ON CHARACTER BIPROJECTIVITY OF BANACH ALGEBRAS

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In this paper, we continue our work [13] in the study of ϕ-biprojective Banach algebra A, with respect to a character ϕ of A. For a locally compact group G, we show that a Segal algebra S(G) is ϕ-biprojective if and only if G is compact.

We also introduce and study character biprojectivity of Banach algebras. We show that the measure algebra M(G) is character biprojective if and only if G is a finite group. For a commutative character biprojective Banach algebra A, we prove that the character space $\Delta(A)$ is discrete. Finally we show that some triangular Banach algebras are never ϕ-biprojective.

Keywords: ϕ-biprojective, Character biprojective, Abstract Segal algebras.

1. Introduction and preliminaries

Let A be a Banach algebra and X a Banach A-bimodule. Then a bounded linear map $D: A \to X$ *is called a derivation if*

$$D(ab) = D(a).b - a.D(b) \qquad (a, b \in A).$$

The set of all derivation from A into X is denoted by $Z^1(A, X)$. *For every* $x \in X$, *we define an inner derivation* $ad_x: A \to X$ *by*

$$ad_x(a) = a.x - x.a \qquad (a \in A)$$

The set of all inner derivations from A into X is denoted by $B^1(A, X)$. *Clearly,* $B^1(A, X)$ *is a subspace of* $Z^1(A, X)$. *We define the first Hochschild cohomology* $\mathcal{H}^1(A, X)$, *as the quotient* $Z^1(A, X)/ B^1(A, X)$ *[12]. A Banach algebra A is called amenable (contractible) if the first Hochschild cohomology group* $\mathcal{H}^1(A, X^*)$ $(\mathcal{H}^1(A, X))$ *vanishes for every Banach A-bimodule X [7]. An alternative approach to Hochschild cohomology is Banach homology with the most important concepts, like biflatness and biprojectivity. Indeed a Banach*

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algebra A is called biflat (biprojective), if there exists a bounded A-bimodule morphism $\rho: A \to (A \otimes_p A)^{**}$ $(\rho: A \to (A \otimes_p A))$ *such that* ${\pi_A}^{**} o\rho(a) = a$ $(\pi_A o\rho(a) = a)$*, respectively, see [12] or [6]. Note that A is an amenable (contractible) Banach algebra if and only if A is biflat (biprojective) and has a bounded approximate identity (identity), respectively.*

Kaniuth et al. in [8] generalized the notion of amenability to a notion of left ϕ*-amenability, where* ϕ *is a Banach algebra character. In fact a Banach algebra A is called left* ϕ *-amenable, if there exists* $m \in A^{**}$ *such that* $am = \phi(a)m$*, and* $m(\phi) = 1$ *for every* $a \in A$*, where* $\phi: A \to \mathbb{C}$ *is a character on A. They also showed that A is left* ϕ*-amenable if and only if* $\mathcal{H}^{1}(A, X^*)$ *vanishes for every Banach A-bimodule X with* $a.x = \phi(a)x$ *for every* $a \in A$ *and* $x \in X$.

A Banach algebra A is called left character amenable, if A is left ϕ*-amenable for every* $\phi \epsilon \Delta(A) \cup \{0\}$*, where* $\Delta(A)$ *is the character space of A, that is, all non-zero multiplicative linear functional on A. see [14]. Nasr Isfahani et al. [9] showed that A is left* ϕ*-contractible if and only if there exists an element* $m \in A$ *such that* $am = \phi(a)m$*,and* $m(\phi) = 1$.

An analogue of the Kaniuth et al. formula for Banach homology have been defined and studied by authors in [13] and the notions like ϕ *-biflatness and* ϕ*-biprojectivity have been introduced. In fact a Banach algebra A is called* ϕ*-biflat(*ϕ*-biprojective) if there exists a bounded A-bimodule morphism* $\rho: A \to (A \otimes_p A)^{**}$ $(\rho: A \to (A \otimes_p A)$ *such that* $\tilde{\phi} o {\pi_A}^{**} o\rho = \phi$ $(\phi o \pi_A o\rho = \phi)$ *, respectively, where* $\tilde{\phi}$ *is an extension of* ϕ *to* A^{**} *which is defined by* $\tilde{\phi}(F) = F(\phi)$ *for every* $F \in A^{**}$*. They showed that for a locally compact group G,* $L^1(G)$ *is* ϕ*-biflat if and only if G is amenable. Also they showed that the Fourier algebra A(G) is* ϕ*-biprojective if and only if G is discrete.*

