ON THE TWO PARAMETER MOTIONS IN THE COMPLEX PLANE

Doğan ÜNAL[1], Muhsin ÇELİK[2], Mehmet Ali GÜNGÖR[3]

In this article, we investigate two-parameter motions in the complex plane. Also, we refer to some definitions, theorems and corollaries related to velocities, accelerations, poles and hodograph of a point in the complex planar motion.

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1. Introduction

We know that the angular velocity vector has an important role in the kinematics of two rigid bodies, especially one rolling on another [1-3]. The investigation of the geometry of the motion of a line or a point in the motion of a plane is important in the study of planar kinematics, planar mechanisms or in physics. Mathematicians and physicists have interpreted rigid body motions in various ways. Nomizu has studied the one-parameter motion of orientable surface M on tangent space along the pole curves, using parallel vector fields at the contact points and he gave some characterizations of the angular velocity vector of rolling without sliding [4]. Hacısalihoğlu showed some of the properties of one-parameter motions in Euclidean space [5]. The geometry of the motion of a point or a line has a number of applications in geometric modeling and model-based manufacturing of mechanical products or in the design of robotic motions. These are specifically used to generate geometric models of shell-type objects and thick surfaces [6-8]. Alternative definitions of the imaginary unit i other than $i^2 = -1$ can give rise to interesting and useful complex number systems. Complex numbers were first discovered by Cardan, who called them "fictitious", during his attempts to find solutions to cubic equations [9]. Müller has introduced one and two parameter planar motions and obtained the relationship between absolute, relative, sliding velocity and pole curves of these motions. Moreover, the relationship between the complex velocities in terms of one-parameter motion in the complex plane were provided by [10]. One-parameter planar homothetic motion was defined in the complex plane [11]. In [12,13], the fluid dynamics in complex sense and the parametric design in architecture with complex geometries were studied. The instantaneous kinematics of a special two-parameter motion and all one-parameter motions obtained from two-parameter motions on the Euclidean plane were investigated in [14,15]. And in [16], these investigations were adapted to Lorentzian plane. Two-parametric motions in the Lobatchevski plane were given in [17]. Two-parameter Lorentzian homothetic motions were defined [18]. Besides, Brownian motions were constructed with two-parameter processes and transformations [19,20]. Two parameter modeling is used for building of a new model of LV wall motion in biomedical area and of effective elastic tensor for cortical bone in biomechanics area [21,22].

[1] Department of Mathematics, Faculty of Arts and Sciences, Sakarya University, Sakarya 54187, Turkey. dunal@sakarya.edu.tr, mcelik@sakarya.edu.tr, agungor@sakarya.edu.tr
[2] Department of Mathematics, Faculty of Arts and Sciences, Sakarya University, Sakarya 54187, Turkey. dunal@sakarya.edu.tr, mcelik@sakarya.edu.tr, agungor@sakarya.edu.tr
[3] Department of Mathematics, Faculty of Arts and Sciences, Sakarya University, Sakarya 54187, Turkey. dunal@sakarya.edu.tr, mcelik@sakarya.edu.tr, agungor@sakarya.edu.tr

Finally, two-parameter notion is applied for uid mixture, boundary crisis, mechanical systems, Newton-Hooke symmetry, cell polarity models, spatial motions and exponential distribution [23-28].

In this paper, two-parameter motions in the complex plane are defined. Sliding velocity, pole lines, hodograph and acceleration poles of two-parameter complex motions at the positions of $\forall(\lambda,\mu)$ are obtained. Some characteristic properties of the velocity vectors, the acceleration vectors and the pole curves are given.

2. Two Parameter Motions in the Complex Plane

Let E, E' be the moving and the fixed complex planes and O, O' be the origin points of their coordinate systems, respectively. If $\overrightarrow{O'O} = C'(\lambda,\mu)$, then

$$Y(\lambda,\mu) = e^{i\theta(\lambda,\mu)}X(\lambda,\mu) + C'(\lambda,\mu) \tag{2.1}$$

where $\theta(\lambda,\mu)$ is the rotation angle of E with respect to E' and $X(\lambda,\mu) = (X_1(\lambda,\mu), X_2(\lambda,\mu))$ and $Y(\lambda,\mu) = (Y_1(\lambda,\mu), Y_2(\lambda,\mu))$ are the coordinate functions of the moving and the fixed plane, respectively, and this motion is shown by B_{II} [10]. If λ and μ are given by the differentiable functions of time parameter t, then the complex motion B_I, which is called the complex motion B_I obtained from the complex motion B_{II} is obtained. Here $Y_1(\lambda,\mu), Y_2(\lambda,\mu), X_1(\lambda,\mu)$, $X_2(\lambda,\mu), A(\lambda,\mu)$ and $B(\lambda,\mu)$ are complex elements. They can be denoted by

$$Y(\lambda,\mu) = [Y_1 \quad Y_2]^T, \quad X(\lambda,\mu) = [X_1 \quad X_2]^T, \quad C'(\lambda,\mu) = [A \quad B]^T.$$

Without losing generality, we can take $\theta(0,0) = A(0,0) = B(0,0) = 0$ to make two complex planes congruent at the position $(\lambda,\mu) = (0,0)$.

