A NOTE ON PROPERTY $\mathcal{F}$ AND APPROXIMATE BIPROJECTIVITY OF THE FOURIER ALGEBRA

Narjes Razi[1], Mehdi Rostami[2], Amir Sahami[3]

Suppose that G is a locally compact group. We show that if the Fourier algebra $A(G)$ has the property $\mathcal{F}$, then either $A(G)$ is pseudo-amenable or G is not amenable and for every open subgroup H in G, $A(H)$ has the property $\mathcal{F}$. This answer the question posed in [23] partially. Furthermore, we obtain the similar result for approximately biprojective Fourier algebra.

Keywords: Approximate biprojectivity, Property $\mathcal{F}$, Fourier algebra
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1. Introduction

Helemskii in [9, 10] introduced two new homological notions, namely biprojectivity and biflatness, for studying and analysing Banach algebras; see also [17]. He examined them for various Banach algebras, including the group algebra $L^1(G)$. In particular, he showed that $L^1(G)$ is biprojective if and only if G is compact [10, Theorem 5.13]. These properties are closely connected to the notions amenability and contractibility, as defined by Johnson [13]. Specifically, a Banach algebra A is contractible if and only if it is both biprojective and unital [11, Theorem VII.1.63]. Additionally, A is amenable if and only if it is biflat and possesses a bounded approximate identity [10, Theorem VII.2.20]. These concepts have been extensively explored in the context of various Banach algebras, including those linked to locally compact groups, Segal algebras, Fourier algebras and semigroup algebras; see [18, 24, 21] for further information. Furthermore, approximate biprojectivity with respect to a closed ideal in a Banach algebra introduced and studied in [19]. Biprojectivity of generalized module extension Banach algebras and second dual of Banach algebras had been studied in [4].

The Fourier algebra $A(G)$, introduced by P. Eymard [5], has been widely studied with regard to its homological and cohomological properties. When G is abelian, by the Fourier transform we may identify $A(G)$ with $L^1(\hat{G})$, where $\hat{G}$ is the dual group of G. Significant contributions in this area were made by H. Leptin [15], B. E. Forrest [6], Ruan [16] and F. Ghahramani [7]. Leptin proved that a locally compact group G is amenable if and only if $A(G)$ admits a bounded approximate identity [15]. Forrest and Runde showed that $A(G)$ is amenable if and only if G is almost abelian, meaning that G contains an abelian subgroup of finite index [6, Theorem 2.3]. In a related result, Z.-J. Ruan demonstrated that $A(G)$ is operator amenable precisely when G is amenable [16]. Biprojectivity and biflatness of Fourier algebras were investigated by V. Runde in [18].

[1]Department of Mathematics Education, Farhangian University, P.O. Box 14665-889, Tehran, Iran, email: Razina@cfu.ac.ir
[2]Department of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), Iran, e-mail (Corresponding Author): mross@aut.ac.ir
[3]Faculty of Basic sciences, Department of Mathematics, Ilam University, P.O.Box 69315-516, Ilam, Iran, e-mail: a.sahami@ilam.ac.ir

The notion of approximate biprojectivity was later introduced and studied by Y. Zhang [25]. A Banach algebra A is said to be approximately biprojective if there exists a net of bounded A-bimodule morphisms $\vartheta_\alpha : A \to A\hat{\otimes}A$ such that

$$\pi_A \circ \vartheta_\alpha(a) \longrightarrow a \quad (a \in A),$$

where $\pi_A : A\hat{\otimes}A \to A$ is the multiplication map defined by $\pi_A(a \otimes b) = ab$ and $\hat{\otimes}$ is the Banach space projective tensor product. This property has been further investigated in works such as [1, 22].

Recently, A. Sahami *et al.* introduced the concept of property $\mathcal{F}$, exploring its relationship with amenability. They showed that a unital Banach algebra is amenable if and only if it satisfies property $\mathcal{F}$ [3, Corollary 3.4]. Sahami *et al.* in [23], studied the relation between Property $\mathcal{F}$ with some other cohomological concepts. Also they investigated property $\mathcal{F}$ for some classes of Banach algebras, such as semigroup algebras, triangular matrix algebras and θ-Lau product algebras. At the end of the paper, they asked a question on property $\mathcal{F}$ for Fourier algebras.

In this paper, we study approximate biprojectivity and property $\mathcal{F}$ concerning the Fourier algebra $A(G)$, where G is a locally compact group and we provide a partial response to the question raised in [23]. We prove that if $A(G)$ satisfies property $\mathcal{F}$, then either $A(G)$ is pseudo-amenable, or G is non-amenable and for every open subgroup $H \subseteq G$, the Fourier algebra $A(H)$ also satisfies property $\mathcal{F}$. A similar result is also established for the approximate biprojectivity of $A(G)$.