In this paper we continue to study the ϕ*-biprojectivity of certain Banach algebras. We investigate the* ϕ*-biprojectivity of* $L^1(G)$*, where G is a locally compact group. More generally we show that every Segal algebra S(G) with respect to* $L^1(G)$ *is* ϕ*-biprojective if and only if G is compact. We introduce the notion of character biprojectivity of Banach algebras and we show that the measure algebra M(G) is character biprojective if and only if G is finite. It will be shown that every commutative character biprojective Banach algebra has a discrete character space. We give some examples which show the differences between these concepts and the classical ones. Finally, we investigate* ϕ*-biprojectivity of triangular Banach algebras.*

2. ϕ-biprojectivity of abstract Segal algebras

In this section we find some conditions under which ϕ*-biprojectivity implies left* ϕ *-contractibility. Then we study* ϕ*-biprojectivity of Segal algebras.*

Proposition 2.1. *Let A be a ϕ-biprojective Banach algebra and let $L \subseteq \ker\phi$ be a closed ideal of A such that $\overline{AL} = L$. Then there exists a non-zero left A-module morphism $\theta: \frac{A}{L} \to A$ such that $\phi o\theta(x+L) = \phi(x)$.*

Proof. *Since A is ϕ-biprojective, there exists a bounded A-bimodule morphism $\rho: A \to A \otimes_p A$ such that $\phi \ o \ \pi_A o\rho(a) = \phi(a)$ for every $a \in A$. Let $q: A \to \frac{A}{L}$ be the quotient map. Then it is easy to see that $\rho_1 = (id_A \otimes q)o\rho: A \to A \otimes_p \frac{A}{L}$ is a bounded left A-module morphism. Since $\overline{AL} = L$, for every $l \in L$, there exist sequences $(l_n{}')$ in L and (a_n) in A such that $a_n l_n{}' \to l$, so continuity of ρ_1 implies that*

$$\rho_1(l) = (id_A \otimes q)o\rho(l) = (id_A \otimes q)o\rho\left(\lim_n a_n l_n{}'\right)$$
$$= \lim_n (id_A \otimes q)(\rho(a_n)l_n') = 0$$

Then ρ_1 induces a bounded left A-module morphism from $\frac{A}{L}$ into $A \otimes_p \frac{A}{L}$ which still is denoted by ρ_1. Let $\rho(x) = \sum_{i=1}^{\infty} a_i{}^x \otimes b_i{}^x$ for some nets $(a_i{}^x)_i$ and $(b_i{}^x)_i$ in A. Hence

$$\phi \otimes \bar{\phi} o\rho_1(x+L) = \sum_i \phi(a_i{}^x)\phi(b_i{}^x) = \phi o\pi_A o\rho(x) = \phi(x) \qquad (2.1),$$

where $\bar{\phi}$ is a character on $\frac{A}{L}$ defined by $\overline{\phi}(a+L) = \phi(a)$. Now define $\theta: (id_A \otimes \bar{\phi})o \ \rho_1: \frac{A}{L} \to A$, where $id_A \otimes \bar{\phi}$ a is defined by $id_A \otimes \bar{\phi}(a \otimes b + L) = \phi(b)$ for every a and b in A. Since ρ_1 is a left A-module morphism, we have

$$\theta(a.x+L) = id_A \otimes \bar{\phi}o \ \rho_1(a.x+L) = id_A \otimes \bar{\phi}o\rho_1(a.x)$$
$$= \sum_i^{\infty} a \ a_i{}^x \phi(b_i{}^x)$$
$$= a.id_A \otimes \bar{\phi}o \ \rho_1(x)$$
$$= a.\theta(x+L).$$

Therefore θ is a bounded left A-module morphism. Also by (2.1) we have

$$\phi \ o \ \theta(x+L) = \phi \otimes \bar{\phi}o \ \rho_1(x+L) = \phi(x).$$

for every $x \in A$. Thus θ is a non-zero left A-module morphism as required.

Let A be a Banach algebra and $\phi\epsilon\Delta(A)$. Then A is left ϕ-contractible (right ϕ-contractible) if and only if there exists an element m in A such that am= $\phi(a)$m (ma= $\phi(a)$m) and ϕ(m)=1 for every $a\epsilon A$, respectively [9].

Corollary 2.2. *Let A be a ϕ-biprojective Banach algebra and let* $\overline{\mathrm{A}\ \ker\phi} = \ker\phi$. *Then A is left ϕ-contractible.*

Proof. Let A be a ϕ-biprojective Banach algebra. By the previous Proposition, there exists a bounded left A-module morphism $\theta: \frac{A}{L} \to A$ such that $\phi o\theta(x + L) = \phi(x)$, where $L = \ker\phi$. Pick $x_0 \epsilon A$ such that $\phi(x_0) = 1$. Since $x_0^2 - x_0 \epsilon L, x_0^2 + x_0 = x_0 + L$. Thus

$$ax_0 + L = (a - \phi(a)x_0 + \phi(a)x_0)x_0 + L = (a - \phi(a)\ x_0^2) + \phi(a)\ x_0^2 + L$$

$$= \phi(a)\ x_0^2 + L$$

$$= \phi(a)x_0 + L.$$

Therefore

$$a.\theta(x_0 + L) = \theta(a.x_0 + L) = \phi\ (a)\ \theta(x_0 + L)$$

and $\phi\ \left(\theta(x_0 + L)\right) = \phi\ (x_0) = 1$. Since $\theta(x_0 + L)\epsilon A$, [9, Theorem 2.1] shows that A is left ϕ -contractible.

