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WAVELETS, PROPERTIES OF THE SCALAR FUNCTIONS

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Pentru a contrui o undină convenabilă Ψ este necesară şi suficientă o analiză multirezoluţie. Analiza multirezoluţie este legată de funcţia de scalare. În acest articol prezentăm câteva proprietăţi ale funcţiei de scalare aplicând transformarea Fourier. Alegerea undinei Ψ este esenţială. Problema este de a adapta Ψ la o anumită clasă de semnale, de exemplu semnale vocale sau muzicale.

To build a convenient wavelet Ψ a multiresolution analysis is necessary and sufficient. The multiresolution analysis is connected to the scalar function. In this article / paper we present some properties of the scalar function applying the Fourier transform. The choice of the wavelet Ψ is essential. The problem is to adapt Ψ to a certain class of signals, for example vocal or musical signals.

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Introduction

These problems have a special role in questions regarding the processing of signals in real time and the identification, optimization and the control of the most diverse systems.

In this paper we determine a scalar function $\varphi \in L^2$, $\varphi \in V_0$ so that the string $\varphi(2t-n)$ is a sampling string.

1. Preliminaries

We recall that a wavelet $\Psi : \mathbb{R} \to \mathbb{C}$ is a function from $L^1 \cap L^2$ such that $\hat{\Psi}(0) = 0$ and Ψ, $\hat{\Psi}$ satisfy the following decreasing-type conditions:

$$\left|\Psi(t)\right| \le C \cdot (1+|t|)^{-1-\varepsilon}, \quad \left|\hat{\Psi}(\omega)\right| \le C(1+|\omega|)^{-1-\varepsilon}, \varepsilon > 0.$$

To determine Ψ, any signal with finite energy $f \in L^2$ has a decomposition of the type

$$f(t) = C_\Psi \int_{\mathbb{R}^2} \left\langle f, \Psi_{a,b} \right\rangle \Psi_{a,b}(t) a^{-2} da\, db \tag{1}$$

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by the Calderon's formula. Such a decomposition is interpreted in this way: the signal f is an overlapping of "blocks of building" $\Psi_{a,b}(t)=|a|^{-1/2}\,\Psi\left(\frac{t-b}{a}\right)$, $a,b\in\mathbb{R}$, $a\neq 0$, b real parameters each well located in time and frequency.

For example if $\Psi(t)=(1-t^2)\exp\left(-\frac{t^2}{2}\right)$ then $\left|\Psi_{a,b}(t)\right|$ is concentrated in $\left[b-2|a|,b+2|a|\right]$, and $\left|\widehat{\Psi_{a,b}}(\omega)\right|=-\sqrt{2\pi}\omega^2\exp\left(-\frac{\omega^2}{2}\right)$ is concentrated in the "crown" $\frac{1}{4}|a|\leq|\omega|\leq 4|a|$.

By portioning the domain of integration of the plane O_{ab} , we can obtain some algorithms for applications of the reconstruction formula (1).

To do this, consider a network in the plane O_{ab}, taking $a=2^{-j},b=k\cdot 2^{-j},j,k\in\mathbb{Z}$. Moreover we can choose the wavelet Ψ such that $\Psi_{j,k}=\Psi_{a,b}$ to form an orthogonal basis for L^2.

Instead of integral representation (1) there is another representation of f:

$$f(t)=\sum_{j,k}\left\langle f,\Psi_{jk}\right\rangle\cdot\Psi_{jk}(t) \tag{2}$$

, where the convergence of series is in the space L^2, endowed with the usual norm.

To build a convenient wavelet Ψ, a multiresolution analysis is necessary and sufficient. To be more specific, we start from the scalar function $\varphi\in L^2$ with $\hat{\varphi}(0)=0$ so that $\varphi(x)=\sum_n c_n\varphi(2x-n)$ and $\Psi(x)=\sum_n d_n\varphi(2x-n)$. Then, for any $f\in L^2$ it follows,

$$f(t)=\lim_{p\to\infty}2^p\int_{-\infty}^{\infty}f(x)\cdot\varphi(2^p\cdot x-t)dx \tag{2'}$$

In particular, the integrals $2^p \int_{-\infty}^{\infty} f(x)\cdot\varphi(2^p x - k)dx$ can be approximated through the samples $f(k/2^p)$.

By recurrent relations one can elaborate an algorithm for the computations of the scalar products $\left\langle f, \Psi_{jk} \right\rangle = 2^{j/2} \int_{-\infty}^{\infty} f(x)\Psi(2^j \cdot x - k)dx$ and for the calculus of the scalars $\left\langle f, \varphi_{jk} \right\rangle$. Such algorithms can be seen in the theory of sub-banda filtration.

