}y(s))ds\\
&\quad+\int_{t_2}^{1}\left|t_1^{2-\alpha}\frac{t_1^{\alpha-1}}{\Gamma(\alpha)}-t_2^{2-\alpha}\frac{t_2^{\alpha-1}}{\Gamma(\alpha)}\right|f(s,s^{\alpha-2}s^{2-\alpha}y(s))ds\\
&\le \int_0^{1}\left[\left|\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}-t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\right|+\frac{|t_1-t_2|}{\Gamma(\alpha)}\right]\phi_l(s)ds\\
&\quad+\frac{2}{\Gamma(\alpha)}\int_{t_1}^{t_2}\phi_l(s)ds+\frac{|t_1-t_2|}{\Gamma(\alpha)}\int_0^1\phi_l(s)ds.
\end{aligned}$$

$$G(t,s)=\begin{cases}-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-1}}{\Gamma(\alpha)}, & s\le t,\\ \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t\le s.\end{cases}$$

As $t_1\to t_2$, the right-hand side of the above inequality tends to zero. Therefore, $\{Ty: y\in M\}$ is equicontinuous on each compact sub-interval of $(0,1]$.

Secondly, we prove that $\{Ty: y\in M\}$ is equicontinuous at zero point. Since

$$\int_0^1 t^{2-\alpha}G(t,s)f(s,y(s))ds\le\frac{1}{\Gamma(\alpha)}\int_0^1\phi_l(s)ds,$$

we get

$$\lim_{t\to 0}t^{2-\alpha}(Ty)(t)=\int_0^1 t^{2-\alpha}G(t,s)f(s,y(s))ds=0$$

uniformly. Then for each $\epsilon>0$, there exists $\delta>0$ such that

$$|t_1^{2-\alpha}(Ty)(t_1)-t_2^{2-\alpha}(Ty)(t_2)|<\epsilon.$$

holds for each $0<t_1,t_2<\delta$. Hence $\{Ty: y\in M\}$ is equicontinuous at zero point.

From above discussion, T is completely continuous. The proof is complete.

3. **Main Results**

In this section, we prove the main results. Let

$$M=\frac{1}{\Gamma(\alpha)},$$

and

$$W=\frac{\beta^3(1-\beta^\alpha)}{\alpha W\Gamma(\alpha)}.$$

Theorem 3.1. Suppose that $f:(0,1]\times[0,\infty)\to[0,\infty)$ is continuous and satisfies that for each $r>0$ there exists $\phi_r\in L^1(0,1]$ such that $|f(t,t^{\alpha-2}x)|\le\phi_r(t)$ for all $t\in(0,1]$ and $|x|\le r$ and there exist constants e_1,e_2 and c such that

$$0<e_1<e_2<\frac{e_2}{\beta^\alpha}<c,\ \ Wc>Me_2,$$

that satisfy

(D1) $f(t,t^{\alpha-2}u)\le\frac{c}{M}$ for $t\in(0,1]$, $u\in[0,c]$;

(D2) $f(t, t^{\alpha-2}u) \le \frac{e_1}{M}$ for $t \in (0,1]$ and $u \in [0, e_1]$;

(D3) $f(t, t^{\alpha-2}u) \ge \frac{e_2}{W}$ for $t \in [\eta, 1]$ and $u \in \left[e_2, \frac{e_2}{\beta^\alpha}\right]$;

then BVP(4) has at least three positive solutions x_1, x_2 and x_3 satisfying

$$\sup_{t\in(0,1]} t^{2-\alpha}x_1(t) < e_1, \quad \min_{t\in[\eta,1]} t^{2-\alpha}x_2(t) > e_2 \tag{8}$$

and

$$\sup_{t\in(0,1]} t^{2-\alpha}x_3(t) > e_1, \quad \min_{t\in[\eta,1]} t^{2-\alpha}x_3(t) < e_2. \tag{9}$$

Proof. Define the functional ψ by

$$\psi(x) = \min_{t\in[\eta,1]} t^{2-\alpha}x(t) \text{ for } x \in P.$$

It is easy to see that ψ is a nonnegative convex continuous functional on the cone P and $\psi(y) \le ||y||$ for all $y \in P$. It follows from Lemma 2.8 that $TP \subseteq P$ and $T : P \to P$ is completely continuous.

To Lemma 2.1, choose

$$d = \frac{e_2}{\beta^\alpha}, \quad b = e_2, \quad a = e_1.$$

Then $0 < a < b < d < c$. We divide the remainder of the proof into four steps.

Step 1. Prove that $T(\overline{P_c}) \subset \overline{P_c}$.

