FLEXIBLE WEIBULL GENERATED FAMILY OF DISTRIBUTIONS: CHARACTERIZATIONS, MATHEMATICAL PROPERTIES AND APPLICATIONS

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We introduce a new class of distributions called the flexible Weibull generated family. We obtain some of its mathematical properties. The special models of this family provide bathtub-shaped, decreasing-increasing, increasing-decreasing-increasing, decreasing-increasing-decreasing, monotone, unimodal and bimodal hazard functions. Some useful characterizations are presented. The maximum likelihood method is adopted for estimating the model parameters. We assess the performance of the maximum likelihood estimators by means of a graphical simulation study. The flexibility and importance of the proposed model are illustrated by means of its application to three real data sets.

Keywords: Order Statistics, Maximum Likelihood Estimation, Quantile function, Generating Function, Moments.

MSC2010: 47N30; 60E05; 62J05; 97K70; 97K80.

1. Introduction

Fitting densities to data sets has a long history. Traditionally, families of curves have been developed to aid in fitting densities. Statistical distributions are very useful in describing and predicting real world phenomena. Numerous classical distributions have been extensively used over the past decades for modeling data in several areas, in particular, in reliability engineering, survival analysis, demography, actuarial study and others. Recent developments address introduction of new families that extend well-known distributions and, at the same time, provide great flexibility in modeling real data.

[1] introduced the flexible Weibull (FW) distribution having two parameters $\alpha > 0$ and $\beta > 0$. A random variable (RV) T is said to have the FW(α, β) distribution if its cumulative distribution function (CDF) is given by $\Pi_{\alpha,\beta}(t) = 1 - \exp\left[-\exp\left(\alpha t - \frac{\beta}{t}\right)\right], t \geq 0$. We define the flexible Weibull generated (FW-G) family of distribution by replacing t with $\frac{G_\psi(x)}{\bar{G}_\psi(x)}$ in $\Pi_{\alpha,\beta}(t)$ to obtain

$$F_{\alpha,\beta,\psi}(x) = 1 - \exp\left\{-\exp\left[\frac{\alpha\, G_\psi(x)}{\bar{G}_\psi(x)} - \frac{\beta\, \bar{G}_\psi(x)}{G_\psi(x)}\right]\right\}, x \in \mathbb{R}, \tag{1.1}$$

where $\alpha, \beta > 0$ are two shape parameters, $\bar{G}_\psi(x) = 1 - G_\psi(x)$ and is the vector of parameters for the baseline CDF $G(.)$. Probability density function (PDF) is given by

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$$f_{\alpha,\beta,\psi}(x) = g_\psi(x)\left[\frac{\alpha}{\bar{G}_\psi(x)^2}+\frac{\beta}{G_\psi(x)^2}\right]\exp\left\{\begin{array}{c}\frac{\alpha\, G_\psi(x)}{\bar{G}_\psi(x)}-\frac{\beta\,\bar{G}_\psi(x)}{G_\psi(x)}\\ -\exp\left[\frac{\alpha\, G_\psi(x)}{\bar{G}_\psi(x)}-\frac{\beta\,\bar{G}_\psi(x)}{G_\psi(x)}\right]\end{array}\right\}, \qquad (\mathbf{1}.2)$$

This paper is organized as follows. In Section 2, some useful characterizations are presented. Two special models are given in Section 3. Section 4 provides useful linear representations for the CDF and PDF of the new family. In Section 5, we assess the performance of the different estimators by the least-squares estimators, weighted least-squares estimators, Cramer-von-Mises Estimators and Anderson-Darling estimators. In Section 6, the potentiality of the proposed model is introduced empirically by means of three real data sets. In Section 7, we offer some concluding remarks.

2. Characterizations results

This section is devoted to the characterizations of the FW-G distribution based on the ratio of two truncated moments. Note that our characterizations can be employed also when the CDF does not have a closed form. We would also like to mention that due to the nature of FW-G distribution, our characterizations may be the only possible ones. We are concerned with the characterizations of FW-G distribution based on a simple relationship between two truncated moments. Our first characterization employs a theorem due to [2], see Theorem 1 of [5] and [4]. The result, however, holds also when the interval H is not closed, since the condition of the Theorem is on the interior of H.

