ON THE BILINEAR DYNAMICAL SYSTEMS

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În acest articol, se dau soluții aproximative pentru anumite clase de sisteme dinamice descrise în ecuațiile (1) şi (5) de mai jos. Soluțiile lor sunt căutate în clase de funcții constante pe porțiuni, de exemplu, funcții Walsh sau exponențiale discrete. În lucrare se utilizează unele concepte tehnice de algebră liniară, cum sunt produsele Kronecker sau pseudosoluțiile studiate în [3]. În partea secundă este studiat un sistem liniar nesingular (6) dat în [1].

In this paper one gives explicit approximative solutions of some classes of dynamical systems described in the equations (1) and (5) below. Their solutions are checked in some classes of functions which are constant on pieces, like the Walsh functions or the discrete exponentials; some practical tools of linear algebra (e.g. Kronecker tensor products, pseudosolutions, etc.) are systematically used.

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1. Introduction

Consider *a* dynamical system with command, where the state parameters $x_1(t), x_2(t), ..., x_n(t)$ verify some relations of the form

$$x'_i\ (t) = \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{k=1}^{p} u_k \left(\sum_{j=1}^{n} p_{ij}^{(k)} x_j \right) + \sum_{k=1}^{p} b_{ik} u_k\ , \tag{1}$$

$1 \le i \le n$, the coefficients a_{ij}, $p_{ij}^{(k)}$, b_{ik}, being real or complex constants. Such a system is called <u>bilinear</u>, with the command parameters $u_1(t),...u_p(t)$.

We now introduce the following matricial notations:

$x = (x_1, x_2, ..., x_n)^T$; $u = (u_1, u_2, ..., u_p)^T$;

$A = (a_{ij}),\ 1 \le i, j \le n$; $B = (b_{ik}),\ 1 \le i \le n$, $1 \le k \le p$ and

$P^{(k)} = (p_{ij}^{(k)})$; $1 \le i, j \le n$, $1 \le k \le p$.

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Keeping these into account, the system (1) can be equivalently written as follows:

$$x' = Ax + \sum_{k=1}^{p} u_k \cdot P^{(k)}x + Bu.$$

Moreover, putting $P = \sum_{k=1}^{p} u_k \cdot P^{(k)}$ (*a n*x*n* matrix), one finally obtains:

$$x' = (A+P)x + Bu \qquad (1')$$

In the case when $P^{(k)} = 0$ for any $1 \le k \le p$, one obtains the classical case of the linear systems with command.

2. Solving the system (1)

Fix a number *N>>1*, $N = 2^k$ and let $T = \{0,1,...,N-1\}$ be considered as a set of *N* moments of discrete time. Any function $x : T \to \mathbf{C}$ can be identified by a function constant on pieces, obtained by blocking its values up to the successive moment; such a function is usually called a discrete signal. Let $B = \{v_0, v_1, ..., v_{N-1}\}$ a standard basis of discrete signals $v_i : T \to \mathbf{C}$, [4], and denote by $V = (v_0 | v_1 | ... | v_{N-1})$ the *N*x*N* –matrix associated to. Consider also a *N*x*N* – matrix *Q* such that

$$\int_0^t V(t)\,dt = Q.V .$$

This is possible in the case when *B* is formed by the discrete exponentials $\{e_k(t)\}$, as well as *B* consists of the Walsh functions $\{w_k(t)\}$. The state parameters and the command parameters can be approximated by discrete signals $T \to \mathbf{C}$ (that is by corresponding functions which are constant on pieces). Under these conditions, we look for a solution *x*(*t*) of the system (1'), with the initial condition $x(0) = x_0$, such that $x' = C.V$, where *C* is a *n*x*N* – matrix with undetermined coefficients; therefore,

$$x(t) = x_0 + C\int_0^t V(t)\,dt . \qquad (2)$$

But

$\int_0^t V(t)\,dt = Q.V$, hence $x(t) = x_0 + C.Q.V,$

where *Q* is a known *N*x*N* - matrix. By substituting in (1'), it follows that $C.V = (A+P)(x_0 + C.Q.V) + Bu.$

