STUDY ON STOCHASTIC DIFFERENTIAL EQUATIONS VIA MODIFIED ADOMIAN DECOMPOSITION METHOD

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In this paper, the well known Adomian decomposition method is modified to solve the one-dimensional stochastic differential equations as an infinite series. The uniqueness of solution for the stochastic differential equation and the convergence of the series obtained by this method are discussed as well. In order to illustrate the efficiency of the proposed method some experiments are presented. The results are compared with the existing exact or analytical methods that confirm the reliability of the improved algorithm.

Keywords: Stochastic differential equation, Modified Adomian decomposition method, Lipschitz continuous, Ito formula

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1. Introduction

The practical applications in mathematical physics and engineering problems have given impetus to the research on stochastic differential equations (SDEs). These equations are widely used to describe processes in biology, seismology, ecology, finance, physics, engineering and etc [1–6]. The stochastic equations are derived from basic principles, i.e. from the changes that occur in a small time interval. Over the last several decades, numerous studies have been developed toward the study of SDEs and stochastic delay differential equations such as attraction, stochastic stability, boundedness, stochastic flow, invariant measure, invariant manifold and numerical approximation [3–5, 7–14].
The Adomian decomposition method (ADM) is an effective and creative scheme for exactly solving functional equations of various kinds. It is important to note that plenty of research studies has been devoted to the application of the ADM to a wide class of differential equations [15–22]. The decomposition method provides the solution as an infinite series in which each term can be easily determined. In this paper, we present a further insight into partial solutions in the decomposition method via modified Adomian decomposition method (MADM) for finding a strong solution of the following SDEs

$$\begin{cases} \dot{X}(t) = F(t, X(t), Y(t)), \ t \in I = [0, T], \\ X(0) = X_0, \end{cases} \tag{1}$$

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in a special case. Where F is a given function, $Y(t)$ and $X(t)$ are known and unknown stochastic processes, respectively, and both are defined on the same given complete filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where X_0 is a $\mathcal{F}_0-$measurable random variable. Modified Adomian decomposition approach is a non-numerical method which can be adapted for solving stochastic differential equations. The general goal of this work is not just to establish the uniqueness and convergence of a solution to the SDE under consideration, but rather to introduce a new algorithm in the context of SDEs that has already proven to be successful in the context of ordinary and partial differential equations. The rest of the paper is organized as follows:

In Section 2 we give some notation and preliminary concepts. Implementation of MADM for solving some special SDEs and analysis of this method are presented in Section 3. Two test problems to evaluate the proposed method are given in Section 4. Finally, the paper is ended with a brief conclusion.

2. **Preliminaries**

In this section, we review some necessary mathematical definitions and lemmas which are used further in this paper.

Definition 2.1. *(*[23, 24]*) A strong solution of the stochastic differential equation*

$$dX(t) = f_1(t, X(t))dt + f_2(t, X(t))dB_t,\ X(0) = X_0;\ t \geq 0, \tag{2}$$

on the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, *with respect to the fixed standard Brownian motion* $\{B_t\}_{t\geq 0}$, *and the independent initial condition* X_0 *over this probability space, is a stochastic process* $X = \{X(t), t \geq 0\}$ *with continuous paths that is adapted to the filtration generated by* B_t *such that:*
i) For all $t \geq 0$ *with probability one,*

$$\int_0^t |f_1(s, X(s))| \mathrm{d}s < \infty, \ \textit{and} \ \int_0^t (f_2(s, X(s)))^2 \, \mathrm{d}B_s < \infty;$$

ii) For all $t \geq 0$,

$$X(t) = X_0 + \int_0^t f_1(s, X(s))\mathrm{d}s + \int_0^t f_2(s, X(s))\mathrm{d}B_s,\ a.s. \tag{3}$$

The last stochastic integral equation corresponds to SDE (2), and characterizes the behavior of the continuous time stochastic process $X(t)$ as the sum of an ordinary Lebesgue integral and an Ito integral.

Definition 2.2. *A weak solution of the SDE* (2) *is a continuous stochastic process* $X(t)$ *defined on some probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that for some Wiener process* B_t *and some admissible filtration* $\mathbb{F}$, *the process* $X(t)$ *is adapted to satisfy the stochastic integral equation* (3).