Proposition 2.1. Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let

$$\begin{aligned} q(x) &= \exp\left[-\left\{\begin{array}{c}\frac{2\alpha G_\psi(x)}{\overline{G}_\psi(x)}-\frac{2\beta\overline{G}_\psi(x)}{G_\psi(x)}\\ -\exp\left(\frac{\alpha G_\psi(x)}{\overline{G}_\psi(x)}-\frac{\beta\overline{G}_\psi(x)}{G_\psi(x)}\right)\end{array}\right\}\right], \text{ and}\\ q_2(x) &= q_1(x)\exp\left(-\frac{\alpha G_\psi(x)}{\overline{G}_\psi(x)}+\frac{\beta\overline{G}_\psi(x)}{G_\psi(x)}\right) \text{ for } x\in\mathbb{R}. \end{aligned}$$

The random variable X has PDF (2) if and only if the function η defined in Theorem 1 is of the form

$$\eta(x)=\frac{1}{2}\exp\left(-\frac{\alpha G_\psi(x)}{\overline{G}_\psi(x)}+\frac{\beta\overline{G}_\psi(x)}{G_\psi(x)}\right), \quad x\in\mathbb{R}.$$

Proof. Suppose the random variable X hasPDF (2), then

$$(1-F(x))\,E[q_1(X)\mid X\geq x]=\exp\left(-\frac{\alpha G_\psi(x)}{\overline{G}_\psi(x)}+\frac{\beta\overline{G}_\psi(x)}{G_\psi(x)}\right), \quad x\in\mathbb{R},$$

$$(1-F(x))\,E[q_2(X)\mid X\geq x]=\frac{1}{2}\exp\left(-\frac{2\alpha G_\psi(x)}{\overline{G}_\psi(x)}+\frac{2\beta\overline{G}_\psi(x)}{G_\psi(x)}\right), \ x\in\mathbb{R},$$

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{q_1(x)}{2}\exp\left(-\frac{\alpha G_\psi(x)}{\overline{G}_\psi(x)}+\frac{\beta\overline{G}_\psi(x)}{G_\psi(x)}\right)<0\ \forall\ x\in\mathbb{R}.$$

Conversely, if η is of the above form, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=g_\psi(x)\left[\frac{\alpha}{\overline{G}_\psi(x)^2}+\frac{\beta}{G_\psi(x)^2}\right], x\in\mathbb{R}.$$

Now, according to Theorem 2, X has density (2). □

Corollary 2.1. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in **2**.1 The random variable X hasPDF (2) if and only if there exist functions q_2 and η defined in Theorem 2 satisfying the following differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = g_\psi(x)\left[\frac{\alpha}{\overline{G}_\psi(x)^2} + \frac{\beta}{G_\psi(x)^2}\right], x \in \mathbb{R}.$$

Corollary 2.2. The general solution of the differential equation in Corollary **2**.1 is

$$\eta(x) = \exp\left(\frac{\alpha G_\psi(x)}{\overline{G}_\psi(x)} - \frac{\beta \overline{G}_\psi(x)}{G_\psi(x)}\right) \times \left[\begin{array}{c} -\int g(x;\psi)\left[\frac{\alpha}{\overline{G}_\psi(x)^2} + \frac{\beta}{G_\psi(x)^2}\right] \times \\ \exp\left(-\frac{\alpha G_\psi(x)}{\overline{G}_\psi(x)} + \frac{\beta \overline{G}_\psi(x)}{G_\psi(x)}\right)(q_1(x))^{-1} q_2(x)\, dx + D \end{array}\right],$$

where D is a constant. We like to point out that one set of functions satisfying the above differential equation is given in Proposition **2**.1 with $D = 0$. Clearly, there are other triplets (q_1, q_2, η) which satisfy conditions of Theorem1.

3. Linear representation

Firstly, using taylor expansion, the CDF of the FW-G family in (**1**.1) can be expressed as

$$F_{\alpha,\beta,\psi}(x) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i!} \exp\left[\frac{\alpha\, G_\psi(x)}{\bar{G}_\psi(x)} - \frac{\beta\, \bar{G}_\psi(x)}{G_\psi(x)}\right].$$

Secondly, we can prove that $\exp\left[-\frac{i\,\beta\,\bar{G}_\psi(x)}{G_\psi(x)}\right] = \sum_{j=0}^{\infty} b_j\, G_\psi(x)^j$, where

$$b_j = \exp(\, i\alpha) \sum_{k=0}^{\infty}\sum_{l=k}^{\infty} \frac{(-1)^{k+l+j}\,(i\,\alpha)^k}{k!}\binom{-k}{l}\binom{l}{j},$$

since

$$\exp\left[\frac{-i\,\alpha\,\bar{G}_\psi(x)}{G_\psi(x)}\right] = \exp\{i\,\alpha\}\, \exp\left[\frac{-i\,\alpha}{G_\psi(x)}\right] = \sum_{j=0}^{\infty} b_j\, G_\psi(x)^j.$$