2.1. Velocities

If we take $Y(\lambda,\mu) = 0$ and $\overrightarrow{OO'} = X(\lambda,\mu) = C(\lambda,\mu)$, then we obtain from Eq. (2.1)

$$C'(\lambda,\mu) = -C(\lambda,\mu)e^{i\theta(\lambda,\mu)}. \tag{2.2}$$

If Eq. (2.2) is substituted into Eq. (2.1), we get

$$Y(\lambda,\mu) = [X(\lambda,\mu) - C(\lambda,\mu)]e^{i\theta(\lambda,\mu)}. \tag{2.3}$$

The relative velocity of the point $X(\lambda,\mu)$ is the velocity of the point $X(\lambda,\mu)$ with respect to the moving plane E and the relative velocity vector of the point $X(\lambda,\mu)$ in the moving plane is given by

$$\overrightarrow{X_r} = \dot{X}(\lambda,\mu) = X_\lambda\dot{\lambda} + X_\mu\dot{\mu}. \tag{2.4}$$

This vector is deduced in the fixed coordinate system as follows;

$$\overrightarrow{Y_r} = \overrightarrow{X_r}e^{i\theta(\lambda,\mu)} = \dot{X}(\lambda,\mu)e^{i\theta(\lambda,\mu)} = (X_\lambda\dot{\lambda} + X_\mu\dot{\mu})e^{i\theta(\lambda,\mu)}. \tag{2.5}$$

The velocity of the point $X(\lambda,\mu)$ with respect to the fixed plane E' is the absolute velocity of the point $X(\lambda,\mu)$. By differentiating Eq. (2.3) with respect to (λ,μ) and simplifying it, we get

$$\begin{aligned}\overrightarrow{Y_a} = &-\left[C_\lambda\dot{\lambda} + C_\mu\dot{\mu} + i\left(\theta_\lambda\dot{\lambda} + \theta_\mu\dot{\mu}\right)C(\lambda,\mu)\right]e^{i\theta(\lambda,\mu)} \\ &+i\left(\theta_\lambda\dot{\lambda} + \theta_\mu\dot{\mu}\right)X(\lambda,\mu)e^{i\theta(\lambda,\mu)} + \overrightarrow{Y_r}.\end{aligned} \tag{2.6}$$

The sliding velocity vector of the point $X(\lambda,\mu)$ is given by

$$\begin{aligned}\overrightarrow{Y_f} = &-\left[C_\lambda\dot{\lambda} + C_\mu\dot{\mu} + i\left(\theta_\lambda\dot{\lambda}+\theta_\mu\dot{\mu}\right)C\left(\lambda,\mu\right)\right]e^{i\theta(\lambda,\mu)}\\ &+i\left(\theta_\lambda\dot{\lambda}+\theta_\mu\dot{\mu}\right)X\left(\lambda,\mu\right)e^{i\theta(\lambda,\mu)}.\end{aligned} \tag{2.7}$$

The expressions of the absolute and the sliding velocity vectors with respect to the coordinate axis of the moving plane are, respectively:

$$\begin{aligned}\overrightarrow{X_a} = Y_{\text{a}}e^{-i\theta(\lambda,\mu)} = &-\left[C_\lambda\dot{\lambda} + C_\mu\dot{\mu} + i\left(\theta_\lambda\dot{\lambda}+\theta_\mu\dot{\mu}\right)C\left(\lambda,\mu\right)\right]\\ &+i\left(\theta_\lambda\dot{\lambda}+\theta_\mu\dot{\mu}\right)X\left(\lambda,\mu\right)+\overrightarrow{X_r}\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\overrightarrow{X_f} = Y_f e^{-i\theta(\lambda,\mu)} = &-\left[C_\lambda\dot{\lambda} + C_\mu\dot{\mu} + i\left(\theta_\lambda\dot{\lambda}+\theta_\mu\dot{\mu}\right)C\left(\lambda,\mu\right)\right]\\ &+i\left(\theta_\lambda\dot{\lambda}+\theta_\mu\dot{\mu}\right)X\left(\lambda,\mu\right).\end{aligned} \tag{2.9}$$