Lemma 2.1. *(The 1-dimensional Ito formula (*[3, 4, 10]*))*
Let $h(t, x) : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ *belongs to* $C^2([0, \infty) \times \mathbb{R})$, *and assume that the stochastic*

process $X(t)$ verifies of the SDE (2). Then we have

$$\begin{aligned} h(t,X(t)) =& h(0,X_0) \\ &+ \int_0^t \Big[\frac{\partial h}{\partial t}(s,X(s)) \qquad (4) \\ &+ f_1(s,X(s))\frac{\partial h}{\partial x}(s,X(s)) + \frac{1}{2} f_2^2(s,X(s))\frac{\partial^2 h}{\partial x^2}(s,X(s))\Big]\mathrm{d}s \\ &+ \int_0^t f_2(s,X(s))\frac{\partial h}{\partial x}(s,X(s))\mathrm{d}B_s. \qquad (5) \end{aligned}$$

Definition 2.3. *(Lipschitz-continuous function ([14, 25]))*
Let $A \subseteq \mathbb{R}^n$ be an open set and $B \subseteq \mathbb{R}^m$, function $f : A \to B$ is called Lipschitz-continuous if there exists a real constant $K \geq 0$ (called the Lipschitz constant of f on A) such that

$$d_B\left(f(x_1), f(x_2)\right) \leq K d_A(x_1, x_2), \ \ \forall x_1, x_2 \in A,$$

where d_A denotes a metric on the set A and d_B is a metric on the set B.

Lemma 2.2. *(Cauchy's formula ([26]))*
Let $f : [a,b] \to \mathbb{R}$ be Lebesgue integrable and $t \in [a,b]$. Then the nth repeated integral of f based at a,

$$f^{(-n)}(t) = \int_a^t \int_a^{t_1} \dots \int_a^{t_{n-1}} f(t_n)\mathrm{d}t_n \dots \mathrm{d}t_2 \mathrm{d}t_1, \ t_1, \dots, t_{n-1} \in [a,b], \qquad (6)$$

is given by single integration

$$f^{(-n)}(t) = \frac{1}{\Gamma(n)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-n}} \mathrm{d}\tau,$$

where $\Gamma(n) = (n-1)!$ denotes the Gamma function.

Lemma 2.3. *Let $q(t)$ be integrable and at least $\dot{r}(t) = \frac{\mathrm{d}r}{\mathrm{d}t}$ exists then*

$$\exp(r(t)) \int \exp(-r(t)) q(t)\mathrm{d}t = \sum_{k=0}^{\infty} \underbrace{\int \dot{r}(t) \dots \int \dot{r}(t)}_{k-times} \int q(t)\mathrm{d}t \underbrace{\mathrm{d}t \dots \mathrm{d}t}_{k-times}.$$

Proof. This is a straightforward consequence of the Theorem 1 in [22]. Applying integration by parts $\int u \mathrm{d}v = uv - \int v \mathrm{d}u$, with $u = \exp(-r(t))$ and

$$\mathrm{d}v = \begin{cases} q(t)\mathrm{d}t, \quad k = 0, \\ \dot{r}(t) \underbrace{\int \dot{r}(t) \dots \int \dot{r}(t)}_{(k-1)-times} \int q(t)\mathrm{d}t \underbrace{\mathrm{d}t \dots \mathrm{d}t}_{k-times}, \quad k = 1, 2, 3, \dots, \end{cases}$$

yields

$$\begin{aligned}
&\int e^{-r(t)}q(t)\mathrm{d}t = e^{-r(t)}\int q(t)\mathrm{d}t + \int \dot{r}(t)e^{-r(t)}\int q(t)\mathrm{d}t\mathrm{d}t,\\
&\int \dot{r}(t)e^{-r(t)}\int q(t)\mathrm{d}t\mathrm{d}t = e^{-r(t)}\int \dot{r}(t)\int q(t)\mathrm{d}t\mathrm{d}t +\\
&\int \dot{r}(t)e^{-r(t)}\int \dot{r}(t)\int q(t)\mathrm{d}t\mathrm{d}t\mathrm{d}t,\\
&\int \dot{r}(t)e^{-r(t)}\int \dot{r}(t)\int q(t)\mathrm{d}t\mathrm{d}t\mathrm{d}t\\
&= e^{-r(t)}\int \dot{r}(t)\int \dot{r}(t)\int q(t)\mathrm{d}t\mathrm{d}t\mathrm{d}t\\
&\qquad\qquad\qquad + \int \dot{r}(t)e^{-r(t)}\int \dot{r}(t)\int \dot{r}(t)\int q(t)\mathrm{d}t\mathrm{d}t\mathrm{d}t\mathrm{d}t,\\
&\quad\vdots\\
&\int \dot{r}(t)e^{-r(t)}\underbrace{\int \dot{r}(t)\ldots\int \dot{r}(t)}_{(m-1)-times}\int q(t)\mathrm{d}t\underbrace{\mathrm{d}t\ldots\mathrm{d}t}_{m-times}\\
&= e^{-r(t)}\underbrace{\int \dot{r}(t)\ldots\int \dot{r}(t)}_{m-times}\int q(t)\mathrm{d}t\underbrace{\mathrm{d}t\ldots\mathrm{d}t}_{m-times}\\
&\qquad\qquad + \int \dot{r}(t)e^{-r(t)}\underbrace{\int \dot{r}(t)\ldots\int \dot{r}(t)}_{m-times}\int q(t)\mathrm{d}t\underbrace{\mathrm{d}t\ldots\mathrm{d}t}_{(m+1)-times}.
\end{aligned}$$