We remind that a multiresolution in L^2 is a rising string $(V_m)_{m\in\mathbb{Z}}$ of closed linear subspaces of L^2 so that $V_m \subset V_{m+1}$ for any $m \in \mathbb{Z}$; the reunion of subspaces V_m is supposed to be dense in L^2 and the intersection is reduced to the null subspace.

Moreover we assume that $f(t)\in V_m \Leftrightarrow f(2t)\in V_{m+1}$ and there exists a function $\varphi \in L^2$ called the scalar function of the multiresolution analysis, such that $\varphi(t-n),\ n\in\mathbb{Z}$ is an orthonormal basis in V_0, then $\left\{\sqrt{2}\varphi(2t-n)\right\}, n\in\mathbb{Z}$ form an orthonormal basis for V_1; as $\varphi\in V_1$, $\varphi(t)=\sum_k c_k\sqrt{2}\varphi(2t-k)$ with $\sum_k |c_k|^2 < \infty$. This relation is called the dilation in time equation for φ; in frequency it follows (under the hypothesis $\varphi \in L^2 \cap L^1$)

$$\hat{\varphi}(\omega) = \sum_k c_k \exp\left(-ik\frac{\omega}{2}\right)\cdot\frac{1}{\sqrt{2}}\hat{\varphi}\left(\frac{\omega}{2}\right) = m_0\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right).$$

where $m_0\left(\frac{\omega}{2}\right) = \sum_k c_k \exp\left(-ik\frac{\omega}{2}\right)\cdot\frac{1}{\sqrt{2}}$.

We wish to determine a wavelet Ψ so that $\Psi(t-n), n\in\mathbb{Z}$ to be an orthonormal basis in W_0, where W_0 is the orthogonal complement of V_0 in V_1 , $V_1 = V_0 \oplus W_0$. If such a wavelet is determined then $(\forall)m$ fixed $\left(\Psi_{mn}\right)_n$ is an orthonormal basis in W_m.

Because $L^2 = \overline{\underset{m\in\mathbb{Z}}{\oplus} W_m}$,($\underset{m\in\mathbb{Z}}{\oplus} W_m$ is dense in L^2) it results that the family with two indices Ψ_{mn}, $m,n\in\mathbb{Z}$ forms an orthonormal basis for L^2.

It is known that $\Psi(t)$ is defined by:

$\Psi(t)=\sqrt{2}\sum_k c_{1-k}(-1)^k\varphi(2t-k)$ or, if $\Psi\in L^1\cap L^2$, then $\hat{\Psi}(\omega)=\exp\left(-\frac{i\omega}{2}\right)\overline{m_0\left(\frac{\omega}{2}+\pi\right)}\cdot\varphi\left(\frac{\omega}{2}\right)$.

2. Examples

1) A classic example is represented by the scalar function φ_H and the wavelet Ψ_H of Haar

$$\varphi(t)=\begin{cases}1, if\ t\in[0,1)\\ 0, otherwise\end{cases}\quad;\quad \Psi(t)=\begin{cases}1,\ t\in\left[0,\frac{1}{2}\right)\\ -1,\ t\in\left[\frac{1}{2},1\right)\\ 0,\ otherwise\end{cases}$$

Another classic example is the Shannon pair, where $\varphi(t)=\frac{\sin\pi t}{\pi t}$ $(t\neq 0);\varphi(0)=1$.

Developping such a function V_0 with respect to the Hilbert base $(\varphi(t-n))_{n\in\mathbb{Z}}$ is exactly the classic formula of sampling of Shannon.

2) It is also known Meyer's wavelet $\varphi(t)$ defined by

$$\hat{\varphi}(\omega) = \begin{cases} 1, |\omega| \le \dfrac{2\pi}{3} \\ \cos\left(\dfrac{\pi}{2} r\left(\dfrac{3|\omega|}{2\pi} - 1\right)\right), \dfrac{2\pi}{3} < |\omega| \le \dfrac{4\pi}{3} \end{cases}$$, $\varphi \in L_1 \cap L_2$, where $r(x)$ is a function C^k so that $r(x) = \begin{cases} 0, for\ x \le 0 \\ 1, for\ x \ge 1 \end{cases}$ and $r(x) + r(1-x) = 1$.

