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COMMON FIXED POINT RESULTS FOR CYCLIC OPERATORS ON COMPLETE METRIC SPACES

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We introduce common fixed point results for cyclic operators satisfying certain nonlinear contraction with a control function, on complete metric space. Examples to illustrate the results are given. This study should be thought as a natural continuation of the research of Karapinar et al. [A common fixed point theorem for cyclic operators on partial metric spaces, Filomat, 26(2012), No. 2, 407-414.]

Keywords: Common fixed point, cyclic operator, metric space.

1. Introduction

Fixed point theory is one of the most interesting area of research in nonlinear analysis. Fixed point theorems give conditions under which we can find a solution of equations, involving certain classes of operators. That is why they found applications in Economics, Theoretical Physics, Engineering. The celebrated Contraction Principle of Banach [6] is extended by scientists such as: Kannan [15], Reich [30], Chatterjea [9] and many others.

Recently, the scientists studied this subject and proved fixed point theorems in ordered metric spaces [3, 4, 10, 13, 33, 37], partial metric spaces [2, 14, 16, 26, 28, 32, 38], convex metric spaces [27], cone metric spaces [23], G-metric spaces [5, 8, 12, 34], quasi-partial metric spaces [36], b-metric spaces [35]. Results on either approximate fixed points or variational inequalities in their relation with the fixed point problem are established [24, 25, 39, 40].

An interesting topic in fixed point theory is the cyclic representation. In 2003, Kirk *et al.* [22] introduced the following notion of cyclic representation.

Definition 1.1. Let X be a nonempty set, $m \in \mathbb{N}$ and $T\colon X \to X$ a mapping. Then $X = \bigcup_{i=1}^{m} A_i$ is called a *cyclic representation* of X with respect to T if

(1) A_i, $i = 1, \ldots, m$ are nonempty subsets of X;

(2) $T(A_1) \subset A_2$, $T(A_2) \subset A_3$, $\ldots$, $T(A_{m-1}) \subset A_m$, $T(A_m) \subset A_1$.

Meantime, other authors obtained results in fixed point theory for cyclic operators; please, see Agarwal *et al.* [1] for fixed point theorems involving mappings which satisfy cyclical generalized contractive conditions in complete partial metric spaces, Păcurar and Rus [29] for fixed point theory for cyclic φ-contractions, Karapinar [17] for a fixed point theory for cyclic weak φ-contractions, Chandok and Postolache [7] for a fixed point theorem for weakly Chatterjea-type cyclic contractions. For more results on this topic, the reader can see Karapinar *et al.* [19], [20].

As a generalization of the previous notion, Karapinar *et al.* [21] introduced the following notion of cyclic representation for two self mappings $T, S\colon X \to X$.

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Definition 1.2. Let X be a nonempty set, $m \in \mathbb{N}$ and $T, S\colon X \to X$ be two mappings. Then $X = \bigcup_{i=1}^{m} A_i$ is called a *cyclic representation* of X with respect to (T, S) if

(1) A_i, $i = 1, \ldots, m$ are nonempty subsets of X;

(2) $T(A_1) \subset S(A_2)$, $T(A_2) \subset S(A_3)$, ..., $T(A_{m-1}) \subset S(A_m)$, and $T(A_m) \subset S(A_1)$.

For common fixed point results for pairs of cyclic operators, we refer the reader to Karapinar *et al.* [18], [21], Shatanawi and Postolache [31].

It is the aim of this paper to introduce common fixed point results for cyclic operators satisfying certain nonlinear contraction with a control function, on complete metric space. Examples to illustrate the results are given.

2. Main result

In this section we will establish some common fixed point theorems concerning certain contractive type mappings.

Let F denote the class of all functions $\varphi\colon [0,\infty) \to [0,\infty)$ nondecreasing and continuous satisfying $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$ and let $\tilde{F}$ be the subset of F contains the function φ such that $\varphi(t) < t$, for each $t > 0$.

Theorem 2.1. *Let (X, d) be a complete metric space, m be a positive integer, $A_1, \ldots, A_m$ nonempty subsets of X and $X = \bigcup_{i=1}^{m} A_i$. Let $T, S\colon X \to X$ be two mappings such that*

(1) $X = \bigcup_{i=1}^{m} A_i$ is cyclic representation of X with respect to (T, S).