From Eq. (2.5), (2.6) and (2.7) it can be written

$$Y_a = Y_f + Y_r. \tag{2.10}$$

Let $\dot{\theta}\left(\lambda,\mu\right) = \theta_\lambda\dot{\lambda}+\theta_\mu\dot{\mu}$ be the angular velocity of the complex motion B_I obtained from the complex motion B_{II}. To avoid the case of pure translation, let us assume that

$$\dot{\theta}\left(\lambda,\mu\right) = \theta_\lambda\dot{\lambda}+\theta_\mu\dot{\mu} \neq 0.$$

Now, let us investigate what happens when the sliding velocity is equal to zero. Such points as these shall be fixed, not only in the moving plane E, but also in the fixed plane $E^{'}$. In this case, we obtain an equation from Eq. (2.7) as follows;

$$\overrightarrow{Y_f} = -(\dot{C}+iC\dot{\theta})e^{i\theta} + i\dot{\theta}Xe^{i\theta} = 0$$

and this gives us

$$P\left(P_1,P_2\right) = C - i\frac{\dot{C}}{\dot{\theta}} \tag{2.11}$$

which is the pole points of the complex motion B_I obtained from the complex motion B_{II}. From Eq. (2.11) $\dot{C}$ can be obtained as follows;

$$\dot{C} = i(P-C)\dot{\theta}.$$

By substituting the equality of $\dot{C}$ into Eq. (2.7), we have the sliding velocity vector of the point $X(\lambda,\mu)$ which is taken into consideration with the pole point $P(P_1,P_2)$ as follows;

$$\overrightarrow{Y_f} = i\dot{\theta}\left(X-P\right)e^{i\theta}. \tag{2.12}$$

Theorem 2.1. *The pole points of the complex motion B_I obtained from the complex motion B_{II} on the moving plane lie on a line at the position of* $\forall\left(\lambda,\mu\right)$.

Proof. *If Eq. (2.11) is written clearly and* $C = \begin{bmatrix} -Ae^{-i\theta} \\ -Be^{-i\theta}\end{bmatrix}$ *is regarded in this equation, we obtain*

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} -Ae^{-i\theta} \\ -Be^{-i\theta}\end{bmatrix} - i\begin{bmatrix} \frac{-\dot{A}e^{-i\theta}+i\dot{\theta}Ae^{-i\theta}}{\dot{\theta}} \\ \frac{-\dot{B}e^{-i\theta}+i\dot{\theta}Be^{-i\theta}}{\dot{\theta}}\end{bmatrix}.$$

Then

$$P_1 = i\frac{e^{-i\theta}}{\dot{\theta}}\dot{A} \tag{2.13}$$

and

$$P_2 = i\frac{e^{-i\theta}}{\dot{\theta}}\dot{B} \tag{2.14}$$

are obtained. Here $\frac{\dot{\lambda}}{\dot{\mu}}$ is taken from the equality of P_2 and then substituted in the equality of P_1, giving us:

$$\frac{ie^{-i\theta}B_\mu - P_2\theta_\mu}{P_2\theta_\lambda - ie^{-i\theta}B_\lambda} = \frac{ie^{-i\theta}A_\mu - P_1\theta_\mu}{P_1\theta_\lambda - ie^{-i\theta}A_\lambda}.$$

Thus the following line equation is obtained

$$(\theta_\lambda B_\mu - B_\lambda\theta_\mu)\,P_1 + (A_\lambda\theta_\mu - \theta_\lambda A_\mu)P_2 = ie^{-i\theta}\,(A_\lambda B_\mu - B_\lambda A_\mu)\,. \tag{2.15}$$

Corollary 2.1. *If $(\lambda,\mu) = (0,0)$ i.e., $A(0,0) = B(0,0) = \theta(0,0) = 0$, then the pole points of the moving plane lie on a line as follows;*

$$(\theta_\lambda B_\mu - B_\lambda\theta_\mu)\,P_1 + (A_\lambda\theta_\mu - \theta_\lambda A_\mu)\,P_2 = i\,(A_\lambda B_\mu - B_\lambda A_\mu)\,. \tag{2.16}$$

Theorem 2.2. *The pole points of the complex motion B_I obtained from the complex motion B_{II} on the fixed plane lie on a line at the position of $\forall(\lambda,\mu)$.*