Note that the previous results also hold when we consider a right module action.

In the following example we show that the condition "$\overline{\mathrm{A}\ \ker\phi} = \ker\phi$" in Corollary 2.2 is a necessary condition.

***Example* 2.3.** *Let $A = \mathbb{C} \oplus \mathbb{C}$ be a two dimensional Banach algebra with product (a,b).(c,d)=(ad,bd). Consider a character $\phi: A \to \mathbb{C}$ defined by $\phi(a, b) = b$. So* $\ker\phi = \{(a, 0): a\epsilon\ \mathbb{C}\}$ *and* $\ker\phi = \{0\}$. *Now define $\rho: A \to\ A \otimes_p A$ by $\rho(a, b) = (a, b) \otimes (0,1)$ for every $(a, b)\epsilon\ A$. It is easy to see that ρ is a bounded A-bimodule morphism and $\phi o\pi_A o\rho(a, b) = \phi(a, b)$. Hence A is ϕ-biprojective. We claim that A is not left ϕ-contractible. Otherwise there exists an element $(m_1, m_2)\epsilon\ A$ such that $(a, b).(m_1, m_2) = \phi(a, b).(m_1, m_2)$ and $\phi(m_1, m_2) = 1$ for every $(a, b)\epsilon\ A$. But, $\phi(a, b).(m_1, m_2) = m_2(a, b) = \phi(m_1, m_2)(a, b) = (a, b)$ which implies that* dim*A=1, which yields a contradiction.*

Let $(A, \|.\|_A)$ be a Banach algebra. We say that a Banach algebra $(B, \|.\|_B)$ is an abstract Segal algebra with respect to A if

(i) B is a dense left ideal in A,

(ii) there exists $M>0$ such that $\|b\|_A \leq M\|b\|_B$ for every $b\epsilon B$,

(iii) there exists $C>0$ such that $\|ab\|_B \leq C\|a\|_A\|b\|_B$ for every $a\epsilon A$ and $b\epsilon B$.

If B is a dense ideal of A and $\|ba\|_B \leq C\|b\|_B\|a\|_A$ for the same C as in (iii), then B is called symmetric abstract Segal algebra. Note that the character space$\Delta(B) = \{\phi|_B: \phi\epsilon\Delta(A)\}$, see [1, Lemma 2.2].

Theorem 2.4. *Let A be a Banach algebra with a left approximate identity and a right approximate identity and let* $\phi\epsilon\Delta(A)$. *Suppose that B is a symmetric abstract Segal algebra with respect to A which has a left approximate identity and a right approximate identity. Then the followings are equivalent*

(i) A is ϕ*-biprojective,*

(ii)A is left and right ϕ*-contractible,*

(iii) B is left and right ϕ*-contractible,*

(iv) B is ϕ*-biprojective.*

Proof. *(i)⇒(ii) Since A has a left approximate identity and a right approximate identity, we have* $\overline{\mathrm{A}\ \ker\phi}=\ker\phi$ *and* $\overline{\ker\phi\mathrm{A}}=\ker\phi$. *Then by Corollary 2.2, A is left and right* ϕ*-contractible.*

(ii)⇒(iii) Let m_1 *and* m_2 *be a left* ϕ*-contraction and a right* ϕ*-contraction for A, respectively. Choose* $i_0\epsilon B$ *such that* $\phi(i_0)=1$. *Since B is an ideal in A, we have* $m_1i_0\epsilon B$. *It is easy to see that* $bm_1i_0=\phi(b)m_1i_0$*,and* $\phi(m_1i_0)=1$ *for every* $b\epsilon B$. *Then B is left* ϕ*-contractible. Also it is easy to see that* $m_2=m_2i_0\epsilon B$ *and* $\phi(m_2)=1$. *Hence B is left and right* ϕ*-contractible.*

(iii)⇒(iv) Let m_1 *and* m_2 *be a left and a right* ϕ*-contraction for B, respectively. Then* $M=m_1\otimes m_2$ *is in* $B\otimes_p B$ *and define* $\rho: B\rightarrow B\otimes_p B$ *by* $\rho(b)=b.M$ *for every* $b\epsilon B$. *It is easy to see that* ρ *is a bounded B-bimodule morphism and* $\phi o\pi_B o\rho=\phi$. *Hence B is* ϕ*-biprojective.*