2. Some homological results among the Fourier algebras

In this section, we study property $\mathcal{F}$ and approximate biprojectivity of the Fourier algebra $A(G)$.

Definition 2.1. *A Banach algebra A has property $\mathcal{F}$ if there is a not necessarily bounded net $(\rho_\alpha)_{\alpha\in I} \subseteq B(A, (A\hat{\otimes}A)^{**})$ of A-bimodule morphisms such that*

$$\pi_A^{**} \circ \rho_\alpha(a) - \kappa_A(a) \longrightarrow 0 \qquad (a \in A),$$

where $\kappa_A : A \longrightarrow A^{**}$ is the canonical embedding.

Lemma 2.1. *Let A be a Banach algebra. If A has property $\mathcal{F}$ then there is a not necessarily bounded net $(\rho_\alpha)_{\alpha\in I} \subseteq B((A\hat{\otimes}A)^*, A^*)$ of A-bimodule morphisms such that*

$$\rho_\alpha \circ \pi_A^*(\varphi) - \varphi \longrightarrow 0 \qquad (\varphi \in A^*).$$

Proof. Suppose that $(\rho_\alpha)_{\alpha\in I} \subseteq B(A, (A\hat{\otimes}A)^{**})$ is a net of A-bimodule morphisms such that

$$\pi_A^{**} \circ \rho_\alpha(a) - \kappa_A(a) \longrightarrow 0 \qquad (a \in A).$$

We define $\tilde{\rho_\alpha} : (A\hat{\otimes}A)^* \longrightarrow A^*$ to be the restriction of ρ_α^* to $(A\hat{\otimes}A)^*$; that is,

$$\tilde{\rho_\alpha} = \rho_\alpha^* \circ \kappa_{(A\hat{\otimes}A)^*}.$$

Certainly, $\tilde{\rho_\alpha}$ is a net of A-bimodule morphisms and for every $a \in A$ and $\varphi \in A^*$, we have

$$\lim_\alpha \langle \tilde{\rho_\alpha}(\pi_A^*(\varphi)), a\rangle = \lim_\alpha \langle \rho_\alpha(a), \pi_A^*(\varphi)\rangle = \lim_\alpha \langle \pi_A^{**}(\rho_\alpha(a)), \varphi\rangle = \langle \varphi, a\rangle.$$

This completes the proof. □

In what follows, for a subgroup $H \subseteq G$, we define the map $\eta_H : A(G) \to A(G)$ by $\eta_H(w) = \chi_H \cdot w$. According to [12, Theorem 1], this map η_H sends $A(G)$ onto $A(H)$, a result known as Herz's restriction theorem. This theorem serves as a fundamental tool in the analysis of the Fourier algebra. The map η_H enjoys the following properties:

(i) η_H is an algebra homomorphism;
(ii) η_H is continuous;

(iii) $\eta_H \circ \pi_{A(G)} = \pi_{A(H)} \circ (\eta_H \otimes \eta_H)$.

We also recall that a Banach algebra A is said to be pseudo-amenable if there exists a net $(m_\alpha) \subseteq A\hat{\otimes}A$ such that

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0 \quad \text{and} \quad \pi_A(m_\alpha)a \to a \quad \text{for all } a \in A.$$

For further details, see [8]. In the sequel, Haar measure on H is the restriction of Haar measure on G to H.

Theorem 2.1. *Suppose that G is a locally compact group and $A(G)$ has property $\mathcal{F}$. Then one of the following holds:*

(i) *$A(G)$ is pseudo-amenable.*
(ii) *G is not amenable and for every open subgroup H in G, $A(H)$ has property $\mathcal{F}$.*

Proof. Since $A(G)$ has property $\mathcal{F}$, by Lemma 2.1 there exists a not necessarily bounded net $(\rho_\alpha)_{\alpha\in I} \subseteq B((A(G)\hat{\otimes}A(G)^*), A(G)^*)$ of $A(G)$-bimodule morphisms such that

$$\rho_\alpha \circ \pi^*_{A(G)}(\varphi) - \varphi \longrightarrow 0 \qquad (\varphi \in A(G)^*).$$