But u=$D.V$, where D is a known pxN – matrix and similarly, $Ax_0 = E_0 V$, $P^{(k)}x_0$ =$E_k V$, where E_0, $E_1, \ldots, E_p$ are known nxN – matrices (this follows from the representation of the discrete signals by means of V). Then

$$(A+P)x_0 = \left(E_0 + \sum_{k=1}^{p} u_k E_k \right) .V$$

and

$$C.V = (A+P)C.Q.V + \left(E_0 + \sum_{k=1}^{p} u_k E_k \right) V + B.D.V.$$

Keeping into account that V is nonsingular (since $v_0, v_1, \ldots, v_{N-1}$ are linearly independent), it follows that

$$C=M.C.Q.+F, \tag{3}$$

where we put $F = E_0 + \sum_{k=1}^{p} u_k E_k + BD$ and M=A+P.

In the relation (3), the matrices M, Q, F are known and the unknown is C.

From all above said, one deduces the following:

Proposition 1. The solution of the system (1) is given by a matricial equation of the form

$$X\text{-}MXQ=F, \tag{4}$$

where $M \in M_n(\mathbf{C})$, $Q \in M_N(\mathbf{C})$, $F \in M_{n,N}(\mathbf{C})$ are given and $X \in M_{n,N}(\mathbf{C})$ is unknown.

A well-known result asserts that whenever M is nonsingular, the equation (4) has solution if and only if the matrices

$$\begin{pmatrix} M^{-1} & M^{-1}F \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} M^{-1} & 0 \\ 0 & Q \end{pmatrix}$$

are similar. Instead of this we give a more general result. Recall by [2] that if $A \in M_{p,q}(\mathbf{C})$, $B \in M_{m,n}(\mathbf{C})$, then their Kronecker tensor product is the following pmxnq – matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \ldots & a_{1q}B \\ \ldots & \ldots & \ldots & \ldots \\ a_{p1}B & a_{p2}B & \ldots & a_{pq}B \end{pmatrix}.$$

Denote by $\widetilde{A}$ the pq-dimensional column-vector obtained by putting in a single column first column of A, followed by the second, third etc. Then $(\widetilde{ADB}) = (B^T \otimes A) \cdot \widetilde{D}$, for any $D \in M_{qm}(\mathbf{C})$ (the Neudecker formula). So

$(\widetilde{MXQ}) = (Q^T \otimes M)\cdot \widetilde{X}$ and the equation (4) becomes $\widetilde{X} - (\widetilde{MXQ}) = \widetilde{F}$, that is $\widetilde{X} - (Q^T \otimes M)\cdot \widetilde{X} = \widetilde{F}$ and so

$$(I_N - Q^T \otimes M)\cdot \widetilde{X} = \widetilde{F} \tag{4'}$$

<u>Propostion 2.</u> Let $\sigma(Q) = \{\lambda_1,...,\lambda_r\}$, $\sigma(M) = \{\mu_1,...,\mu_S\}$ be the matrix spectrums and suppose that $\lambda_i \cdot \mu_j \neq 1$ for any $1 \le i \le r$, $1 \le j \le s$. Then the equation (3) has a unique solution, namely

$$\widetilde{C} = (I_N - Q^T \otimes M)^{-1}.\widetilde{F}.$$

<u>Proof</u>. The eigen values of the matrix $Q^T \otimes M$ are just the products $\lambda_i \cdot \mu_j$ and those of the matrix $I_N - Q^T \otimes M$ will be $1 - \lambda_i \cdot \mu_j$; the latter being nonzero (by hypothesis), it will follow that the square matrix $I_N - Q^T \otimes M$ is invertible and apply (4').

<u>Remark</u>. From (4') we can deduce without any restrictions the pseudosolution of (3) and (4'), namely: $(\widetilde{C^+}) = (I_N - Q^T \otimes M)^+ \cdot \widetilde{F}$.

<u>Application</u>. Suppose now *B*=0*, p*=1, hence an homogeneous linear system by the form $x' = Ax + Pxu$, $P \in M_n(\mathbf{C})$, with the initial condition $x(0) = x_0$ and we check the command u which minimizes a quadratic functional of the form

$$J(x,u) = \frac{1}{2}\int_0^a (x^T \cdot S \cdot x + u^2)dt.$$

In order to determine the command u^* which drives the system from the state $x_0 = x(0)$ to the state x(a) and minimizes *J,* one considers the hamiltonian $H = \frac{1}{2}x^T Sx + \frac{1}{2}u^2 + \lambda^T \cdot (Ax + Pxu)$. By the Pontreagyne principle, $\frac{\partial}{\partial u}(H^T) = 0$, hence $u^* = -x^T \cdot P^T$. Then $x' = \frac{\partial}{\partial \lambda}(H^T) = Ax + PxU$ and $\lambda' = -\frac{\partial}{\partial x}(H^T) = -S^T x - A^T \cdot \lambda - P^T \lambda u$, with $x(0) = x_0$ and λ (a)=0.