(2) $d(Tx, Ty) \le \varphi(M(x, y))$, for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, \ldots, m$, where

$$M(x,y) = \max\Big\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \frac{1}{2}[d(Sx,Ty) + d(Sy,Tx)]\Big\},$$

$A_{m+1} = A_1$, $\varphi \in \tilde{F}$, and each $S(A_i)$ is closed.

(3) Mapping S is one to one.

Then there exists $z \in \bigcap_{i=1}^{m} A_i$ such that $Tz = Sz$.

Proof. Let $x_1 \in A_1$, we choose a point x_2 in A_2 such that $Tx_1 = Sx_2$. For this point x_2 there exists a point x_3 in A_3 such that $Tx_2 = Sx_3$, and so on. Hence we obtain a sequence $\{x_n\}$ such that $Tx_n = Sx_{n+1}$, for $n = 1, 2, \ldots$.

If there exists $n_0 \in \mathbb{N}$ such that $Sx_{n_0} = Sx_{n_0+1}$, then $Sx_{n_0+1} = Tx_{n_0} = Sx_{n_0}$ and therefore x_{n_0} is the coincidence point of T and S.

Suppose we have $Sx_{n+1} \ne Sx_n$, for all $n = 0, 1, 2, \ldots$.

Since $X = \bigcup_{i=1}^{m} A_i$ and $T(A_i) \subset S(A_{i+1})$, for any $n > 0$, there exists an index $i_n \in \{1, \ldots, m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_n+1}$.

By the assumption of the theorem we have:

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &= d(Tx_{n-1}, Tx_n) \le \varphi(M(x_{n-1}, x_n)) \\ &= \varphi\big(\max\{d(Sx_{n-1}, Sx_n), d(Sx_{n-1}, Tx_{n-1}), d(Sx_n, Tx_n), \\ &\qquad \frac{1}{2}\left(d(Sx_{n-1}, Tx_n) + d(Sx_n, Tx_{n-1})\right)\}\big) \\ &= \varphi\big(\max\{d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1}), \\ &\qquad \frac{1}{2}d(Sx_{n-1}, Sx_{n+1})\}\big). \end{aligned} \tag{1}$$

Using the triangle inequality, we obtain

$$d(Sx_{n-1}, Sx_{n+1}) \le d(Sx_{n-1}, Sx_n) + d(Sx_n, Sx_{n+1}).$$

Hence, relation (1) becomes

$$d(Sx_n, Sx_{n+1}) \le \varphi(\max\{d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1})\}),$$

and there are two cases to be studied.

CASE I. $M(x_{n-1}, x_n) = d(Sx_n, Sx_{n+1})$.

This relation leads us to the conclusion that

$$d(Sx_n, Sx_{n+1}) \leq \varphi(d(Sx_n, Sx_{n+1})),$$

which contradicts the fact that $t > \varphi(t)$, for all $t > 0$.

CASE II. $M(x_{n-1}, x_n) = d(Sx_{n-1}, Sx_n)$

In this case, we have

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &\leq \varphi(d(Sx_{n-1}, Sx_n)) \\ &= \varphi(d(Tx_{n-2}, Tx_{n-1})) \\ &\leq \varphi^2(M(x_{n-2}, x_{n-1})) = \varphi^2(d(Sx_{n-2}, Sx_{n-1})). \end{aligned}$$

This implies that $d(Sx_n, Sx_{n+1}) \leq \varphi^{n-1}(d(Sx_1, Sx_2))$. Letting $n \to \infty$, and using the properties of function φ, we get

$$\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = 0.$$

We prove that $\{Sx_n\}$ is Cauchy sequence.

First, we show that for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that, if $p, q \geq n$ with $p - q \equiv 1 \pmod{m}$, then $d(Sx_p, Sx_q) < \varepsilon$.

Suppose that our claim does not hold.

Therefore there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ we can find $p_n > q_n \geq n$ with $p_n - q_n \equiv 1 (\text{mod } m)$ satisfying $d(Sx_{q_n}, Sx_{p_n}) \geq \varepsilon$.