For $x \in \overline{P_c}$, one has $||x|| \le c$. Then

$$0 \le t^{2-\alpha}x(t) \le c, \; t \in (0,1].$$

It follows from (D1) that

$$f(t, x(t)) = f(t, t^{\alpha-2}t^{2-\alpha}x(t)) \le \frac{c}{M}, t \in (0,1].$$

Then $Tx \in P$ implies that

$$\begin{aligned} ||Tx|| &= \sup_{t\in(0,1]} t^{2-\alpha}(Tx)(t) \\ &= \sup_{t\in(0,1]} \int_0^1 t^{2-\alpha}G(t,s)f(s,x(s))ds \\ &\le \sup_{t\in(0,1]} \int_0^1 t^{2-\alpha}G(t,s)\frac{c}{M}ds \\ &\le \frac{1}{\Gamma(\alpha)}\frac{c}{M} \\ &= c. \end{aligned}$$

Then $Tx \in \overline{P_c}$, Hence $T(\overline{P_c}) \subseteq \overline{P_c}$. This completes the proof of Step 1.

Step 2. Prove that

$$\{y \in P(\psi; b, d) | \psi(y) > b\} = \{y \in P(\psi; e_2, e_2/\beta^\alpha) | \psi(y) > e_2\} \ne \emptyset$$

and $\psi(Ty) > b = e_2$ for $y \in P(\psi; e_2, e_2/\beta^\alpha)$.

It is easy to see that $\{x \in P(\psi, e_2, e_2/\beta^\alpha), \psi(x) > e_2\} \ne \emptyset$. For $x \in P(\psi, e_2, e_2/\beta^\alpha)$, then $\psi(x) \ge e_2$ and $||x|| \le e_2/\beta^\alpha$. Then

$$\min_{t\in[\eta,1]} t^{2-\alpha}x(t) \ge e_2, \quad \sup_{t\in(0,1]} x(t) \le e_2/\beta^\alpha.$$

Hence

$$e_2 \le t^{2-\alpha}x(t) \le \frac{e_2}{\beta^\alpha},\ t \in [\eta, 1].$$

Hence (D3) implies that

$$f(t, x(t)) = f(t, t^{\alpha-2}t^{2-\alpha}x(t) \ge \frac{e_2}{W}, t \in [\eta, 1].$$

Since $Ty \in P$, we get $\psi(Ty) = \min_{t\in[\eta,1]} t^{2-\alpha}(Ty)(t) \ge \beta^\alpha \sup_{t\to(0,1]} t^{2-\alpha}(Tx)(t)$. We get

$$\begin{aligned}
\psi(Tx) &\ge \beta^\alpha \sup_{t\to(0,1]} \int_0^1 t^{2-\alpha}G(t,s)f(s,x(s))ds \\
&> \beta^\alpha \sup_{t\to(0,1]} \int_\beta^1 t^{2-\alpha}G(t,s)f(s,x(s))ds \\
&\ge \beta^3 \sup_{t\to(0,1]} \int_\beta^1 G(s,s)f(s,x(s))ds \\
&\ge \beta^3 \int_\beta^1 \frac{s^{\alpha-1}}{\Gamma(\alpha)}\frac{e_2}{W}ds \\
&\ge e_2.
\end{aligned}$$

Thus $\psi(Tx) > e_2$ for all $x \in P(\psi; e_2, e_2/\beta^\alpha)$. This completes the proof of Step 2.

Step 3. Prove that $||Ty|| < a = e_1$ for $y \in P$ with $||y|| \le a$.

For $x \in \overline{P_{e_1}}$, we have

$$\sup_{t\in(0,1]} t^{2-\alpha}x(t) \le e_1 = a.$$

It follows from (D2) and $Tx \in P$ that

$$f(t, x(t)) = f(t, t^{\alpha-2}t^{2-\alpha}x(t)) \le \frac{e_1}{M},\ t \in (0, 1].$$

The proof is similar to that of Step 1. Then $||Ty|| < e_1$ for $||y|| \le e_1$. This completes that proof of Step 3.

Step 4. Prove that $\psi(Ty) > b$ for $y \in P(\psi; b, c)$ with $||Ty|| > d$.