Hence upon substitution of the results by the successive integration by parts, we obtain

$$\begin{aligned}
\int e^{-r(t)}q(t)\mathrm{d}t = \sum_{k=0}^{m} e^{-r(t)}&\underbrace{\int \dot{r}(t)\ldots\int \dot{r}(t)}_{k-times}\int q(t)\mathrm{d}t\underbrace{\mathrm{d}t\ldots\mathrm{d}t}_{k-times}\\
&+ \int \dot{r}(t)e^{-r(t)}\underbrace{\int \dot{r}(t)\ldots\int \dot{r}(t)}_{m-times}\int q(t)\mathrm{d}t\underbrace{\mathrm{d}t\ldots\mathrm{d}t}_{(m+1)-times}.
\end{aligned}$$

As $m \to \infty$,

$$\int e^{-r(t)}q(t)\mathrm{d}t = e^{-r(t)}\sum_{k=0}^{\infty}\underbrace{\int \dot{r}(t)\ldots\int \dot{r}(t)}_{k-times}\int q(t)\mathrm{d}t\underbrace{\mathrm{d}t\ldots\mathrm{d}t}_{k-times},$$

and consequently the desired result can be achieved. □

3. Analysis of the MADM

Let us consider a special case of SDE (1) as the following form

$$\begin{cases} \dot{X}(t) = f(t, X(t)) + g(t, Y(t)), \ t \in I = [0, T], \\ X(0) = X_0, \end{cases} \tag{7}$$

where g is a function that includes terms of the second side Eq. (1) which are devoid of unknown $X(t)$ and f is the remaining term of the function F. We present the modified Adomian decomposition approach for solving Eq. (7) by the following recursive formula

$$\begin{cases} X_0(t) = X_0 + \displaystyle\int_0^t g(s, Y(s))ds, \\ X_{n+1}(t) = \displaystyle\int_0^t f(s, X_n(s))ds, \ n = 0, 1, 2, \dots. \end{cases} \tag{8}$$

The ADM expresses the unknown function $X(t)$ by an infinite series [17, 18],

$$X(t) = \sum_{i=0}^{\infty} X_i(t).$$

Introducing the notation $S_n(t) = \sum_{i=0}^{n} X_i(t)$ an approximate solution of Eq. (7) with MADM may be obtained by $X(t) = \lim_{n\to\infty} S_n(t)$.

Theorem 3.1. *Let $f(t, X(t))$ corresponding to the SDE (7) be a Lipschitz-continuous function with respect to the second component with Lipschitz constant K_1 and $0 \leq K_1 T < 1$. Then SDE (7) has a unique solution.*

Proof. Let $X(t)$ and $X^*(t)$ be two different solutions of Eq. (7), so

$$\dot{X}(t) - \dot{X}^*(t) = f(t, X(t)) - f(t, X^*(t)). \tag{9}$$

By applying the ordinary Lebesgue integration on both sides of the above equation on the interval $[0, t]$, we obtain

$$X(t) - X^*(t) = \int_0^t (f(s, X(s)) - f(s, X^*(s)))\, \mathrm{d}s. \tag{10}$$

On the other hand, since the function f is Lipschitz-continuous and by considering Euclidean norm as a metric on the related metric space, one gets

$$\|X(t) - X^*(t)\| \leq K_1 \|X(t) - X^*(t)\| \int_0^t ds \leq K_1 T \|X(t) - X^*(t)\|,$$

therefore

$$(1 - K_1 T)\|X(t) - X^*(t)\| \leq 0.$$

Since $0 \leq K_1 T < 1$ then $\|X(t) - X^*(t)\| = 0$, implies $X(t) = X^*(t)$ and this completes the proof. □