Let $n > 2m$. For $q_n \geq n$ we can choose p_n such that p_n is the smallest integer greater than q_n satisfying $p_n - q_n \equiv 1 (\text{mod } m)$ and $d(Sx_{q_n}, Sx_{p_n}) \geq \varepsilon$. Hence $d(Sx_{q_n}, Sx_{p_n - m}) < \varepsilon$. Using this fact we have

$$\begin{aligned} \varepsilon &\leq d(Sx_{q_n}, Sx_{p_n}) \\ &\leq d(Sx_{q_n}, Sx_{p_n-m}) + \sum_{i=1}^{m} d(Sx_{p_n-i}, Sx_{p_n-i+1}) \\ &< \varepsilon + \sum_{i=1}^{m} d(Sx_{p_n-i}, Sx_{p_n-i+1}). \end{aligned}$$

Now, taking the limit for $n \to \infty$ in the last inequality, and having in mind that $\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = 0$, we obtain

$$\lim_{n\to\infty} d(Sx_{q_n}, Sx_{p_n}) = \varepsilon. \tag{2}$$

Using the triangular inequality, we have

$$\begin{aligned} \varepsilon &\leq d(Sx_{q_n}, Sx_{p_n}) \\ &\leq d(Sx_{q_n}, Sx_{q_n+1}) + d(Sx_{q_n+1}, Sx_{p_n+1}) + d(Sx_{p_n+1}, Sx_{p_n}) \\ &\leq d(Sx_{q_n}, Sx_{q_n+1}) + d(Sx_{q_n+1}, Sx_{q_n}) + d(Sx_{q_n}, Sx_{p_n}) \\ &+ d(Sx_{p_n}, Sx_{p_n+1}) + d(Sx_{p_n+1}, Sx_{p_n}) \qquad (3) \\ &= 2d(Sx_{q_n}, Sx_{q_n+1}) + d(Sx_{q_n}, Sx_{p_n}) + 2d(Sx_{p_n}, Sx_{p_n+1}) \end{aligned}$$

Taking the limit for $n \to \infty$ in (3), and having in mind that $\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = 0$ and (2), we get

$$\lim_{n\to\infty} d(Sx_{q_n+1}, Sx_{p_n+1}) = \varepsilon.$$

Since x_{q_n} and x_{p_n} lie in different adjacently labeled sets A_i and A_{i+1} for certain $1 \leq i \leq m$, using property (2) of the theorem, we have

$$d(Sx_{q_n+1}, Sx_{p_n+1})) = d(Tx_{q_n}, Tx_{p_n}) \leq \varphi(M(x_{q_n}, x_{p_n})). \tag{4}$$

There are several cases to be studied.

Case I. $M(x_{q_n}, x_{p_n}) = d(Sx_{q_n}, Sx_{p_n})$.

In this situation, relation (1) becomes

$$d(Sx_{q_n+1}, Sx_{p_n+1})) \leq \varphi(d(Sx_{q_n}, Sx_{p_n})).$$

Using equalities (2), and (4), letting $n \to +\infty$, we obtain $\varepsilon \leq \varphi(\varepsilon)$, therefore $\varepsilon = 0$, false.

Case II. $M(x_{q_n}, x_{p_n}) = d(Sx_{q_n}, Sx_{q_n+1})$. Hence,

$$d(Sx_{q_n+1}, Sx_{p_n+1}) \quad \leq \quad \varphi(d(Sx_{q_n}, Sx_{q_n+1})).$$

Considering $n \to +\infty$, the previous relation implies $\varepsilon \leq \varphi(0)$, which is a contradiction.

Case III. This is $M(x_{q_n}, x_{p_n}) = d(Sx_{p_n}, Sx_{p_n+1})$. In this case, it follows

$$d(Sx_{q_n+1}, Sx_{p_n+1}) \leq \varphi(d(Sx_{p_n}, Sx_{p_n+1})).$$

Using relation (4), and letting $n \to +\infty$, we get $\varepsilon \leq \varphi(0) = 0$. This leads us to the conclusion that $\varepsilon = 0$, false.

Case IV. $M(x_{q_n}, x_{p_n}) = \frac{1}{2}\left(d(Sx_{q_n}, Sx_{p_n+1}) + d(Sx_{p_n}, Sx_{q_n+1})\right)$.