(iv)⇒(i) Since B have a left and a right approximate identity, $\overline{\ker\phi\mathrm{B}}=\ker\phi$ *and* $\overline{\mathrm{B}\ \ker\phi}=\ker\phi$. *Hence by Corollary 2.2, B is left and right* ϕ*-contractible. Let* m_1 *and* m_2 *be a left and a right* ϕ*-contraction for B, respectively. Since B is a symmetric abstract Segal algebra,* m_1 *and* m_2 *are a left and a right* ϕ *-contraction for A, respectively. Using the similar arguments as in the proof of (iii)⇒(iv) one can see that A is φ-biprojective.*

We remind that A is a biprojective Banach algebra, if there exists a bounded A-bimodule morphism $\rho: A\rightarrow A\otimes_p A$ such that $\pi_A o\rho(a)=a$ for every $a\epsilon A$ [6]. It is easy to see that if A is biprojective, then A is ϕ -biprojective for every $\phi\epsilon\Delta(A)$.

We recall that, for a locally compact group G, a linear subspace $S(G)$ of $L^1(G)$ is said to be a Segal algebra on G if it satisfies the following conditions.

(i) $S(G)$ is dense in $L^1(G)$,

(ii)$S(G)$ with the norm $\|.\|_{S(G)}$ is a Banach space and $\|f\|_{L^1(G)}\leq\|f\|_{S(G)}$ for every $f\epsilon S(G)$

(iii) for every $f\epsilon S(G)$ and $y\epsilon G$we have $L_y f\epsilon S(G)$ and the map $y \longmapsto L_y f$ of G into $S(G)$ is continuous, where $L_y f(y) = f(y^{-1}x)$,

(iv) $\|L_y f\|_{S(G)} = \|f\|_{S(G)}$ for every $f\epsilon S(G)$ and $y\epsilon G$.

It is well-known that $S(G)$ has a left approximate identity. Also every Segal algebra is an abstract Segal algebra with respect to $L^1(G)$. For more information on Segal algebras see [11].

Theorem 2.5. *Let G be a locally compact group. Suppose that S(G) is a Segal algebra with respect to $L^1(G)$. Then the following statements are equivalent*

(i) $L^1(G)$ is ϕ -biprojective,

(ii)$L^1(G)$. is left and right ϕ -contractible,

(iii) S(G) is left and right ϕ -contractible,

(iv) S(G) is ϕ -biprojective,

(v) G is compact.

Proof. By the same arguments as in the proof of Theorem 2.4 the implications (i)⇒(ii)⇒(iii)⇒(iv) are clear.

(iv)⇒(v) Since S(G) has a left approximate identity, $\overline{S(G) \ker\phi} = \ker\phi$. *So by Corollary 2.2, S(G) is left ϕ-contractible. Since S(G) is a dense left ideal in $L^1(G)$, it is easy to see that $L^1(G)$ is also left ϕ-contractible, then G is compact, see [9, Theorem 6.1].*

(v)⇒(i) If G is compact, then by [6, Theorem 5.13] $L^1(G)$ is biprojective. Hence $L^1(G)$is ϕ-biprojective, for every $\phi\epsilon\Delta(L^1(G))$.

3. Character biprojectivity of some Banach algebras

In this section we study the notion of character biprojectivity for some Banach algebras.

Definition 3.1. *A Banach algebra A is called character biprojective, if for every $\phi\epsilon\Delta(A)$ there exists a bounded A-bimodule morphism $\rho_\phi: A \rightarrow A \otimes_p A$ such that $\phi o\pi_A o\rho_\phi(a) = a$ for each $a\epsilon A$.*

Theorem 3.2. *Let A be a Banach algebra and $\phi\epsilon\Delta(A)$. If A is ϕ-biprojective and $Z(A) \cap (A - \ker\phi)$ is a non-empty set, then $\{\phi\}$ is open in $\Delta(A)$ with respect to the w^*-topology.*

Proof. Suppose that A is ϕ-biprojective. Then there exists a bounded A-bimodule morphism $\rho: A \rightarrow A \otimes_p A$ such that $\phi o\pi_A o\rho = \phi$. Pick $a_0\epsilon Z(A) \cap (A - \ker\phi)$ such that $\phi(a_0) = 1$. Set $m = \rho(a_0)\epsilon A \otimes_p A$. Since ρ is a bounded A-bimodule morphism, we have$a.m = m.a$ and $\phi o\pi_A(m) = \phi(a_0) =$