Theorem 3.2. *Assume that $\widetilde{f}(t, S_n(t)) = \sum_{i=0}^{n} f(t, X_i(t))$, is a Lipschitz-continuous function with Lipschitz constant K_2, and $0 \leq \kappa = K_2 T < 1$. Then the series solution $\sum_{i=0}^{\infty} X_i(t)$ which is obtained by (8) as a solution of Eq. (7), converges to the exact solution $X(t)$.*

Proof. Denote the Banach space $(L_2[0,T], \|.\|)$, the space of all square-integrable functions with the Euclidean norm. Let for $n, m \in N$ and $m \leq n$, S_m and S_n be arbitrary partial sums. Now we show that $\{S_i(t)\}_{i=0}^{\infty}$ is a Cauchy sequence in Banach space $L_2[0,T]$. For this purpose we have

$$\begin{aligned} S_n(t) - S_m(t) = \sum_{i=m+1}^{n} X_i(t) &= \sum_{i=m+1}^{n} \int_0^t f(s, X_{i-1}(s))ds \\ &= \int_0^t \sum_{i=m}^{n-1} f(s, X_i(s))ds \\ &= \int_0^t \left(\widetilde{f}(s, S_{n-1}(s)) - \widetilde{f}(s, S_{m-1}(s)) \right) ds. \end{aligned}$$

Because the function $\widetilde{f}$ is Lipschitz-continuous and taking Euclidean norm on both sides of the above relation, we obtain

$$\|S_n(t) - S_m(t)\| \leq \kappa \|S_{n-1}(t) - S_{m-1}(t)\|$$

Putting $n = m + 1$ and repeating this process

$$\begin{aligned} &\|S_{m+1}(t) - S_m(t)\| \leq \kappa \|S_m(t) - S_{m-1}(t)\| \\ &\Rightarrow \|X_{m+1}(t)\| \leq \kappa \|X_m(t)\| \leq \kappa^2 \|X_{m-1}(t)\| \leq \cdots \leq \kappa^{m+1} \|X_0\|. \end{aligned}$$

Also, from the triangle inequality for integer $p \geq 1$

$$\begin{aligned} \|S_{m+p}(t) - S_m(t)\| &\leq \|S_{m+p}(t) - S_{m+p-1}(t)\| + \|S_{m+p-1}(t) - S_{m+p-2}(t)\| + \\ &\quad \cdots + \|S_{m+1}(t) - S_m(t)\| \\ &\leq \left(\kappa^{m+p} + \kappa^{m+p-1} + \cdots + \kappa^{m+1} \right) \|X_0\| \\ &= \frac{\kappa^{m+1} - \kappa^{m+p+1}}{1 - \kappa} \|X_0\|. \end{aligned}$$

Since $0 \leq \kappa < 1$, as $m \to \infty$, $\|S_{m+p}(t) - S_m(t)\| \to 0$. Therefore $\{S_i(t)\}_{i=0}^{\infty}$ is a Cauchy sequence in Banach space which provides convergence of the series solution. □

4. **Applications and results**

In this section, we will use the modified Adomian decomposition scheme to solve some initial stochastic differential equation. These examples are selected because closed form solutions are available for them; this allows one to compare the results obtained using this scheme with the analytical solution or the solutions obtained using other schemes.

First test problem. Following Kloeden and Platen [10], we consider SDE

$$\begin{cases} \dot{X}(t) = -\alpha X(t) + \sigma W(t), \ t \in [0,T], \\ X(0) = X_0, \end{cases} \tag{11}$$

where $W(t)$ is a 1-dimensional white noise, and α and σ are constant coefficients. Let $B(t)$ be the Brownian motion starting at origin. The correlated iterative process of the MADM for (11) is

$$\begin{cases} X_0(t) = X_0 + \sigma B(t), \\ X_{n+1}(t) = -\alpha \int_0^t X_n(s)\mathrm{d}s, \ n = 0, 1, 2, \dots. \end{cases}$$