Here, using the triangle inequality, we get

$$\begin{aligned} d(Sx_{q_n+1}, Sx_{p_n+1}) &\leq \varphi\Big(\frac{1}{2}\left(d(Sx_{q_n}, Sx_{p_n+1}) + d(Sx_{p_n}, Sx_{q_n+1})\right)\Big) \\ &\leq \varphi\Big(\frac{1}{2}\left(d(Sx_{q_n}, Sx_{p_n}) + d(Sx_{p_n}, Sx_{p_n+1})\right. \\ &\quad \left. + d(Sx_{p_n}, Sx_{q_n}) + d(Sx_{q_n}, Sx_{q_n+1})\right)\Big). \end{aligned}$$

Taking $n \to +\infty$, we obtain $\varepsilon \leq \varphi(\varepsilon)$, therefore $\varepsilon = 0$, false.

In conclusion, our claim has been proved.

In the following, we will show that $\{Sx_n\}$ is a Cauchy sequence.

Fix $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that if $p, q \geq n_0$ with $p - q \equiv 1 \pmod{m}$,

$$d(Sx_p, Sx_q) \leq \varepsilon/2. \tag{5}$$

Since $\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = 0$, there exists $n_1 \in \mathbb{N}$ such that

$$d(Sx_n, Sx_{n+1}) \leq \varepsilon/2m, \tag{6}$$

for each $n \geq n_1$.

Suppose that $r, s \geq \max\{n_0, n_1\} = N$ and $s > r$. Then there exists $k \in \{1, \ldots, m\}$ such that $s - r \equiv k \pmod{m}$.

Since $m + 1 \equiv 1 \pmod{m}$, we have $(s + j) - r \equiv 1 \pmod{m}$ for $j = m - k + 1$. So

$$d(Sx_r, Sx_s) \leq d(Sx_r, Sx_{s+j}) + d(Sx_{s+j}, Sx_{s+j-1}) + \cdots + d(Sx_{s+1}, Sx_s).$$

By (5), (6) and from the above inequality, we have

$$d(Sx_r, Sx_s) \leq \varepsilon/2 + j\varepsilon/2m \leq \varepsilon/2 + m\varepsilon/2m = \varepsilon.$$

Thus for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $r, s \geq N$ implies that $d(Sx_r, Sx_s) \leq \varepsilon$. This means that $\{Sx_n\}$ is Cauchy sequence.

Sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1, \ldots, m\}$ we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$ for each $k \in \mathbb{N}$. Hence $\{Sx_{n_k}\}$ is a subsequence of $\{S(x_n)\}$ such that $Sx_{n_k} \in S(A_{i-1})$ for each $k \in \mathbb{N}$.

Now, by the second assumption of the theorem, we have

$$\begin{aligned} d(Sx_{n_k+1}, Tx)) &= d(Tx_{n_k}, Tx) \leq \varphi(M(x_{n_k}, x)) \\ &= \varphi(\max\{d(Sx_{n_k}, Sx), d(Sx_{n_k}, Sx_{n_k+1}), d(Sx, Tx), \\ &\qquad \frac{1}{2}\left(d(Sx_{n_k}, Tx) + d(Sx, Sx_{n_k+1})\right)\}). \end{aligned}$$

Taking the limit $k \to \infty$, we get $d(Sx, Tx) = 0$, therefore $Sx = Tx$.

Since X is a complete metric space, there exists $z \in X$ such that $\lim_{n\to\infty} Sx_n = z$. Since $\lim_{n\to\infty} Sx_n = z$ and, as $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to (T, S), sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1, \dots, m\}$, and because $S(A_i)$ is closed for each i we conclude that $z \in \bigcap_{i=1}^{m} S(A_i)$. Hence, there exists $x_i \in A_i$ such that $Sx_i = z$. Since S is one to one, we have $x_1 = x_2 = \cdots = x_m = x$. Therefore, $\lim_{n\to\infty} Sx_n = Sx = z$, for $x \in \bigcap_{i=1}^{m} A_i$. □

Example 2.1. Let $X = [-1, 1]$ as a subspace of $\mathbb{R}$, endowed with the usual metric. Suppose that $A_1 = [-1, 0] = A_3$ and $A_2 = [0, 1] = A_4$. Define $T, S\colon X \to X$ respectively by the formulas $T(x) = -\dfrac{1}{6}x$, and $S(x) = \dfrac{1}{2}x$, for all $x \in X$. Consider $\varphi\colon [0, \infty) \to [0, \infty)$ with $\varphi(t) = \dfrac{1}{2}t$.