For $x \in P(\psi; b, c) = P(\psi, e_2, c)$ and $||Tx|| > d = \frac{e_2}{\beta^\alpha}$, then

$$\min_{t\in[\eta,1]} t^{2-\alpha}x(t) \ge e_2,\quad \sup_{t\in(0,1]} t^{2-\alpha}(Tx)(t) \ge \frac{e_2}{\beta^\alpha} \text{ and } ||x|| = \sup_{t\in(0,1]} t^{2-\alpha}x(t) \le c.$$

Hence we have from $Tx \in P$ that

$$\begin{aligned}
\psi(Tx) &= \min_{t\in[\eta,1]} t^{2-\alpha}(Tx)(t) \\
&= \beta^\alpha \sup_{t\in(0,1]} t^{2-\alpha}(Tx)(t) \\
&> \beta^\alpha \frac{e_2}{\beta^\alpha} \\
&= b.
\end{aligned}$$

Thus $\psi(Tx) > b$ for all $x \in P(\psi; b, c)$ with $||Tx|| > d$. This completes the proof of Step 4.

From above steps, (E1), (E2) and (E3) of Lemma 2.1 are satisfied. Then, by Lemma 2.1, T has three fixed points x_1, x_2 and $x_3 \in \overline{P_c}$ such that

$$||x_1|| < a,\ \psi(x_2) > b,\ ||x_3|| \geq a,\ \psi(x_3) \leq b,$$

i.e., x_1, x_2 and x_3 satisfy (8) and (9). Hence BVP(4) has at least three positive solutions that may be unbounded positive solutions since $\lim_{t\to 0} t^{2-\alpha}x(t) = 0$. The proof is complete.

Theorem 3.2. Let $W = \frac{\beta^3(1-\beta^\alpha)}{\alpha W\Gamma(\alpha)}$ and $M = \frac{1}{\Gamma(\alpha)}$. Suppose that $f : (0,1] \times [0,\infty) \to [0,\infty)$ is continuous and satisfies that for each $r > 0$ there exists $\phi_r \in L^1(0,1]$ such that $|f(t,t^{\alpha-2}x)| \leq \phi_r(t)$ for all $t \in (0,1]$ and $|x| \leq r$, and there exist positive numbers $a < b < c$ such that $Wb > Ma$, and

(E1) $f(t,t^{\alpha-2}u) \geq \frac{c}{W}$ for $t \in [\eta,1]$, $u \in [c, c/\beta^\alpha]$;

(E2) $f(t,t^{\alpha-2}u) \leq \frac{b}{M}$ for $t \in (0,1]$ and $u \in [0,b]$;

(E3) $f(t,t^{\alpha-2}u) \geq \frac{a}{W}$ for $t \in [\eta,1]$ and $u \in [\beta^\alpha a, a]$.

Then BVP(4) has at least two positive solutions x_1 and x_2 satisfying

$$\sup_{t\in(0,1]} t^{\alpha-2}x_1(t) > a,\quad \sup_{t\in(0,1]} t^{\alpha-2}x_1(t) < b,\quad \sup_{t\in(0,1]} t^{\alpha-2}x_2(t) > b,\quad \min_{t\in[\eta,1]} t^{\alpha-2}x_2(t) < c. \tag{10}$$

Proof. Define the nonnegative, increasing and continuous functionals $\gamma, \theta, \phi : P \to [0\infty)$ by

$$\begin{aligned}
\gamma(x) &= \min_{t\in[\eta,1]} t^{\alpha-2}x(t),\ \ x \in P,\\
\theta(x) &= \sup_{t\in(0,1]} t^{\alpha-2}x(t),\ \ x \in P,\\
\phi(x) &= \sup_{t\in(0,1]} t^{\alpha-2}x(t),\ \ x \in P.
\end{aligned}$$

It is easy to see that $\theta(0) = 0$ and

$$\gamma(x) \leq \theta(x) \leq \phi(x),\ \ x \in P$$

and for $x \in P$ we have $\gamma(x) \geq \beta^\alpha ||x||$, $\theta(\nu x) \leq \nu\theta(x)$ for all $\nu \in [0,1]$ and $x \in P$. From Lemma 2.8, we have $TP \subset P$ and T is completely continuous. Hence (i)-(iii) in Lemma 2.2 hold. To obtain two positive solutions of BVP(4), it suffices to show that the condition (iv) in Lemma 2.2 holds.