Hence for variables $t_1, t_2, \dots, t_{n-1} \in [0, T]$,

$$\begin{aligned} X_1(t) =& - X_0\alpha t - \alpha\sigma \int_0^t B(t_1)\mathrm{d}t_1, \\ X_2(t) =& X_0\alpha^2 \frac{t^2}{2!} + \alpha^2\sigma \int_0^t \int_0^{t_1} B(t_2)\mathrm{d}t_2\mathrm{d}t_1, \\ X_3(t) =& - X_0\alpha^3 \frac{t^3}{3!} - \alpha^3\sigma \int_0^t \int_0^{t_1} \int_0^{t_2} B(t_3)\mathrm{d}t_3\mathrm{d}t_2\mathrm{d}t_1, \\ &\vdots \\ X_n(t) =& (-1)^n X_0\alpha^n \frac{t^n}{n!} + (-1)^n\alpha^n\sigma \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} B(t_n)\mathrm{d}t_n \dots \mathrm{d}t_2\mathrm{d}t_1, \end{aligned}$$

and the series solution is

$$X(t) = X_0 \sum_{i=0}^{\infty} \frac{(-\alpha t)^i}{i!} + \sigma B(t) + \sigma \sum_{i=1}^{\infty} (-\alpha)^i \underbrace{\int_0^t \int_0^{t_1} \dots \int_0^{t_{i-1}}}_{i-times} B(t_i)\mathrm{d}t_i \dots \mathrm{d}t_2\mathrm{d}t_1.$$

Maclaurin expansion of e^x and Lemma 2.2 yield

$$X(t) = X_0 \exp(-\alpha t) + \sigma B(t) - \alpha\sigma \int_0^t \exp(\alpha s - \alpha t)B(s)\mathrm{d}s,$$

that is in perfect agreement with the exact solutions of SDE (11),

$$X(t) = \mathrm{e}^{-\alpha t} \left(X_0 + \sigma \int_0^t \mathrm{e}^{\alpha s}\mathrm{d}B(s) \right).$$

Second test problem. Let us consider the following SDE [27, 28],

$$\begin{cases} \dot{X}(t) = a_1(t)X(t) + a_2(t) + b(t)\mathcal{W}(t), \ t \in I = [0, T], \\ X(0) = X_0, \end{cases} \tag{12}$$

where the coefficients $a_1(t)$, $a_2(t)$ and $b(t)$ are specified functions of time t or constants, and $\frac{d\mathcal{B}(t)}{dt} = \mathcal{W}(t)$, independent of X_0 for each $t \in I$. Also, $\mathcal{B}(t)$ and $X(t)$ are respectively known and unknown stochastic processes defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

To solve Eq. (12) or

$$\begin{cases} dX(t) = a_1(t)X(t)dt + a_2(t)dt + b(t)d\mathcal{B}(t), \ t \in I = [0, T], \\ X(0) = X_0, \end{cases} \tag{13}$$

the corresponding homogeneous equation is an ordinary differential equation of the form [4, 10],

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = a_1(t)X(t),$$

and its fundamental solution is

$$\xi_{t,0} = \exp\left(\int_0^t a_1(s)\mathrm{d}s\right).$$

Applying the Ito formula to the transformation $J(t,x) = \xi_{t,0}^{-1}x$ and the solution $X(t)$ of (12), we obtain

$$\begin{aligned}\mathrm{d}\left(\xi_{t,0}^{-1}X(t)\right) &= \left(\frac{\mathrm{d}\xi_{t,0}^{-1}}{\mathrm{d}t}X(t) + \left(a_1(t)X(t) + a_2(t)\right)\xi_{t,0}^{-1}\right)\mathrm{d}t + b(t)\xi_{t,0}^{-1}\mathrm{d}\mathcal{B}(t),\\ &= a_2(t)\xi_{t,0}^{-1}\mathrm{d}t + b(t)\xi_{t,0}^{-1}\mathrm{d}\mathcal{B}(t),\end{aligned} \tag{14}$$

since

$$\frac{\mathrm{d}\xi_{t,0}^{-1}}{\mathrm{d}t} = -a_1(t)\xi_{t,0}^{-1}.$$

Consequently, in view of the fact that $\xi_{0,0} = 1$, and by writing Eq. (14) in integrated form, the solution of (12) is obtained as follows:

$$X(t) = \xi_{t,0}\left(X_0 + \int_0^t a_2(s)\xi_{s,0}^{-1}\mathrm{d}s + \int_0^t b(s)\xi_{s,0}^{-1}\mathrm{d}\mathcal{B}(s)\right). \tag{15}$$