In the following, we will show that the hypotheses of Theorem 2.1 are satisfied.

First, we remark that $X = \bigcup\limits_{i=1}^{4} A_i$ is a cyclic representation of X with respect to (T, S). This because $T(A_1) \subset S(A_2)$, $T(A_2) \subset S(A_3)$, $T(A_3) \subset S(A_4)$ and $T(A_4) \subset S(A_1)$.

Now we have to prove that

$$d(Tx, Ty) \le \varphi(M(x, y)), \tag{7}$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, \dots, m$, where

$$M(x, y) = \max\left\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), \frac{1}{2}[d(Sx, Ty) + d(Sy, Tx)]\right\},$$

$A_{m+1} = A_1$.

Suppose $x \in A_1$ and $y \in A_2$; other cases should be studied by analogy.

In this case,

$$d(Tx, Ty) =\mid Tx - Ty \mid= \left|-\frac{1}{6}x + \frac{1}{6}y\right| = \frac{1}{6} \mid y - x \mid= \frac{1}{6}(y - x).$$

and we have to examine the cases as follows.

Case 1. This is $M(x, y) = d(Sx, Sy)$.

We have $d(Sx, Sy) =\mid Sx - Sy \mid= \left|\dfrac{1}{2}x - \dfrac{1}{2}y\right| = \dfrac{1}{2} \mid x - y \mid= \dfrac{1}{2}(y - x)$.

Condition (7) becomes

$$\frac{1}{6} \mid x - y \mid\le \frac{1}{2} \cdot \frac{1}{2} \mid x - y \mid,$$

and this is true since

$$\frac{1}{6} \mid x - y \mid\le \frac{1}{4} \mid x - y \mid .$$

Case 2. In this case, $M(x, y) = d(Sx, Tx)$.

Since $d(Sx, Tx) =\mid Sx - Tx \mid= \left|\dfrac{1}{2}x + \dfrac{1}{6}x\right| = \dfrac{4}{6} \mid x \mid$, condition (7) becomes

$$\frac{1}{6} \mid x - y \mid\le \frac{4}{6} \cdot \frac{1}{2} \mid x \mid .$$

We easily get

$$\mid x - y \mid\le 2 \mid x \mid, \text{ where } x \in [-1, 0] \text{ and } y \in [0, 1]. \tag{8}$$

Because $x \in [-1, 0]$ and $y \in [0, 1]$, we have

$$\mid x - y \mid= |-(y - x)| = y - x. \tag{9}$$

Now, from (8) and (9), we obtain that $y - x \le 2(-x)$, that is $y \le -x$. This is true from the definition of $M(x,y)$, where we have that $\frac{4}{6} \mid x \mid \geq \frac{4}{6} \mid y \mid$, therefore we obtain $\mid x \mid \geq \mid y \mid$.

CASE 3. In this case, $M(x,y) = \frac{1}{2}\left[d(Sx,Ty) + d(Sy,Tx)\right]$.

We have

$$\begin{aligned}\frac{1}{2}[d(Sx,Ty)+d(Sy,Tx)] &= \frac{1}{2}\left(\mid Sx - Ty \mid + \mid Sy - Tx \mid\right) \\ &= \frac{1}{2}\left(\left|\frac{1}{2}x + \frac{1}{6}y\right| + \left|\frac{1}{2}y + \frac{1}{6}x\right|\right).\end{aligned}$$

Using this form, condition (7) becomes

$$\frac{1}{6} \mid x - y \mid \leq \frac{1}{2} \cdot \frac{1}{2}\left(\left|\frac{1}{2}x + \frac{1}{6}y\right| + \left|\frac{1}{2}y + \frac{1}{6}x\right|\right),$$

and, more convenient

$$\mid x - y \mid \leq \frac{3}{2}\left(\left|\frac{1}{2}x + \frac{1}{6}y\right| + \left|\frac{1}{2}y + \frac{1}{6}x\right|\right). \tag{10}$$