Consider the 2n - dimensional column vector $z = \begin{pmatrix} x \\ \lambda \end{pmatrix}$ and it follows that $z' = A_1 z - P_1 z z^T P_2 z$, where

$$A = \begin{pmatrix} A & 0 \\ -S^T & -A^T \end{pmatrix}, \quad P_1 = \begin{pmatrix} P & 0 \\ 0 & -P^T \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & P^T \\ 0 & 0 \end{pmatrix}$$

Relatively to the standard basis *B*, we have *z=G.V*, where G is a *2n*x*N* – matrix which is unknown. Then $C_1 V = A_1 GV - P_1 GVV^T G^T P_2 GV$ and *V* being nonsingular, it follows that $A_1 G - P_1 GVV^T G^T P_2 G = C_1$,
with A_1, P_1, P_2, C_1 given. If *G* is thus determined, then *z* and a fortiori *x(t)*, *λ(t)* will be determined and in the same time, the checked optimal command will be $u^* = -x^T \cdot P^T \cdot \lambda$.

In the case of the Walsh basis (or that of the discrete exponentials), one can give explicit formulas.

3. A nonsingular linear system

Consider a dynamical system from [1], by the form

$$\sum_{j=1}^{n} b_{ij} x_j^{'}(t) = \sum_{j=1}^{n} a_{ij} x_j(t) + \varphi_i(x_1(t),...,x_n(t)), \quad 1 \le i \le n, t \in [0,1], \tag{5}$$

with the initial condition $x(0) = x_0$ Put $B = (b_{ij})$, $1 \le i, j \le n$; $A = (a_{ij})$, $1 \le i, j \le n$; $\varphi = (\varphi_1,...,\varphi_n)^T$ supposed continuos, it follows the matricial form of the above system:

$$Bx' = Ax + \varphi(x(t)). \tag{6}$$

Let $\{w_k(t)\}_{k \ge 0}$ the sequence of Walsh functions on the interval [0,1]; any function $f \in L^2_{[0,1]}$ can be represented as a series $f(t) = \sum_{k=1}^{\infty} c_k w_k(t)$, $t \in [0,1]$. In particular, $\varphi(x(t))$ is approximated by a partial sum of the type $\sum_{k=1}^{N} c_k w_k(t)$, with $N = 2^k$ convenient. Denote $c = (c_0, c_1,...,c_{N-1})^T$ and $V = (w_0(t) | w_1(t) | ... | w_{n-1}(t))$ hence $\varphi(x(t)) \approx c^T \cdot V(t)$.

By reasoning like in § 1, one gets from (6) the matricial equation $BCV = A(x_0 + CQV) + c^T \cdot V$, with unknown C.

Then

$BCV - ACQV = Ax_0 + c^T \cdot V$,

hence

$(BCV\widetilde{)} - (ACQV\widetilde{)} = (Ax_0 + c^T \cdot V\widetilde{)}$

and by the Neudecker relation,

$(V^T \otimes B) \cdot \widetilde{C} - ((QV)^T \otimes A) \cdot \widetilde{C} = (Ax_0 + c^T V\widetilde{)}$,

with the pseudosolution $(V^T \otimes B - (QV)^T \otimes A)^+ \cdot (Ax_0 + c^T V\widetilde{)}$.

Example. Consider the system $x'_2(t) = -x_1, \quad 2x_2 = x_1^2$. Put $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \varphi = \begin{pmatrix} 0 \\ -x_1^2 \end{pmatrix}$ and the system can be simply written $Bx' = Ax + \varphi(x(t))$. If the initial condition is $x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the exact solution is $x_1(t) = -t, \quad x_2(t) = \frac{t^2}{2}, \quad t \in [0,1]$. The above presented method gives good numerical results.

R E F E R E N C E S

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