1. *Define* $T: A \to A \otimes_p A$ *by* $T(a \otimes b) = \phi(b)a$ *for every* $a, b\epsilon A$. *It is easy to see that*

$$aT(m) = T(am) = T(ma) = \phi(a)T(m) \qquad (a\epsilon A\ )$$

and

$$\phi o T(m) = \phi o \pi_A(m) = \phi o \pi_A o \rho(a_0) = 1$$

Set $n = T(m) \in A \hookrightarrow A^{**}$. *Let* ψ *be an arbitrary element of* $\Delta(A)$ *such that* $\psi \neq \phi$. *Fix* a_ψ *in A with* $\phi(a_\psi) = 1$ *and* $\psi(a_\psi) = 0$. *So*

$$n(\psi) = a_\psi n(\psi) = n(\psi . a_\psi) = \psi(a_\psi)n(\psi) = 0 \qquad , n(\phi) = \phi(n) = 1$$

It follows that $n = \chi_{\{\phi\}}$, *where* χ *identifies the characteristic function at* $\{\phi\}$. *Therefore* $n\epsilon C(\Delta(A))$ *which imlpies that* $\{\phi\}$ *is an open set in* $\Delta(A)$.

Corollary 3.3. *Let A be a character biprojective Banach algebra with a left approximate identity. Then* $\Delta(A)$ *is discrete with respect to the* w^**-topology.*

Proof. Suppose that A is character biprojective. Since A has a left approximate identity, we have

$\overline{\mathrm{A}\ \ker\phi} = \ker\phi$ *for every* $\phi\epsilon\Delta(A)$. *Then by Corollary 2.2. character biprojectivity of A implies the left* ϕ*-contractiblity of A. Then there exists an element n* $\in$ *A such that* $an = \phi(a)n$*,and* $\phi(n) = 1$. *By the same argument as in the proof of Theorem 3.2. one can show that* $\{\phi\}$ *is an open set in* $\Delta(A)$, *that is*$\Delta(A)$*is discrete.*

Corollary 3.4. *Every commutative character biprojective Banach algebra has a discrete character space.*

Corollary3.5. *Let G be a locally compact group. Then the following statements are equivalent*

(i) $C_0(G)$ *is character biprojective,*

(ii)$C_0(G)$ *is* ϕ*-biprojective,*

(iii) G is discrete.

Proof. (i)⇒(ii) is clear.

(ii)⇒(iii) Let $C_0(G)$ *be* ϕ*-biprojective. By Theorem 3.2.* $\{\phi\}$ *is open in* $\Delta(C_0(G))$. *We may identify G with*$\Delta(C_0(G))$. *Therefore* $\{\phi\}$ *is open in G which implies that G is discrete.*

(iii)⇒(i) is clear.

Remark 3.6. *Consider the semigroup* $\mathbb{N}_\vee$, *with the semigroup operation* $m \vee \mathrm{n} = \max\{\mathrm{m}, \mathrm{n}\}$, *where m,n*$\in\mathbb{N}$. *Let* $\omega: \mathbb{N}_\vee \to [1, \infty)$ *be a weight function, that is, a function which satisfies* $\omega(m \vee \mathrm{n}) \leq \omega(m)\omega(n)$, *for every m and n in* $\mathbb{N}_\vee$,. *We denote by* $\ell^1(\mathbb{N}_\vee, \omega)$ *the set of all functions* $f: \mathbb{N}_\vee \to \mathbb{C}$ *such that* $\sum_{i=1}^{\infty} |f(i)|\omega(i) < \infty$. *With the norm*

$$\|f\|_{\omega} = \sum_{i=1}^{\infty}|f(i)|\omega(i) < \infty$$

and with the convolution product, $\ell^1(\mathbb{N}_{\vee}, \omega)$ *is a Banach algebra. The character space* $\Delta\left(\ell^1(\mathbb{N}_{\vee}, \omega)\right)$ *consists precisely of all functions* $\phi_{\mathrm{n}}: \ell^1(\mathbb{N}_{\vee}, \omega) \to \mathbb{C}$*defined by* $\phi_{\mathrm{n}}(\sum_{i=1}^{\infty}\alpha_{\mathrm{i}}\delta_{\mathrm{i}}) = \sum_{i=1}^{n}\alpha_{\mathrm{i}}$ *for every* $n \in \mathbb{N} \cup \{\infty\}$*. Hence* $\Delta\left(\ell^1(\mathbb{N}_{\vee}, \omega)\right) = \mathbb{N} \cup \{\infty\}$ *is compact, because* $\ell^1(\mathbb{N}_{\vee}, \omega)$ *is a commutative unital Banach algebra. For more information see [3].*

Lemma 3.7. $\ell^1(\mathbb{N}_{\vee}, \omega)$ *is not character biprojective.*

Proof. Assume towards a contradiction that $\ell^1(\mathbb{N}_{\vee}, \omega)$is character biprojective. Then Corollary 3.4 shows that the character space $(\ell^1(\mathbb{N}_{\vee}, \omega)\ \ )$ is discrete. But in the previous remark we showed that it is compact. Therefore $\Delta\left(\ell^1(\mathbb{N}_{\vee}, \omega)\right) = \mathbb{N} \cup \{\infty\}$ is finite, a contradiction.