Proof. *If the equation $P(P_1,P_2)$ is substituted into Eq. (2.1), then $\bar{P}(\bar{P}_1,\bar{P}_2)$ pole point of the fixed plane is obtained. Then the pole point on the fixed plane is*

$$\bar{P}_1 = i\frac{\dot{A}}{\dot{\theta}} + A \tag{2.17}$$

and

$$\bar{P}_2 = i\frac{\dot{B}}{\dot{\theta}} + B. \tag{2.18}$$

Here $\frac{\dot{\lambda}}{\dot{\mu}}$ is taken from the equality of $\bar{P}_2$ and then substituted in the equality of $\bar{P}_1$ and we get

$$\frac{iB_\mu + B\theta_\mu - \bar{P}_2\theta_\mu}{\bar{P}_2\theta_\lambda - iB_\lambda - B\theta_\lambda} = \frac{iA_\mu + A\theta_\mu - \bar{P}_1\theta_\mu}{\bar{P}_1\theta_\lambda - iA_\lambda - A\theta_\lambda}.$$

Thus, the following line equation is obtained

$$\begin{aligned}(i\theta_\lambda B_\mu - iB_\lambda\theta_\mu)\bar{P}_1 + (iA_\lambda\theta_\mu - i\theta_\lambda A_\mu)\bar{P}_2 = {} & BA_\lambda\theta_\mu + A\theta_\lambda B_\mu - AB_\lambda\theta_\mu \\ & - B\theta_\lambda A_\mu + i\,(A_\lambda B_\mu - B_\lambda A_\mu)\,.\end{aligned} \tag{2.19}$$

Corollary 2.2. *If $(\lambda,\mu) = (0,0)$ i.e., $A(0,0) = B(0,0) = \theta(0,0) = 0$, then the pole points of the complex motion B_I obtained from the complex motion B_{II} on the fixed plane lie on a line as follows;*

$$(\theta_\lambda B_\mu - B_\lambda\theta_\mu)\bar{P}_1 + (A_\lambda\theta_\mu - \theta_\lambda A_\mu)\,\bar{P}_2 = i\,(A_\lambda B_\mu - B_\lambda A_\mu)\,. \tag{2.20}$$

Corollary 2.3. *The pole lines of the complex motion B_I obtained from the complex motion B_{II} on the moving and the fixed plane at the position of $\lambda = \mu = 0$ are congruent.*

If the pole line of the complex motion B_I obtained from the complex motion B_{II} is y-axis on the moving plane, then the equation $\dot{A}\,(\lambda,\mu) = A_\lambda\dot{\lambda} + A_\mu\dot{\mu}$ vanishes. Since $\dot{\lambda}$ and $\dot{\mu}$ are independent motion parameters, they never vanish. So A_λ and A_μ should be equal to zero at the position of $\lambda = \mu = 0$. Then, we obtain

$$P_1{=}0, \tag{2.21}$$

$$P_2{=}i\frac{\dot{B}}{\dot{\theta}}. \tag{2.22}$$

Because of this special case, the pole line and y-axis of the fixed plane are congruent at the position of $(\lambda,\mu) = (0,0)$. Therefore, we obtain

$$\bar{P}_1{=}0, \tag{2.23}$$

$$\bar{P}_2{=}i\frac{\dot{B}}{\dot{\theta}}. \tag{2.24}$$

If the y-axis is chosen as the pole axis, that is, $A_\lambda = A_\mu = 0$ is taken, then the sliding velocity of any fixed point $Q(X_1,X_2)$ on the moving plane at the position of $\lambda = \mu = 0$ is equal to the absolute velocity.

Theorem 2.3. *In the complex motion B_I obtained from the complex motion B_{II}, let the y-axis be the pole axis at the position of $\lambda = \mu = 0$. Then, the pole ray $\overrightarrow{PQ} = (Q - P)e^{i\theta}$ going from the pole point $P(0, P_2)$ to the point $Q(X_1, X_2)$ and the sliding velocity $\overrightarrow{Y_f}$ of the point $Q(X_1, X_2)$ are perpendicular.*

Proof. *If the pole axis is the y-axis,*

$$\overrightarrow{PQ} = (X_1, X_2 - P_2)$$

and

$$\overrightarrow{Y_f} = \left[-\dot{\theta}(X_2 - P_2), \dot{\theta}X_1\right]. \tag{2.25}$$

Hence,

$$\left\langle \overrightarrow{Y_f}, \overrightarrow{PQ} \right\rangle = 0. \tag{2.26}$$

Thus, the proof is completed.

Theorem 2.4. *The length of the sliding velocity vector $\overrightarrow{Y_f}$ of the complex motion B_I obtained from the complex motion B_{II} is given by*

$$\left\|\overrightarrow{Y_f}\right\| = \left|\dot{\theta}\right| \left\|\overrightarrow{PQ}\right\|. \tag{2.27}$$

at the position of $\forall(\lambda, \mu)$.