Note that $\exp\left[\frac{i\,\alpha\,G_\psi(x)}{\bar{G}_\psi(x)}\right] = \sum_{j=0}^{\infty} a_j\, G_\psi(x)^j$, where

$$a_j = \sum_{(l,k)\in I_j} \frac{(i\,\alpha)^k\,(-1)^l}{k!}\binom{-k}{l}|_{(I_j=\{(l,k)\in 0,1,2,\cdots|j=k+l\})},$$

then the CDF of the FW-G family can be expressed as

$$F_{\alpha,\beta,\psi}(x) = \sum_{j=0}^{\infty} d_j\, G_\psi(x)^j = \sum_{j=0}^{\infty} d_j\, H_j(x;\psi), \qquad (\mathbf{3}.1)$$

where $H_\gamma(x)$ is the CDF of the exp-G family with power parameter γ and $c_j = \sum_{m=0}^{j} a_m\, b_{j-m}$, for any $j \geq 0$ and $d_j = \sum_{i=1}^{\infty}\left[(-1)^{i+1}\, c_j/i!\right]$ .Upon diffrentiating (**3**.1), the density function of the FW-G family can be expressed as

$$f_{\alpha,\beta,\psi}(x) = \sum_{j=0}^{\infty} d_{j+1}\, h_{j+1}(x;\psi), \qquad (\mathbf{3}.2)$$

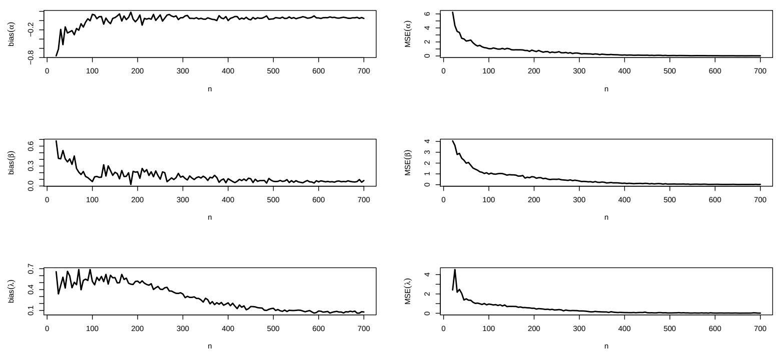

FIGURE 1. Simulation results of the special FWL distribution

where $h_{\gamma+1}(x) = (\gamma+1)\, g(x)\, G(x)^{\gamma}$ is the PDF of the exp-G family with power parameter γ. Equations (**3**.1) and (**3**.2) are the main results of this section. Many mathematical properties can be derived using (**3**.2) such as moments, moment generating function, incomplete moments, moment of the residual life and moment of the reversed residual life.

4. Simulation study

In this section a graphical Monte Carlo simulation study is conducted to compare the performance of the different estimators of the unknown parameters for the FWL(α, β, λ) distribution. All the computations in this section are done by R program. We generate $N = 700$ samples samples of size $n = 20, 25, ..., 700$ from FWL distribution with true parameters values $\alpha = 2$, $\beta = 1.5$ and $\lambda = 1.2$. We also calculate the bias and mean square error (MSE) of the MLEs empirically. We give results of this simulation study in Figure 1. Figure 1 indicates that the biases and MSEs appear reasonably small when $n \geq 500$. So, statistical inferences based on the proposed distribution can be considered accurate enough for $n \geq 500$. For smaller samples sizes, inferences based on the the proposed distribution may not be accurate.

5. Applications

In this Section, we provide two applications to show empirically the potentiality of the new family. We compare the flexible Weibull-Logistic (FWL) distribution with those of the Logistic (L), beta Logistic (BL), exponentiated Logistic (EL), Kumaraswamy Logistic (KwL) and Transmuted Weibull Logistic (TWL).

Data set I: consists of 63 observations of the strengths of 1.5 cm glass fibers which obtained by workers at the UK National Physical Laboratory (see), the MLEs and some statistics of the models for data set **I** are presented in Tables 1, 2 respectively.

Data set II: (see Murthy et al (2004)), the MLEs and some statistics of the models for data set II are presented in Tables 3, 4 respectively.

Table 1: The MLEs for data set I.