First, we verify that

$$\gamma(Tx) > c \text{ for alll } x \in \partial P(\gamma, c). \tag{11}$$

Since $x \in \partial P(\gamma, c)$, we get $\min_{t\in[\eta,1]} t^{2-\alpha}x(t) = c$. Then $||x|| \leq \frac{1}{\beta^\alpha}\gamma(x) \leq \frac{c}{\beta^\alpha}$. Then $c \leq t^{2-\alpha}x(t) \leq \frac{c}{\beta^\alpha}$ for all $t \in [\eta,1]$. Hence (E1) implies

$$f(t,x(t)) = f(t,t^\alpha t^{2-\alpha}x(t)) \geq \frac{c}{W},\ \ t \in [\eta,1]\,.$$

So we get from $Tx \in P$ that

$$\gamma(Tx) = \min_{t\in[\eta,1]} t^{2-\alpha}(Tx)(t) \geq \beta^\alpha \sup_{t\in(0,1]} t^{2-\alpha}(Tx)(t).$$

We find

$$\begin{aligned}\gamma(Tx)(t) &\geq \beta^\alpha \int_0^1 t^{2-\alpha}G(t,s)f(s,x(s))ds \\ &> \beta^2 \int_\beta^1 \beta\frac{s^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds \\ &\geq \beta^3 \int_\beta^1 \frac{s^{\alpha-1}}{\Gamma(\alpha)}\frac{c}{W}ds \\ &\geq c.\end{aligned}$$

Secondly, we prove that

$$\theta(Tx) < b \text{ for all } x \in \partial P(\theta, b). \tag{12}$$

Since $\theta(x) = b$, we get $\sup_{t\in(0,1]} t^{2-\alpha}x(t) = b$. Then

$$t^{2-\alpha}x(t) \leq b \text{ for all } t \in (0,1].$$

Hence (E2) implies

$$f(t,x(t)) = f(t, t^{\alpha-2}t^{2-\alpha}x(t)) \leq \frac{b}{M}, t \in (0,1].$$

So the definition of T imply

$$\begin{aligned}\theta(Tx) &= \sup_{t\in(0,1]} t^{2-\alpha}(Tx)(t) \\ &< \sup_{t\in(0,1]} \int_0^1 t^{2-\alpha}G(t,s)f(s,x(s))ds \\ &\leq \frac{1}{\Gamma(\alpha)}\frac{b}{M} \\ &= b.\end{aligned}$$

Finally, we prove that

$$P(\phi, a) \neq \emptyset, \ \ \phi(Tx) > a \text{ for all } x \in \partial P(\phi, a). \tag{13}$$

It is easy to see that $P(\phi,a) \neq \emptyset$. For $x \in \partial P(\phi, a)$, we have $\sup_{t\in(0,1]} t^{2-\alpha}x(t) = a$. Then

$$\beta^\alpha a \leq t^{2-\alpha}x(t) \leq a \text{ for all } t \in [\eta, 1].$$

Then (E3) implies

$$f(t,x(t)) = f(t, 2^{\alpha-2}t^{2-\alpha}x(t)) \geq \frac{a}{W}, t \in [\eta, 1].$$

Similarly to the first step, we can prove that $\alpha(Tx) > a$. It follows from above discussion that all conditions in Lemma 2.2 are satisfied. Then T has two fixed points x_1, x_2 in P. So BVP(4) has two positive solutions x_1 and x_2 satisfying (10). The proof is complete.

Theorem 3.3. Let W, M be defined in Theorem 3.2. Suppose that $f : (0,1] \times [0,\infty) \to [0,\infty)$ is continuous and satisfies that for each $r > 0$ there exists $\phi_r \in L^1(0,1]$ such that $|f(t, t^{\alpha-2}x)| \leq \phi_r(t)$ for all $t \in (0,1]$ and $|x| \leq r$, and there exist positive numbers $a < \beta^\alpha b < b < c$ such that $Wc > Mb$, and

(E4) $f(t, t^{\alpha-2}u) \leq \frac{c}{M}$ for $t \in (0,1]$, $u \in [0, c/\beta^\alpha]$;

(E5) $f(t, t^{\alpha-2}u) \geq \frac{b}{W}$ for $t \in [\eta, 1]$ and $u \in [\beta^\alpha b, b]$;

(E6) $f(t, t^{\alpha-2}u) \leq \frac{a}{M}$ for $t \in (0, 1]$ and $u \in [0, a]$.

Then BVP(4) has at least two positive solutions x_1 and x_2 satisfying

$$\sup_{t\in(0,1]} t^{\alpha-2}x_1(t) > a, \quad \sup_{t\in(0,1]} t^{\alpha-2}x_1(t) < b, \quad \sup_{t\in(0,1]} t^{\alpha-2}x_2(t) > b, \quad \min_{t\in[\eta,1]} t^{\alpha-2}x_2(t) < c. \tag{14}$$

Proof. Let the nonnegative, increasing and continuous functionals $\gamma, \theta, \phi : P \to [0, \infty)$ be defined in the proof of Theorem 3.2. The remainder of the proof is similar to that of the proof of Theorem 3.2 and is omitted.

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