Proof. *It is known that*

$$\begin{aligned}
\left\langle \overrightarrow{a}e^{i\theta}, \overrightarrow{b}e^{i\theta} \right\rangle &= \langle (a_1 + ia_2)(\cos\theta + i\sin\theta), (b_1 + ib_2)(\cos\theta + i\sin\theta) \rangle \\
&= \left\langle \begin{matrix} (a_1\cos\theta - a_2\sin\theta, a_1\sin\theta + a_2\cos\theta), \\ (b_1\cos\theta - b_2\sin\theta, b_1\sin\theta + b_2\cos\theta) \end{matrix} \right\rangle \\
&= a_1b_1 + a_2b_2 \\
&= \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle.
\end{aligned}$$

Therefore, the length of the sliding velocity vector $\overrightarrow{Y_f}$ with the pole point is

$$\left\|\overrightarrow{Y_f}\right\| = \left|\dot{\theta}\right| \sqrt{(X_2 - P_2)^2 + (X_1 - P_1)^2}$$

and

$$\left\|\overrightarrow{Y_f}\right\| = \left|\dot{\theta}\right| \left\|\overrightarrow{PQ}\right\|.$$

Theorem 2.5. *For the complex motion B_I obtained from the complex motion B_{II}, let Ψ be an angle between the pole ray $\overrightarrow{PQ} = (X - P)e^{i\theta}$going from the pole point $P = (P_1, P_2)$ to the point $Q(X_1, X_2)$ and the sliding velocity vector $\overrightarrow{Y_f}$. Then, the following relation holds*

$$\Psi(\lambda, \mu) = \frac{\pi}{2} + 2k\pi, \quad k = 0, 1, 2, ... \tag{2.28}$$

at the position of $\forall(\lambda, \mu)$.

Proof. *Since*

$$\overrightarrow{Y_f} = \left[-\dot{\theta}e^{i\theta}(X_2 - P_2), \dot{\theta}e^{i\theta}(X_1 - P_1)\right]$$

and

$$\overrightarrow{PQ} = \left[e^{i\theta}(X_1 - P_1), e^{i\theta}(X_2 - P_2)\right],$$

if we get the inner product, then

$$\left\langle \overrightarrow{PQ}, \overrightarrow{Y_f} \right\rangle = \langle [(X_1 - P_1), (X_2 - P_2)], [-(X_2 - P_2), (X_1 - P_1)] \rangle \dot{\theta} = 0.$$

On the other hand, it is known that

$$\left\langle \overrightarrow{PQ}, \overrightarrow{Y_f} \right\rangle = \left\|\overrightarrow{PQ}\right\| \left\|\overrightarrow{Y_f}\right\| \cos\Psi(\lambda, \mu).$$

By comparing these last two equations the proof of the theorem is completed.

Definition 2.1. *When the sliding velocity vectors of the fixed points are carried to the initial point without changing their directions, then the locus of the end points of these vectors specify a curve called a hodograph.*

Now, we investigate any points (X_1, X_2) of the locus of the hodographs in all the complex motion B_I obtained from the complex motion B_{II} at the position of $\forall(\lambda, \mu)$. For this let $\dot{\lambda}^2 + \dot{\mu}^2 = 1$. By differentiating Eq. (2.3) with respect to λ and μ, we have

$$Y_a = Y_f = i\dot{\theta}Xe^{i\theta} - (\dot{C} + i\dot{\theta}C)e^{i\theta}$$

and

$$Y_f = \left(i\dot{\theta}X_1e^{i\theta} + \dot{A}, i\dot{\theta}X_2e^{i\theta} + \dot{B}\right).$$

Then we obtain

$$\begin{aligned}\dot{Y}_1 &= \left(i\theta_\lambda X_1 e^{i\theta} + A_\lambda\right)\dot{\lambda} + \left(i\theta_\mu X_1 e^{i\theta} + A_\mu\right)\dot{\mu},\\ \dot{Y}_2 &= \left(i\theta_\lambda X_2 e^{i\theta} + B_\lambda\right)\dot{\lambda} + \left(i\theta_\mu X_2 e^{i\theta} + B_\mu\right)\dot{\mu}.\end{aligned}$$

If the equations are written as follows;

$$\begin{aligned}\dot{Y}_1 &= m_1\dot{\lambda} + m_2\dot{\mu},\\ \dot{Y}_2 &= m_3\dot{\lambda} + m_4\dot{\mu}\end{aligned}$$

and Cramer's method is applied to

$$\Gamma = \begin{vmatrix} m_1 & m_2 \\ m_3 & m_4 \end{vmatrix} = m_1m_4 - m_2m_3$$

at the position of $\lambda = \mu = 0$ and after it is substituted into the equation of $\dot{\lambda}^2 + \dot{\mu}^2 = 1$, we obtain

$$\Gamma = i\theta_\lambda X_1 B_\mu + iA_\lambda X_2\theta_\mu - iB_\lambda X_1\theta_\mu - i\theta_\lambda X_2 A_\mu + A_\lambda B_\mu - B_\lambda A_\mu \tag{2.29}$$

and

$$\dot{\lambda} = \frac{\begin{vmatrix} \dot{Y}_1 & m_2 \\ \dot{Y}_2 & m_4 \end{vmatrix}}{\Gamma}\ ,\ \dot{\mu} = \frac{\begin{vmatrix} m_1 & \dot{Y}_1 \\ m_3 & \dot{Y}_2 \end{vmatrix}}{\Gamma}\ ,\ \dot{\lambda}^2 + \dot{\mu}^2 = 1.$$