We recall that *A* is left character contractible (left character amenable) if *A* is left ϕ-contractible (left ϕ-amenable) for every $\phi \epsilon \Delta(A) \cup \{0\}$., respectively. Right character contractibility and right character amenability are defined similarly. For more details we refer the reader to [9] and [14].

Proposition 3.8. *Let G be a locally compact group. Then M(G) is character biprojective if and only if G is finite.*

Proof. Sufficiency is clear in view of Theorem 2.5 and the fact that if G is finite, then M(G) =$\ell^1(G)$*). To show the converse statement, we note that by [9, Proposition 3.4] M(G) is 0-contractible. Using Corollary 2.2 character biprojectivity of M(G) implies the left character contractibility of M(G). Now apply [9, Corollary 6.2] to show that G is finite.*

Proposition 3.9. *Let G be a locally compact group. If* $M(G)^{**}$ *is character biprojective, then G is discrete and amenable.*

Proof. Since M(G) has a unit element e and since two maps $x \longmapsto xe$ and $x \longmapsto ex$ are w^*-continuous on $M(G)^{**}$,one can easily see that e is a unit for $M(G)^{**}$. Hence $M(G)^{**}$ is 0-contractible. Since $M(G)^{**}$is character biprojective, by [13, Lemma 3.2], there exists an element $m \epsilon M(G)^{**} \otimes_p M(G)^{**}$, such that $a.m = m.a$ and $\tilde{\phi} o \pi_{A^{**}}(m) = 1$ for every $a \epsilon$M(G), where $\phi \epsilon \Delta(M(G))$. Using [5, Lemma 1.7], one can assume that $m \epsilon (\mathrm{M(G)} \otimes_p \mathrm{M(G)})^{**}$ such that $a.m = m.a$ and $\tilde{\phi} o \pi_A{}^{**}(m) = 1$ for every $a \epsilon$M(G). Applying the same argument as in the proof of [13, Proposition 2.2] one can easily see that M(G) is left character amenable and right character amenable. Hence by [14, Corollary 2.5] G is discrete and amenable.

In the following example we show that

(i) there exists a ϕ-biprojective Banach algebra A which is not character biprojective,

(ii) there exists a character biprojective Banach algebra which is neither left ϕ-contractible nor is right ϕ-contractible for some character ϕ,

(iii) there exists a character biprojective Banach algebra which is not biprojective.

Example 3.10. *(i) Let G be an infinite compact group and* $\phi\epsilon\Delta(L^1(G))$. *Using Theorem 2.5,* $L^1(G)$ *has a left and a right* ϕ*-contraction. Since* $L^1(G)$ *is a closed ideal of M(G), we can assume that* $\phi\epsilon\Delta(M(G))$. *So one can easily see that M(G) has a left and a right* ϕ*-contraction. Using the similar argument as in the proof (the implication (iii)⇒(iv)) of Theorem 2.4, M(G) is* ϕ*-biprojective. But by Proposition 3.8, M(G) is not character biprojective.*

(ii) Let A and B be Banach algebras such that dimA>1 and dimB>1. For every a,b□A and x,y□B and for fix $\phi\epsilon\Delta(A)$ *and fix* $\psi\epsilon\Delta(B)$*,we define*

$$ab = \phi(a)b, \qquad xy = \psi(y)x.$$

With these products A and B are Banach algebras such that $\Delta(A) = \{\phi\}$ *and* $\Delta(A) = \{\psi\}$*, respectively.*

Pick $a_0\epsilon A$ *and* $x_0\epsilon B$ *such that* $\phi(a_0) = \psi(x_0) = 1$ *Define a bounded A-bimodule morphism*$\rho: A \rightarrow A \otimes_p A$ *by* $\rho(a) = a_0 \otimes a(a\epsilon A)$ *and define a bounded B-bimodule morphism* $g: B \rightarrow B \otimes_p B$ *by* $\rho(x) = x \otimes x_0(x\epsilon B)$.