To prove (10), we have to consider the definition of $M(x,y)$. From this definition, we know that $d(Sx,Sy) \leq \frac{1}{2}[d(Sx,Ty)+d(Sy,Tx)]$. So, we have

$$\frac{1}{2} \mid x - y \mid \leq \frac{1}{2}\left(\left|\frac{1}{2}x + \frac{1}{6}y\right| + \left|\frac{1}{2}y + \frac{1}{6}x\right|\right), \tag{11}$$

and

$$\mid x - y \mid \leq \left|\frac{1}{2}x + \frac{1}{6}y\right| + \left|\frac{1}{2}y + \frac{1}{6}x\right|. \tag{12}$$

Looking at (12), we see that (10) holds.

The hypotheses of Theorem 2.1 are satisfied, therefore mappings T and S have a common coincidence point $x = 0$.

Recall that the maps S and T are called *weakly compatible* [11] if they commute at their coincidence points, i.e., $STx = TSx$, for each x in X such that $Tx = Sx$.

Corollary 2.1. *Suppose the assumptions in Theorem* 2.1 *are satisfied. In addition, suppose the pair* (T,S) *is weakly compatible.*

Then T *and* S *have a unique common fixed point* $z \in \bigcap_{i=1}^{m} A_i$.

Proof. For proving this statement, say $z = Sx = Tx$. Since S and T are two weakly compatible mappings, we have $TTx = TSx = STx = SSx$. Hence $Tz = Sz$. Since $z \in X$, there exists some i such that $z \in A_i$.

On the other hand $x \in \bigcap_{i=1}^{m} A_i$, therefore $x \in A_{i-1}$, and using the contractive condition we obtain

$$\begin{aligned} d(Tx,Tz) &\leq \varphi(M(x,z)) \\ &= \varphi(d(Sx,Sz)) = \varphi(d(Tx,Tz)),\end{aligned}$$

therefore $d(Tx,Tz) = 0$, so $Tx = Tz = z = Sx = Sz$. Since S is one to one, we obtain $x = z$.

Next we prove that this fixed point is unique.

Let $y, z \in \bigcap_{i=1}^{m} A_i$ be common fixed points of T and S. We have

$$\begin{aligned} d(y,z) &= d(Ty,Tz) \leq \varphi(M(y,z)) \\ &= \varphi(d(Sy,Sz)) = \varphi(d(y,z)).\end{aligned}$$

therefore $d(y,z) = 0$, thus $y = z$. □

Considering $S = Id$, we get

Corollary 2.2. *Let (X, d) be a complete metric space, m be a positive integer, $A_1, \ldots, A_m$ nonempty subsets of X and $X = \bigcup_{i=1}^m A_i$. Let $T\colon X \to X$ be a mapping such that*

(1) $X = \bigcup_{i=1}^m A_i$ is cyclic representation of X with respect to (T, Id).

(2) $d(Tx, Ty) \leq \varphi(M(x, y))$, for any $x \in A_i, y \in A_{i+1}, i = 1, \ldots, m$ where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}\left\{d(x, Ty) + d(y, Tx)\right\}\right\},$$

$A_{m+1} = A_1$, $\varphi \in \tilde{F}$, and each A_i is closed.

Then, there exist $z \in \bigcap_{i=1}^m A_i$ such that $Tz = z$.

Example 2.2. Let $X = [-1, 1]$ as a subspace of $\mathbb{R}$ with the usual metric. Suppose that $A_1 = [-1, 0] = A_3$ and $A_2 = [0, 1] = A_4$. Define $T\colon X \to X$ by the formula $T(x) = -\frac{1}{3}x$, for all $x \in X$, $S = Id$, and let $\varphi\colon [0, \infty) \to [0, \infty)$ with $\varphi(t) = \frac{1}{2}t$. The hypotheses of Corollary 2.2 are satisfied, so T has a unique fixed point $x = 0$.

3. Conclusion

In this article, we introduced common fixed point results for cyclic operators satisfying certain nonlinear contraction with a control function, on complete metric space. Illustrative examples for the new results are given. This study is as a natural continuation of the research of Karapinar *et al.* [21], Păcurar and Rus [29] and many others.

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