Then, we get

$$\frac{\left[\left(i\theta_\mu X_2 + B_\mu\right)\dot{Y}_1 - \left(i\theta_\mu X_1 + A_\mu\right)\dot{Y}_2\right]^2}{\Gamma^2} + \frac{\left[\left(i\theta_\lambda X_1 + A_\lambda\right)\dot{Y}_2 - \left(i\theta_\lambda X_2 + B_\lambda\right)\dot{Y}_1\right]^2}{\Gamma^2} = 1.$$

From this last equation, we obtain

$$\begin{aligned}\Gamma^2 = &\left[\left(i\theta_\mu X_2 + B_\mu\right)^2 + \left(i\theta_\lambda X_2 + B_\lambda\right)^2\right]\dot{Y}_1^2 + \left[\left(i\theta_\mu X_1 + A_\mu\right)^2 + \left(i\theta_\lambda X_1 + A_\lambda\right)^2\right]\dot{Y}_2^2\\ &-2\left[\left(i\theta_\mu X_2 + B_\mu\right)\left(i\theta_\mu X_1 + A_\mu\right) + \left(i\theta_\lambda X_1 + A_\lambda\right)\left(i\theta_\lambda X_2 + B_\lambda\right)\right]\dot{Y}_1\dot{Y}_2\end{aligned} \tag{2.30}$$

and this is the equation of the hodograph at the position of $\forall(\lambda, \mu)$.

Theorem 2.6. *The hodograph of any points (X_1, X_2) in the complex motion B_I obtained from the complex motion B_{II} at the position of $\lambda = \mu = 0$ is an ellipse.*
Proof. *Taking the general conic form*

$$KX^2 + 2LXY + MY^2 + 2DX + 2EY + F = 0$$

we obtain

$$\begin{aligned}K &= \left[\left(i\theta_\mu X_2 + B_\mu\right)^2 + \left(i\theta_\lambda X_2 + B_\lambda\right)^2\right],\\ L &= -\left[\left(i\theta_\mu X_2 + B_\mu\right)\left(i\theta_\mu X_1 + A_\mu\right) + \left(i\theta_\lambda X_1 + A_\lambda\right)\left(i\theta_\lambda X_2 + B_\lambda\right)\right],\\ M &= \left[\left(i\theta_\mu X_1 + A_\mu\right)^2 + \left(i\theta_\lambda X_1 + A_\lambda\right)^2\right].\end{aligned}$$

From here, we have

$$\begin{vmatrix} K & L \\ L & M \end{vmatrix} = \left[\left(i\theta_\lambda X_2 + B_\lambda \right) \left(i\theta_\mu X_1 + A_\mu \right) - \left(i\theta_\mu X_2 + B_\mu \right) \left(i\theta_\lambda X_1 + A_\lambda \right) \right]^2 > 0$$

and this indicates the equation of an ellipse.

2.2. Accelerations

The relative acceleration vector of the point $X(\lambda, \mu)$ is the acceleration vector of the point $X(\lambda, \mu)$ with respect to the moving plane. When the vectorial velocity $\overrightarrow{X_r}$ is derived with respect to λ and μ, then the relative acceleration vector is obtained. Therefore, from Eq. (2.4) it is written that

$$\overrightarrow{b_r} = \dot{X}_r = \ddot{X}(\lambda, \mu) = X_{\lambda\lambda}\dot{\lambda}^2 + X_{\lambda\mu}\dot{\lambda}\dot{\mu} + X_\lambda \ddot{\lambda} + X_{\mu\mu}\dot{\mu}^2 + X_{\mu\lambda}\dot{\lambda}\dot{\mu} + X_\mu \ddot{\mu}. \tag{2.31}$$

This vector is expressed with respect to the fixed plane as follows;

$$\overrightarrow{b_r}' = \overrightarrow{b_r} e^{i\theta} = \ddot{X} e^{i\theta}. \tag{2.32}$$