Suppose that G is amenable. Then [15] implies that $A(G)$ has a bounded approximate identity which is central and we denote it by $(e_\lambda)_{\lambda\in\Lambda}$. Thus, for every $w \in A(G)$ we have

$$w \cdot \rho^*_\alpha \circ \kappa_{A(G)}(e_\lambda) - \rho^*_\alpha \circ \kappa_{A(G)}(e_\lambda) \cdot w = \rho^*_\alpha \circ \kappa_{A(G)}(w \cdot e_\lambda - e_\lambda \cdot w) = 0,$$

and for every $\varphi \in A(G)^*$

$$\begin{aligned}\lim_\alpha \lim_\lambda \langle \pi^{**}_{A(G)}(\rho^*_\alpha(\kappa_{A(G)}(e_\lambda))) \cdot w, \varphi\rangle &= \lim_\alpha \lim_\lambda \langle \rho_\alpha(\pi^*_{A(G)}(w \cdot \varphi)), e_\lambda\rangle \\ &= \lim_\alpha \langle \rho_\alpha(\pi^*_{A(G)}(\varphi)), w\rangle \\ &= \langle \varphi, w\rangle.\end{aligned}$$

Let $E = \Lambda \times I^\Lambda$ be directed by the product order, defined as

$$(\lambda_1, f_1) \leq_E (\lambda_2, f_2) \quad \Leftrightarrow \quad \lambda_1 \leq_\Lambda \lambda_2 \quad \text{and} \quad f_1 \leq_{I^\Lambda} f_2 \quad (\lambda_1, \lambda_2 \in \Lambda,\ f_1, f_2 \in I^\Lambda),$$

where I^Λ denotes the set of all functions from Λ into I, and $f_1 \leq_{I^\Lambda} f_2$ means that $f_1(\lambda) \leq_I f_2(\lambda)$ for every $\lambda \in \Lambda$. For each $\beta = (\lambda, (\alpha_{\lambda'})) \in E$, define

$$m_\beta = \rho^*_{\alpha_\lambda} \circ \kappa_{A(G)}(e_\lambda) \in (A(G)\hat{\otimes}A(G))^{**}.$$

Then for all $w \in A(G)$, we have

$$w \cdot m_\beta = m_\beta \cdot w.$$

Moreover, by the iterated limit theorem [14, p. 69], we may assume that

$$\pi^{**}_{A(G)}(m_\beta)w \longrightarrow \kappa_{A(G)}(w),$$

in the weak*-topology of $A(G)^{**}$. By Goldstein's theorem, we can approximate each m_β by elements in $A(G)\hat{\otimes}A(G)$, replacing the weak* convergence above with weak convergence. Then, by Mazur's lemma, it follows that $A(G)$ is pseudo-amenable; see [8, Proposition 2.3].

(ii) Assume that G is not amenable and H is an open subgroup of G. Since $A(G)$ has property $\mathcal{F}$, there is a not necessarily bounded net $(\rho_\alpha)_{\alpha\in I} \subseteq B(A(G), (A(G)\hat{\otimes}A(G))^{**})$ of $A(G)$-bimodule morphisms such that

$$\pi^{**}_{A(G)} \circ \rho_\alpha(w) - \kappa_{A(G)}(w) \longrightarrow 0 \qquad (w \in A(G)).$$

Define the map $\theta : A(H) \to A(G)$ by

$$\theta(u)(g) = \begin{cases} u(g) & g \in H, \\ 0 & g \notin H, \end{cases}$$

for $u \in A(H)$ and $g \in G$. Certainly, θ is an algebra continuous homomorphism [5]. Now, we can define a net $\mu_\alpha : A(H) \to (A(H)\hat{\otimes}A(H))^{**}$ by $\mu_\alpha = (\eta_H \otimes \eta_H)^{**} \circ \rho_\alpha \circ \theta$. On the other hand, we have the following diagram which commutes.

$$\begin{array}{ccc} A(H)^{**} & \xleftarrow{\pi^{**}_{A(H)}} & (A(H)\hat{\otimes}A(H))^{**} \\ \Big\uparrow \eta_H^{**} & & \Big\uparrow (\eta_H \otimes \eta_H)^{**} \\ A(G)^{**} & \xleftarrow{\pi^{**}_{A(G)}} & (A(G)\hat{\otimes}A(G))^{**} \end{array}$$

So we imply that

$$\pi^{**}_{A(H)} \circ (\eta_H \otimes \eta_H)^{**} = \eta_H^{**} \circ \pi^{**}_{A(G)}.$$

Since $\eta_H \circ \theta = id_{A(H)}$, for every $u \in A(H)$ we obtain

$$\begin{aligned} \lim_\alpha \pi^{**}_{A(H)} \circ \mu_\alpha(u) =& \lim_\alpha \pi^{**}_{A(H)} \circ (\eta_H \otimes \eta_H)^{**} \circ \rho_\alpha \circ \theta(u) \\ =& \lim_\alpha \eta_H^{**} \circ \pi^{**}_{A(G)} \circ \rho_\alpha \circ \theta(u) \\ =& \eta_H^{**} \circ \kappa_{A(G)} \circ \theta(u) \\ =& \eta_H \circ \theta(u) = u. \end{aligned}$$