Now we try to solve the initial value equation (12) via MADM; therefore,

$$\begin{cases} X_0(t) = X_0 + \displaystyle\int_0^t \Big(a_2(s) + b(s)\mathcal{W}(s)\Big)\mathrm{d}s,\\ X_{n+1}(t) = \displaystyle\int_0^t a_1(s)X_n(s)\mathrm{d}s, \ n = 0,1,2,\dots. \end{cases} \tag{16}$$

From (16), we get

$$\begin{aligned}
X_1(t) &= X_0\int_0^t a_1(t_1)\mathrm{d}t_1 + \int_0^t a_1(t_1)\int_0^{t_1}\Big(a_2(t_2) + b(t_2)\mathcal{W}(t_2)\Big)\mathrm{d}t_2\mathrm{d}t_1,\\
X_2(t) &= X_0\int_0^t a_1(t_1)\int_0^{t_1} a_1(t_2)\mathrm{d}t_2\mathrm{d}t_1\\
&\quad + \int_0^t a_1(t_1)\int_0^{t_1} a_1(t_2)\int_0^{t_2}\Big(a_2(t_3) + b(t_3)\mathcal{W}(t_3)\Big)\mathrm{d}t_3\mathrm{d}t_2\mathrm{d}t_1,\\
&\ \vdots\\
X_n(t) &= X_0\int_0^t a_1(t_1)\int_0^{t_1} a_1(t_2)\dots\int_0^{t_{n-1}} a_1(t_n)\mathrm{d}t_n\dots\mathrm{d}t_2\mathrm{d}t_1\\
&\quad + \int_0^t a_1(t_1)\int_0^{t_1} a_1(t_2)\dots\int_0^{t_{n-1}} a_1(t_n)\int_0^{t_n}\Big(a_2(t_{n+1})\\
&\quad + b(t_{n+1})\mathcal{W}(t_{n+1})\Big)\mathrm{d}t_{n+1}\dots\mathrm{d}t_2\mathrm{d}t_1.
\end{aligned}$$

Consequently, the corresponding series solution of SDE (12) is

$$
\begin{aligned}
X(t) =& X_0(t) + \sum_{i=1}^{\infty} X_0 \frac{1}{i!} \left(\int_0^t a_1(s)\mathrm{d}s \right)^i \\
&+ \sum_{i=1}^{\infty} \int_0^t a_1(t_1) \int_0^{t_1} a_1(t_2) \dots \int_0^{t_{i-1}} a_1(t_i) \\
&\int_0^{t_i} \Big(a_2(t_{i+1}) + b(t_{i+1})\mathcal{W}(t_{i+1}) \Big) \mathrm{d}t_{i+1} \dots \mathrm{d}t_2 \mathrm{d}t_1.
\end{aligned}
$$

Finally from Lemma 2.3 with

$$
r(t) = \int_0^t a_1(s)\mathrm{d}s, \text{ and } q(t) = a_2(t) + b(t)\mathcal{W}(t),
$$

and reminding that $\xi_{t,0} = \exp(r(t))$, we can write

$$
X(t) = \xi_{t,0} \left(X_0 + \int_0^t \xi_{s,0}^{-1} \Big(a_2(s) + b(s)\mathcal{W}(s) \Big) \mathrm{d}s \right), \tag{17}
$$

which is the exact solution (15). It is observed that by the presented method in this paper, we achieved the exact analytical solutions of some special SDEs. According to [27, 28], putting $a_1(t) = 2t, a_2(t) = \exp(-t), b(t) = 1$ and applying Eq. (17) the exact solution of SDE

$$
\begin{cases} \dot{X}(t) = 2tX(t) + \exp(-t) + \mathcal{W}(t), \ t \in I = [0, 1], \\ X(0) = X_0, \end{cases} \tag{18}
$$

is obtained as follows:

$$
X(t) = \exp(t^2) \left(X_0 + \int_0^t \exp(-s^2) \Big(\exp(-s) + \mathcal{W}(s) \Big) \mathrm{d}s \right). \tag{19}
$$

5. **Conclusion**

The modified Adomian decomposition method is considered in application to stochastic differential equations. We explained some necessary and sufficient conditions for uniqueness of the solution, and convergence of the series obtained by modified Adomian decomposition method for solving some special cases of stochastic differential equations. Implementation results of the modified Adomian decomposition method are found the validity and efficiency of this technique to obtain analytical solution in comparison to other numerical methods. The advantage of the presented approach is that the argument only requires some basic knowledge about functional and stochastic analysis. Finally, several examples were given to check the reliability of the presented method.

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