The absolute acceleration vector of the point $X(\lambda, \mu)$ is the acceleration vector of the point $X(\lambda, \mu)$ with respect to the fixed plane. By taking Eq. (2.5) and (2.12) in Eq. (2.10), we have the absolute velocity as follows

$$\overrightarrow{Y_a} = \overrightarrow{Y_f} + \overrightarrow{Y_r} = i\dot{\theta}(X - P) e^{i\theta} + \dot{X} e^{i\theta}.$$

When this absolute velocity is derived with respect to λ and μ, then the absolute acceleration vector of the point $X(\lambda, \mu)$ is obtained. Therefore,

$$\overrightarrow{b_a}' = (X - P)\left(i\ddot{\theta} - \dot{\theta}^2 \right) e^{i\theta} - i\dot{\theta}\dot{P} e^{i\theta} + 2i\dot{\theta}\dot{X} e^{i\theta} + \ddot{X} e^{i\theta}. \tag{2.33}$$

Here the sliding acceleration vector of the point $X(\lambda, \mu)$ is

$$\overrightarrow{b_f}' = (X - P)\left(i\ddot{\theta} - \dot{\theta}^2 \right) e^{i\theta} - i\dot{\theta}\dot{P} e^{i\theta} \tag{2.34}$$

and the Coriolis acceleration vector of the point $X(\lambda, \mu)$ is

$$\overrightarrow{b_c}' = 2i\dot{\theta}\dot{X} e^{i\theta}. \tag{2.35}$$

Hence, the sliding acceleration vector is the acceleration of the fixed point in the moving system with respect to the fixed system. Therefore, composition of these accelerations can be given from Eq. (2.31), (2.33), (2.34) and (2.35) with the following theorem.

Theorem 2.7. *The following relation holds between the acceleration vectors of any points in composition of two complex motions.*

$$\overrightarrow{b_a}' = \overrightarrow{b_f}' + \overrightarrow{b_c}' + \overrightarrow{b_r}' \tag{2.36}$$

where

$$\overrightarrow{b_a} = \overrightarrow{b_a}' e^{-i\theta} = (X - P)\left(i\ddot{\theta} - \dot{\theta}^2 \right) - i\dot{\theta}\dot{P} + 2i\dot{\theta}\dot{X} + {b_r}', \tag{2.37}$$

$$\overrightarrow{b_f} = \overrightarrow{b_f}' e^{-i\theta} = (X - P)\left(i\ddot{\theta} - \dot{\theta}^2 \right) - i\dot{\theta}\dot{P} \tag{2.38}$$

and

$$\overrightarrow{b_c} = \overrightarrow{b_c}' e^{-i\theta} = 2i\dot{\theta}\dot{X} \tag{2.39}$$

are the equations of the absolute, the sliding and the Coriolis acceleration vectors with respect to the moving system, respectively.

Corollary 2.4. *The Coriolis acceleration vector* $\overrightarrow{b_c}'$ *and the relative velocity vector* $\overrightarrow{Y_r}$ *are perpendicular.*
Proof. *Since* $\overrightarrow{Y_r} = \dot{X}e^{i\theta}$ *and* $\overrightarrow{b_c}' = 2i\dot{\theta}\dot{X}e^{i\theta}$, *if we get the inner product, then*

$$\begin{aligned} \left\langle \overrightarrow{Y_r}, \overrightarrow{b_c}' \right\rangle &= \left\langle \dot{X}e^{i\theta}, 2i\dot{\theta}\dot{X}e^{i\theta} \right\rangle \\ &= 2\dot{\theta}\left(-\dot{X}_1\dot{X}_2 + \dot{X}_2\dot{X}_1\right) = 0 \end{aligned} \tag{2.40}$$

and the proof is completed.

If the Coriolis acceleration of the point $X(\lambda,\mu)$ on the moving plane is equal to zero, then the following relationship holds between accelerations for velocities:

$$\overrightarrow{b_a}' = \overrightarrow{b_f}' + \overrightarrow{b_r}'. \tag{2.41}$$

Corollary 2.5. *If the Coriolis acceleration of the point* $X(\lambda,\mu)$ *on the moving plane is equal to zero, then the complex motion* B_I *obtained from the complex motion* B_{II} *is a translation motion and the opposite explanation is also true.*
Proof. *If the Coriolis acceleration vector of the point* $X(\lambda,\mu)$ *is*

$$\overrightarrow{b_c}' = 2i\dot{\theta}\dot{X}e^{i\theta} = 0.$$

Then, $\dot{X}(\lambda,\mu) \neq 0$. *This is because the point* $X(\lambda,\mu)$ *is not a fixed point. Therefore,* $\dot{\theta}(\lambda,\mu) = \theta_\lambda\dot{\lambda} + \theta_\mu\dot{\mu} = 0$ *and it shows that the motion is only a translation motion. On the other hand, if the motion is only a translation motion, then* $\theta(\lambda,\mu)$ *is constant. Here*