Let $u, v \in A(H)$, $w \in A(G)$ and $\Phi \in (A(H)\widehat{\otimes}A(H))^*$. One can easily check that $\eta_H(w) \cdot u = \eta_H(w \cdot \theta(u))$ and $u \cdot \eta_H(w) = \eta_H(\theta(u) \cdot w)$ and this implies that $(\eta_H \otimes \eta_H)^*(\Phi \cdot u) = (\eta_H \otimes \eta_H)^*(\Phi) \cdot \theta(u)$ and $(\eta_H \otimes \eta_H)^*(u \cdot \Phi) = \theta(u) \cdot (\eta_H \otimes \eta_H)^*(\Phi)$. Therefore, we obtain

$$\begin{aligned} \langle \mu_\alpha(uv), \Phi \rangle =& \langle (\eta_H \otimes \eta_H)^{**} \circ \rho_\alpha \circ \theta(uv), \Phi \rangle \\ =& \langle \rho_\alpha \circ \theta(uv), (\eta_H \otimes \eta_H)^*(\Phi) \rangle \\ =& \langle \rho_\alpha(\theta(u)\theta(v)), (\eta_H \otimes \eta_H)^*(\Phi) \rangle \\ =& \langle \theta(u) \cdot \rho_\alpha \circ \theta(v), (\eta_H \otimes \eta_H)^*(\Phi) \rangle \\ =& \langle \rho_\alpha \circ \theta(v), (\eta_H \otimes \eta_H)^*(\Phi \cdot u) \rangle \\ =& \langle (\eta_H \otimes \eta_H)^{**} \circ \rho_\alpha \circ \theta(v), \Phi \cdot u \rangle \\ =& \langle \mu_\alpha(v), \Phi \cdot u \rangle \\ =& \langle u \cdot \mu_\alpha(v), \Phi \rangle. \end{aligned}$$

Thus, $\mu_\alpha(uv) = u \cdot \mu_\alpha(v)$. Similarly, we may obtain $\mu_\alpha(uv) = \mu_\alpha(u) \cdot v$. Moreover, since the maps θ, ρ_α and η_H are continuous, therefore $(\mu_\alpha)_{\alpha \in I}$ is a net of continuous $A(H)$-bimodule morphisms. Thus $A(H)$ has property $\mathcal{F}$. □

Since approximate biprojectivity is a stronger condition than property $\mathcal{F}$, the conclusions of the aforementioned theorem also holds when $A(G)$ is approximately biprojective. Moreover, if $A(G)$ is approximately biprojective, this necessarily implies that G is discrete.

We also recall that a Banach algebra A is said to be pseudo-contractible if there exists a net $(m_\alpha) \subseteq A\hat{\otimes}A$ such that

$$a \cdot m_\alpha = m_\alpha \cdot a \quad \text{and} \quad \pi_A(m_\alpha)a \to a \quad \text{for all } a \in A.$$

For further information, see [8].

Corollary 2.1. *Assume that G is a locally compact group such that $A(G)$ is approximately biprojective. Then G is discrete and one of the following holds:*

(i) *$A(G)$ is pseudo-contractible.*
(ii) *G is not amenable and for every subgroup H in G, $A(H)$ has property $\mathcal{F}$.*

Proof. Suppose that $\varphi \in \Delta(A(G))$. Since $A(G)$ is approximately biprojective, $A(G)$ is left φ-contractible [20, Lemma 2.5]. Now, [2, Theorem 3.5] implies that G is discrete. Since approximate biprojectivity implies property $\mathcal{F}$, applying Theorem 2.1, either $A(G)$ is pseudo-amenable or G is not amenable and for every (open) subgroup H in G, $A(H)$ has property $\mathcal{F}$. If $A(G)$ is pseudo-amenable then it has a (central) approximate identity and so applying [8, Proposition 3.8] implies that $A(G)$ is pseudo-contractible. □

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4. Conclusions

In this paper we studied property two notions of $\mathcal{F}$ and approximate biprojectivity for a class of Banach algebras, namely the Fourier algebras. As a main result, we show that if the Fourier algebra $A(G)$ has the property $\mathcal{F}$, then either $A(G)$ is pseudo-amenable or G is not amenable and $A(H)$ has the property $\mathcal{F}$ for every open subgroup H in G. This partially addresses the question raised in [23].

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