Since for every a∈A, $\pi_A o\rho(a) = a$ *and for every x∈B,* $\pi_A og(x) = x$*, A and B are biprojective, respectively. Now [10, Proposition 2.4] implies that* $A \otimes_p B$ *is biprojective and so* $A \otimes_p B$ *is character biprojective. We claim that* $A \otimes_p B$ *is not left* $\phi \otimes \psi$*-contractible. Otherwise by [9, Theorem 3.14] A is left* ϕ*-contractible and B is left* ψ *-contractible. So there exists* $m\epsilon B$ *such that* $x.m = \psi(x).m$ *and* $\psi(m) = 1$ *for every x∈B which leads to* $x = xm = \psi(x)m$*, that is, dimB=1, a contradiction. With the similar method working with A instead of B one can show that* $A \otimes_p B$ *is not right* $\phi \otimes \psi$*-contractible.*

(iii) Consider the semigroup $\mathbb{N}_\wedge$ *with the semigroup operation m∧n=*min*{m,n} for evey m,n∈N. Using exactly the same argument of [, Example 5.3] we can show that* $\ell^1(\mathbb{N}_\wedge)$ *is character biprojective but this algebra is not biprojective, if* $\ell^1(\mathbb{N}_\wedge)$ *is biprojective, then* $\ell^1(\mathbb{N}_\wedge)$ *is biflat. Since* $\ell^1(\mathbb{N}_\wedge)$ *has a bounded approximate identity, biflatness of* $\ell^1(\mathbb{N}_\wedge)$ *implies the amenability of* $\ell^1(\mathbb{N}_\wedge)$. *Hence by [4, Theorem 2] the set of idempotents of* $\mathbb{N}_\wedge$ *is finite which is impossible.*

4. ϕ-biprojectivity of triangular Banach algebras

Let A be a Banach algebra and $\phi \epsilon \Delta(A)$. Suppose that X is a Banach left A-module. A non-zero linear functional $\psi \in X^*$ is called left ϕ-character if $\psi(a.x) = \phi(a)\ \ \psi(x)$ and it is called right ϕ-character if if $\psi(x.a) = \phi(a)\ \ \psi(x)$. A left and a right ϕ-character is called ϕ -character. Note that if A is a Banach algebra and $\phi \epsilon \Delta(A)$, then $\phi \otimes \phi$ on $A \otimes_p A$ and $\tilde{\phi}$ on A^{**} are ϕ-characters.

Note that if a Banach left A-module X has a left ϕ-character, then $A\cdot X \leq 0$. Since if $A\cdot X=\{0\}$, then for every a in A and x in X, we have $a\cdot x=0$, so $0 = \psi(ax) = \phi(a)\ \ \psi(x)$ which implies that $\psi(x) = 0$ for every x in X, which is a contradiction.

In this section we focus on triangular Banach algebras. We will present a number of examples of triangular Banach algebras which is not ϕ-biprojective.

Let A and B be Banach algebras and let X be a Banach A,B-module, that is, X is a Banach left A-module and a Banach right B-module that satisfy $(a\cdot x)\cdot b=a\cdot(x\cdot b)$ and $\|a\cdot x\cdot b\|\leq\|a\|\ \|x\|\ \|b\|$ for every $a\epsilon A,\ b\epsilon B\ \ and\ \ x\epsilon X$. Consider

$$T = Tri\ \ (A,B,X) = \left\{\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} : a\epsilon A, b\epsilon B\ \ , x\epsilon X\right\}$$

with the usual matrix operations and

$$\left\|\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\right\| = \|a\| + \|x\| + \|b\| \qquad (\ a\epsilon A, b\epsilon B\ \ , x\epsilon X)$$

T becomes a Banach algebra which is called triangular Banach algebra. Let $\phi \epsilon \Delta(B)$. We define a character $\psi_\phi \epsilon \Delta(T)$ via $\psi_\phi\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \phi(b)$ for every $a\epsilon A,\ b\epsilon B\ \ and\ \ x\epsilon X$.

Theorem 4.1. *Let T=Tri(A,B,X) be a triangular Banach algebra such that* $\overline{A^2} = A$ *and A·X=X·B=X. Suppose that* ϕ□Δ(B) *such that* $\overline{B\ \ \ker\phi} = \ker\phi$. *If one of the following holds*

(i) B is not left ϕ*-contractible,*

(ii)X has a right ϕ*-character,*

then T is not ψ_ϕ*-biprojective.*

Proof. *Assume towards a contradiction that T is a* ψ_ϕ*-biprojective Banach algebra. One can easily see that* $\overline{T\ \ ker\,\psi_\phi} = T$. *Hence by Corollary 2.2, T is left* ψ_ϕ*-contractible. Clearly* $I = \begin{pmatrix} 0 & X \\ 0 & B \end{pmatrix}$ *is a closed ideal of T and* $\psi_\phi\big|_I \neq 0$, *then by [9, Proposition 3.8] I is left* ψ_ϕ $-$*contractible. Thus there exists an* $m\epsilon I$ *such that* $am = \psi_\phi\big|_I(a)m$ *and* $\psi_\phi\big|_I(m) = 1$, *where* $a\epsilon I$. *Let* $x_0\epsilon X$ *and* $b_0\epsilon B$ *be such that* $m = \begin{pmatrix} 0 & x_0 \\ 0 & b_0 \end{pmatrix}$. *Then we have* $\psi_\phi\begin{pmatrix} 0 & x_0 \\ 0 & b_0 \end{pmatrix} = \phi(b_0) = 1$ *and*