$$\dot{\theta}(\lambda,\mu) = \theta_\lambda\dot{\lambda} + \theta_\mu\dot{\mu} = 0$$

and the following equation is obtained

$$\overrightarrow{b_c}' = 2i\dot{\theta}\dot{X}e^{i\theta} = 0.$$

Theorem 2.8. *The acceleration pole at the position of* $\forall(\lambda,\mu)$, *whose angular velocity in the complex motion* B_I *obtained from the complex motion* B_{II} *is different from zero, is*

$$X = P + \frac{\dot{\theta}\ddot{\theta}\dot{P}}{\dot{\theta}^4 + \ddot{\theta}^2} - i\frac{\dot{\theta}^3\dot{P}}{\dot{\theta}^4 + \ddot{\theta}^2}\ . \tag{2.42}$$

Proof. *Let us search the points where the sliding accelerations are zero at the position of* $\forall(\lambda,\mu)$. *From Eq. (2.34), the sliding acceleration vector of the point* $X(\lambda,\mu)$ *can be written as follows;*

$$\overrightarrow{b_f}' = (X - P)\left(i\ddot{\theta} - \dot{\theta}^2\right)e^{i\theta} - i\dot{\theta}\dot{P}e^{i\theta} = 0$$

and from here

$$X = P + \frac{\dot{\theta}\ddot{\theta}\dot{P}}{\dot{\theta}^4 + \ddot{\theta}^2} - i\frac{\dot{\theta}^3\dot{P}}{\dot{\theta}^4 + \ddot{\theta}^2}$$

is obtained.

Theorem 2.9. *If* $\dot{\lambda} = \dot{\mu} = 0$, *then the acceleration poles of the complex motion* B_I *obtained from the complex motion* B_{II} *at the position of* $\lambda = \mu = 0$ *lie on the following line*

$$(\theta_\lambda B_\mu - B_\lambda\theta_\mu)P_{i_1} + (A_\lambda\theta_\mu - \theta_\lambda A_\mu)P_{i_2} = i(A_\lambda B_\mu - B_\lambda A_\mu). \tag{2.43}$$

Proof. *From the differentiation of Eq. (2.7), we obtain*

$$\overrightarrow{b_f}' = -\left(\ddot{C} + i\ddot{\theta}C + i\dot{\theta}\dot{C}\right)e^{i\theta} - i\dot{\theta}\left(\dot{C} + i\dot{\theta}C\right)e^{i\theta} + i\ddot{\theta}Xe^{i\theta} + i\dot{\theta}\dot{X}e^{i\theta} - \dot{\theta}^2Xe^{i\theta}$$

and if $(\lambda,\mu) = (0,0)$ *and* $\dot{\lambda} = \dot{\mu} = 0$ *are substituted into the equation and we simplify it, then*

$$\overrightarrow{b_f}' = -\ddot{C} + i\ddot{\theta}X$$

is obtained. Hence, the acceleration pole is

$$P_{i_1} = X_1 = i\frac{A_\lambda\ddot{\lambda} + A_\mu\ddot{\mu}}{\theta_\lambda\ddot{\lambda} + \theta_\mu\ddot{\mu}}, \tag{2.44}$$

$$P_{i_2} = X_2 = i\frac{B_\lambda\ddot{\lambda} + B_\mu\ddot{\mu}}{\theta_\lambda\ddot{\lambda} + \theta_\mu\ddot{\mu}} \ . \tag{2.45}$$

Here, if $\frac{\ddot{\lambda}}{\ddot{\theta}}$ *is taken from the equality of* P_{i_2} *and substituted into the equality of* P_{i_1}*, then we obtain*

$$\frac{iB_\mu - P_{i_2}\theta_\mu}{P_{i_2}\theta_\lambda - iB_\lambda} = \frac{iA_\mu - P_{i_1}\theta_\mu}{P_{i_1}\theta_\lambda - iA_\lambda}$$

and from here we get the following line equation:

$$(\theta_\lambda B_\mu - B_\lambda\theta_\mu) P_{i_1} + (A_\lambda\theta_\mu - \theta_\lambda A_\mu) P_{i_2} = i (A_\lambda B_\mu - B_\lambda A_\mu).$$

Corollary 2.6. *This acceleration pole and the pole lines of the moving plane given by Eq. (2.16) and the fixed plane given by Eq. (2.20) are congruent.*

3. **Conclusion**

The results we have presented deal with complex motions in which the position of the moving object depend on two parameters. Hodograph of two-parameter complex motions was obtained. The hodograph is the locus of the end points of the velocity of a particle and it is the solution of the first order equation which is Newton's Law. The locus of the hodograph of complex motion was found as an ellipse in this study.

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