We propose to determine a scalar function $\varphi \in L^2$, $\varphi \in V_0$ such that the string $\varphi(2t-n)$ is a sampling string for $f \in V_0$ that is for any $f \in V_0$,

$$f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n}{2}\right) \varphi(2t-n) , \tag{3}$$

the convergence being in L^2.

In particular,

$$\varphi(t) = \sum_{n \in \mathbb{Z}} \varphi\left(\frac{n}{2}\right) \varphi(2t-n) \tag{3'}$$

If $\varphi \in L^1 \cap L^2$, then, applying the Fourier operator, one obtains:

$$\hat{\varphi}(\omega) = \frac{1}{2} \sum_{n} \varphi\left(\frac{n}{2}\right) \exp\left(-i\omega \frac{n}{2}\right) \hat{\varphi}\left(\frac{\omega}{2}\right).$$

3. Theorem 1

Let $\varphi \in V_0$ be a fixed scalar function.
The following statements are equivalent:
a) for any $f \in V_0$, we have

$$f(t) = \sum_{k \in 2\mathbb{Z}+1} f\left(\frac{k}{2}\right) \varphi(2t-k)$$ (the convergence is in L^2);

b) $$\varphi(t) = \sum_{j \in 2\mathbb{Z}+1} \varphi\left(\frac{j}{2}\right) \varphi(2t-j)$$ (in L^2)

<u>Proof:</u> (a)$\Rightarrow$(b) is obvious, since $\varphi \in V_0$.

(b)$\Rightarrow$(a). Since $\left(\varphi(t-n)\right)_{n\in\mathbb{Z}}$ is an orthonormal base in $L^2 \supset V_0$, $f \in V_0$, we have:

$$f(t) = \sum_{n\in\mathbb{Z}} \beta_n \varphi(t-n)\text{, where}$$

$$\sum_{n\in\mathbb{Z}} \beta_n^2 = \|f\|^2 < \infty \tag{4}$$

On the other hand, from (b) we have:

$$\varphi(t-n) = \sum_{j\in 2\mathbb{Z}+1} \varphi\left(\frac{j}{2}\right)\varphi(2t-2n-j)\text{, where } \sum_{j\in 2\mathbb{Z}+1} \varphi^2\left(\frac{j}{2}\right) < \infty$$

One obtains

$$f(t) = \sum_{n\in\mathbb{Z}} \beta_n \left(\sum_{j\in 2\mathbb{Z}+1} \varphi\left(\frac{j}{2}\right)\varphi(2t-2n-j) \right) \overset{k:=2n+j}{=}$$

$$= \sum_{n\in\mathbb{Z}} \beta_n \left(\sum_{k\in 2\mathbb{Z}+1} \varphi\left(\frac{k}{2}-n\right)\varphi(2t-k) \right) = \sum_{k\in 2\mathbb{Z}+1} \varphi(2t-k)\left[\sum_{n\in\mathbb{Z}} \beta_n \varphi\left(\frac{k}{2}-n\right) \right]$$

$$\overset{(4)}{=} \sum_{k\in 2\mathbb{Z}+1} \varphi\left(2t-k\right) f\left(\frac{k}{2}\right).$$

The convenient grouping of terms in the series in the computation from above is permitted, since the series is absolutely convergent in L^2, because t is commutative convergent (sumable). Precisely, we have:

$$\sum_{\substack{n\in\mathbb{Z}\\ k\in 2\mathbb{Z}+1}} \left| |\beta_n| \varphi\left(\frac{k}{2}-n\right) \right| \|\varphi(2t-k)\| = \sum_{\substack{n\in\mathbb{Z}\\ k\in 2\mathbb{Z}+1}} |\beta_n| \left|\varphi\left(\frac{k}{2}-n\right)\right| \cdot \frac{1}{\sqrt{2}}$$

Thus $f(t) = \sum_{k\in 2\mathbb{Z}+1} f\left(\frac{k}{2}\right)\varphi(2t-k) = \sum_{n\in\mathbb{Z}} \beta_n \varphi(t-n)$

both series being convergent in L^2. It follows:

$$\|f\|^2 = \sum_{k\in 2\mathbb{Z}+1} f^2\left(\frac{k}{2}\right)\cdot\frac{1}{2} = \sum_{n\in\mathbb{Z}} \beta_n^2 < \infty \Rightarrow \sum_{k\in 2\mathbb{Z}+1} f^2\left(\frac{k}{2}\right) < \infty .$$

The proof is complete.