$$\begin{pmatrix} 0 & x \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & x_0 \\ 0 & b_0 \end{pmatrix} = \psi_\phi\begin{pmatrix} 0 & x \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & x_0 \\ 0 & b_0 \end{pmatrix} = \phi(b)\begin{pmatrix} 0 & x_0 \\ 0 & b_0 \end{pmatrix} \qquad (4.1),$$

for each $x\epsilon X$ *and* $b\epsilon B$. *Using (4.1) we obtain* $bb_0 = \phi(b)b_0$,*and since* $\phi(b_0) = 1$, *we see that B is left* ϕ*-contractible, which contradicts (i).*

Now suppose that the statement (ii) holds. Then from (4.1) we have $xb_0 = \phi(b)x_0$ *and from (ii) there exists a right* ϕ*-character* $\eta \in X^*$ *such that* $\eta(xb_0) = \eta(x) = \phi(b)\eta(x_0)$ *for every* $b\epsilon B$ *and* $x\epsilon X$, *which is impossible (take* $b\epsilon \ker\phi$*), that is, (ii) does not hold.*

Corollary 4.2. *Suppose that A is a unital Banach algebra and* $\phi\epsilon\Delta(A)$. *Let* $T = Tri(A, A, A \otimes_p A)$ *and* $\psi_\phi\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \phi(b)$. *Then T is not* ψ_ϕ*-biprojective.*

Proof. Since $\phi \otimes \phi\epsilon\Delta(A \otimes_p A)$*is a* ϕ*-character, by Theorem 4.1 the proof is complete.*

Corollary 4.3. *Let A be a Banach algebra with a left identity and* $\phi\epsilon\Delta(A)$. *If T=Tri(A,A,A) and* ψ_ϕ $\epsilon\Delta(T)$ *is defined by* $\psi_\phi\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \phi(b)$ *for every a,x and b in A. Then T is not* ψ_ϕ*-biprojective.*

Similarly one can show that for a Banach algebra *A* with a right identity, *T=Tri*(*A,A,A*) is not ψ_ϕ-biprojective, where $\psi_\phi\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \phi(a)$ and $\phi\epsilon\Delta(A)$.

Example 4.4. *Let G be a locally compact group. It is well-known that* $L^1(G)^{**}$ *has a right identity. We show that* $Tri(L^1(G)^{**}, L^1(G)^{**}, L^1(G)^{**})$ *is not biprojective. Our proof proceeds by contradiction. Suppose that T is biprojective. Then T is* ψ_ϕ *-biprojective, where* $\psi_\phi\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \phi(a)$ *and* $\phi\epsilon\Delta(L^1(G)^{**})$. *Since* $\overline{\ker\psi_\phi T} = \ker\psi_\phi$, *by the same argument as in the proof of Corollary 2.2, T is right* ψ_ϕ*-contractible. Set* $I = \begin{pmatrix} L^1(G)^{**} & L^1(G)^{**} \\ 0 & 0 \end{pmatrix}$, *it is easy to see that I is a closed ideal of T and it is right* ψ_ϕ*-contractible. A similar argument as in the proof of Theorem 4.1 yields a contradiction.*

Example 4.5. *Let S be a right-zero semigroup with |S|≥2, that is, st=t for every s,t∈S. Let* ϕ *be the augmentation character on* $\ell^1(S)$, *that is, the map* $\phi\colon \ell^1(S) \to \mathbb{C}$ *given by*

$$\phi\left(\sum_{s\in S}\alpha(s)\delta_s\right) = \sum_{s\in S}\alpha(s),$$

where $(\alpha(s))_{s\in S}$ *is a net in* $\mathbb{C}$ *and* δ_s *is the characteristic function at {s}. It is easy to check that f*g=* ϕ*(f)g for every* $f, g\epsilon\ell^1(S)$. *Hence* $\ell^1(S)$ *has a left identity. One can easily see that* $\ell^1(S)$ *is biprojective. But Corollary 4.3 shows that* $T = Tri(\ell^1(S), \ell^1(S), \ell^1(S)$ *is not* ψ_ϕ*-biprojective and so it is not biprojective.*

Corollary 4.6. *Let G be a discrete group. Then* $T = Tri(\ell^1(G), \ell^1(G), \ell^1(G)^{**})$ *is not* ψ_ϕ*-biprojective for every* $\phi \epsilon \Delta(\ell^1(G))$.

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