Distribution	Estimates with standard error in parenthesis			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{t}$
FWL	3.5416214	9.8500825	0.3092401	-
	(4.9771815)	(4.5962141)	(0.2857689)	
BL	119.6737781	57.2125339	0.4927973	-
	(90.0040120)	(63.06649)	(0.2413815)	
KL	21.021409	38.783896	1.018039	-
	(2.6093964)	(23.98600)	(0.1458867)	
GL	99.9432295	86.0386434	0.5188749	-
	(96.3688437)	(107.04687)	(0.3155779)	
TWL	0.003845322	1.642500707	2.067838533	0.00015
	(0.002212)	(4.2344)	(5.26034)	(0.001344)
EL	2.699803	37.607975	-	-
	(0.221463)	(10.366132)		

Table 2: Some statistics for the models fitted to data set I.

Distribution	Goodness of fit criteria					
	A^*	W^*	L	KS	P-value	AIC
FWL	0.8520827	0.1454573	14.88532	0.12483	0.2801	35.47065
BL	2.037812	0.3706532	18.43717	0.18529	0.02644	42.87434
KL	1.360935	0.2477292	15.39133	0.15498	0.09696	36.78266
GL	2.005401	0.3647971	18.28275	0.18448	0.02746	42.5655
TWL	0.8766299	0.1499408	15.32979	0.15782	0.08671	38.65958
EL	3.990389	0.7279398	29.86841	0.21813	0.004982	63.73681

Table 3: The MLEs for data set II.

Distribution	Estimates with standard error in parenthesis			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{t}$
FWL	0.01093515	3.47742271	0.21194267	-
	(0.01068807)	(0.71522996)	(0.04253401)	
BL	11.57323809	5.37909832	0.07832496	-
	(34.16601)	(23.48995)	(0.16440)	
KL	4.7030798	1.2654707	0.1648382	-
	(1.8357255)	(2.1653108)	(0.1487055)	
GL	1.6043781	0.1068959	1.4165800	-
	(0.328170332)	(0.025619065)	(0.006375468)	
TWL	0.08365499	0.36385403	0.44447994	0.38157711
	(0.041832)	(0.08339)	(0.028223)	(0.44411300)
EL	0.1872034	4.4803852	-	-
	(0.0217403)	(0.8167213)		

Table 4: Some statistics for the models fitted to data set II.

Distribution	Goodness of fit criteria					
	A^*	W^*	L	KS	P-value	AIC
FWL	0.1884926	0.02281417	128.7806	0.071504	0.9776	263.5613
BL	0.6569334	0.09324409	134.0233	0.11815	0.5904	274.0465
KL	0.632201	0.08504553	134.1713	0.1126	0.6497	274.3427
GL	0.7216431	0.1004488	133.372	0.10035	0.7783	272.7441
TWL	1.048186	0.1726038	136.3169	0.12789	0.4903	280.6338
EL	0.6301666	0.08408219	134.181	0.10906	0.6875	272.3621

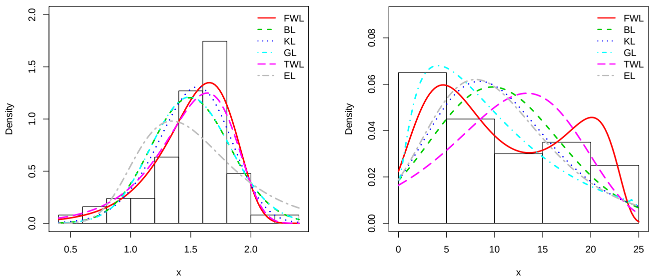

Figure 2. Estimated PDF for data set **I, II**

Based on Tables 2 and 4 the FWL lifetime model provides adequate fits as compared to other L models with small values for A^*, W^*, KS and largest P-values among all fitted models . The FWL lifetime model is better than the BL, KL, GL, TML and EL models in modeling the data sets I and II. The estimated PDFs plots are displayed in Figures 2. It is clear from Figures 2 that the FWL distribution provides the best fits to all data sets.

6. Conclusions

We introduce a new class of distributions called the flexible Weibull generated family. We obtain some of its mathematical properties. The special models of this family provide bathtub-shaped, decreasing-increasing, increasing-decreasing-increasing, decreasing-increasing-decreasing, monotone, unimodal and bimodal hazar functions. Some useful characterizations are presented. The maximum likelihood method is adopted for estimating the model parameters. We assess the performance of the maximum likelihood estimators by means of a graphical simulation study. The flexibility and importance of the proposed model are illustrated by means of three real data sets.

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