Remark a) The implication (b) $\Rightarrow$(a) says that we can "know" f somehow, only knowing the samples $f\left(\frac{k}{2}\right), k \in \mathbb{Z}$, k odd integer, if this is possible for the scalar function φ.

b) It is well known that the convergence of a sequence of functions in L^p, $1 \le p < \infty$, implies the pointwise convergence almost everywhere of a subsequence. In our case, the sequence is the sequence of the partial sums of the series $\sum_{k \in 2\mathbb{Z}+1} f\left(\frac{k}{2}\right)\varphi(2t-k), (p=2)$.

4. Theorem 2

Assume that

$$\hat{\varphi}(\omega) = \left(\int_{\omega-\pi}^{\omega+\pi} |u(x)|\, dx\right)^{1/2}, \textit{where } u \in L^1(\mathbb{R}) \tag{5}$$

$\operatorname{supp} u \subset [-\varepsilon, \varepsilon]$ where $0 < \varepsilon < \pi$.

Let $\hat{\varphi}_1(\omega) := \sum_{k \in \mathbb{Z}} \hat{\varphi}(\omega + 4k\pi), \omega \in \mathbb{R}$. The following statements are equivalent:

(a) $\hat{\varphi}(\omega) = \hat{\varphi}_1(\omega) \cdot \hat{\varphi}\left(\frac{\omega}{2}\right)$, $(\forall)\, \omega \in \operatorname{supp} \hat{\varphi}$

(b) $\hat{\varphi}\left(\frac{\omega}{2}\right) = 1, (\forall)\, \omega \in \operatorname{supp} \hat{\varphi}$

Proof

From $\operatorname{supp} u \subset [-\varepsilon, \varepsilon]$ and by the assumption (5), one can deduce that $\operatorname{supp} \hat{\varphi} \subset [-\pi - \varepsilon, \pi + \varepsilon]$.This leads to:

$$\operatorname{supp} \hat{\varphi}_1 \subset \bigcup_{k \in \mathbb{Z}} \left[-\varepsilon - (4k+1)\pi, \varepsilon + (1-4k)\pi\right].$$

These facts imply: $\operatorname{supp} \hat{\varphi}_1 \cap \operatorname{supp} \hat{\varphi} \subset \operatorname{supp} \hat{\varphi} \subset [-\pi - \varepsilon, \pi + \varepsilon]$ and, for $\omega \in \operatorname{supp} \hat{\varphi}_1 \cap \operatorname{supp} \hat{\varphi}$, only one of the terms of the sum $\hat{\varphi}_1(\omega)$ is nonzero, and this term is equal to $\hat{\varphi}(\omega)$. To prove (a) $\Rightarrow$ (b) , observe that

$\omega \in \operatorname{supp}\hat{\varphi} \Rightarrow \omega \in \operatorname{supp}\hat{\varphi}_1$, hence $\hat{\varphi}(\omega) = \hat{\varphi}_1(\omega) \neq 0 \overset{(a)}{\Rightarrow} \hat{\varphi}\left(\frac{\omega}{2}\right) = 1\ \forall \omega \in \operatorname{supp}\hat{\varphi}$, i.e. (b).

(b) $\Rightarrow$ (a) is obvious. The proof is complete.

Example Let $u(x) = \begin{cases} 1, x \in [0,1) \\ 0, otherwise \end{cases}$.

Let $\hat{\varphi}(\omega) := \left(\int_{\omega-\pi}^{\omega+\pi} u(x)\,dx \right)^{1/2} = \left(\int_{\omega-\pi}^{\omega+\pi} \chi_{[0,1)}(x)\,dx \right)^{1/2} =$

$$= \begin{cases} (\omega+\pi)^{1/2}, -\pi \leq \omega < 1-\pi \\ 1, 1-\pi \leq \omega \leq \pi \\ (1-\omega+\pi)^{1/2}, \pi < \omega < 1+\pi \\ 0, \ otherwise \end{cases} \tag{6}$$

Thus $\operatorname{supp}\hat{\varphi} = [-\pi, 1+\pi]$ and $\omega \in \operatorname{supp}\hat{\varphi} \Leftrightarrow \frac{\omega}{2} \in \left[-\frac{\pi}{2}, \frac{1}{2} + \frac{\pi}{2} \right] \subset [1-\pi, \pi] \Rightarrow \hat{\varphi}\left(\frac{\omega}{2}\right) = 1$ by (6).

It follows that (a) holds for any $\omega \in [-\pi, 1+\pi]$.

Conclusion

Instead of looking for a sampling function f in V_0, this can be searched in V_1 and its recovery can be done through samples $f\left(\frac{n}{2}\right)$, where $n \in \mathbb{Z}$ are odd integers.

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