CONCERNING A PROBLEM OF E. BOREL

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Este cunoscut rolul deosebit al bazelor ortonormale în spatii concrete de functii; acestea permit o abordare eleganta si eficienta a conversiei analogic - digitala a semnalelor.

În anul 1910 E. Borel a considerat o serie de functii f_n *din* $L^2(a,b)$ *care sa formeze o baza ortonormala si în plus fiecare functie* f_n *sa ia cel mult doua valori.*

Primul exemplu a fost dat de Walsh. În aceasta lucrare se da o alta descriere a functiilor Walsh si o demonstratie noua a completitudinii sistemului Walsh. Totodata se arata legatura cu functiile Rademacher si Haar folosite în studiul semnalelor discrete.

It is widely known the special role of orthonormal basis in certain spaces of functions; they allow an elegant and efficient approach of analogic - digital conversion of signals.

In 1910 E. Borel considered a series of functions f_n *from* $L^2(a,b)$ *which will form an orthonormal basis and, moreover each function* f_n *will have at most two values.*

The first example was given by Walsh. In this paper we have another description of Walsh functions and a new proof of the complexity of Walsh system. In the same time the connection between Rademacher and Haar functions used in the study of discrete signals is shown.

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Introduction

The Walsh functions are used in Electronics at connecting problems in data transmissions. This paper gives an original proof that Walsh functions form an orthonormal base in $L^2_{(0,1)}$.

1. Preliminaries

Let H be a Hillbert space (complex); $(e_n)_{n\in\mathbb{N}}$ an orthonormal string in H and $(a_n)_{n\in\mathbb{N}}$ a string of complex numbers. The series $\sum_n a_n e_n$ is known to converge in H with the sum s if and only if $(a_n)\in l_2$; moreover $a_n = \langle s, e_n\rangle$.

If (e_n) is an orthonormal basis and if there is $t\in H$ so that $a_n = \langle t, e_n\rangle$ for any n, then $t = s$.

The set of indexes can be replaced with any other countable set.

We also remind the following fact:

2. Lemma

Let $B = (e_n)$ an orthonormal string in H. The following 5 assertions are equivalent:

a) The subspace generated by B is dense in H.

b) If $x\in H$ and $x \perp e_n, (\forall) n$ then $x = 0$.

c) $\forall\ x\in H$, $\sum_n |a_n|^2 = \|x\|^2$, where $a_n = \langle x, e_n\rangle$

d) $\forall\ x\in H$, $x = \sum_n c_n e_n$, where $c_n = \langle x, e_n\rangle$

e) $\forall x, y\in H, < x, y >= \sum_n c_n \overline{d_n}$, where $c_n = \langle x, e_n\rangle$, $d_n = \langle y, e_n\rangle$.

Any string of vectors from H satisfying one of these 5 conditions is called orthonormal basis in H.

3. Examples

1) Let $H = L^2_{(0,1)}$ with scalar product $\langle f, g \rangle = \int_0^1 f(x) \cdot \overline{g(x)}\, dx$. The string $e_n = \sqrt{2} \sin np\, x\ (n \geq 1)$ is an orthonormal basis.

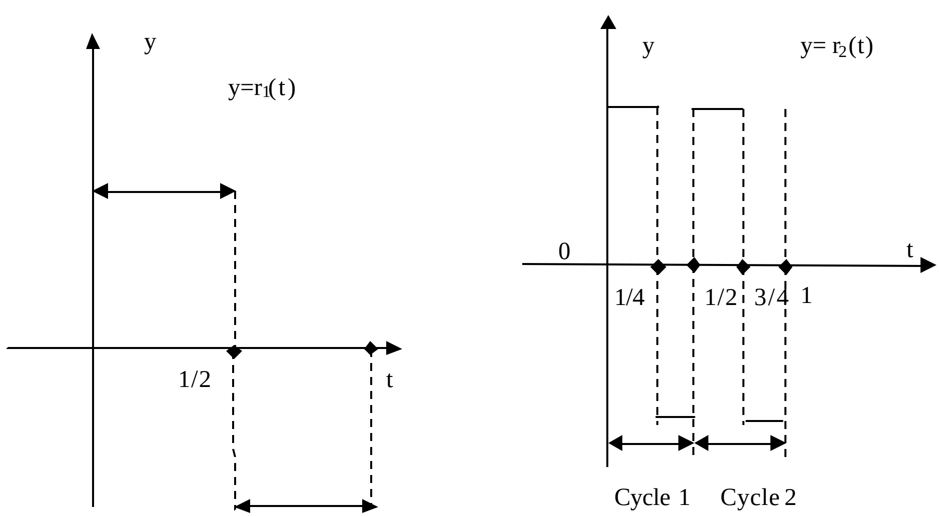
We prove that the assertion c) from the above lemma is true.

$\sum_n |a_n|^2 = \|x\|^2$ where $a_n = \langle x, e_n \rangle$.

$a_n = \langle x, e_n \rangle = \int_0^1 x \cdot \sqrt{2} \cdot \sin np\, x\, dx$ which through integration through parts is $\frac{(-1)^{n+1} \cdot \sqrt{2}}{np}$; $\sum_n |a_n|^2 = \frac{2}{p^2} \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k^2} = \frac{2}{p^2} \cdot \frac{p^2}{6} = \frac{1}{3} = \|x\|^2$

2) The discontinuous functions $j_n : (0,1) \to \mathbb{R}$ $j_n(t) = \operatorname{sgn}(\sin np\, t), n \geq 1$ are considered. This string is not orthonormal in $L^2_{(0,1)}$, but satisfies the condition b) from the previous lemma (if $f \in L^2_{(0,1)}$ and $f \perp j_n, \forall n$ then $f = 0$ a.e.).

This string (j_n) has a substring that is $r_n : (0,1) \to \mathbb{R}$, $r_n = j_{2^n}, n \geq 1$ called Rademacher functions. The graphs of the first three functions Rademacher are shown in fig. 1.



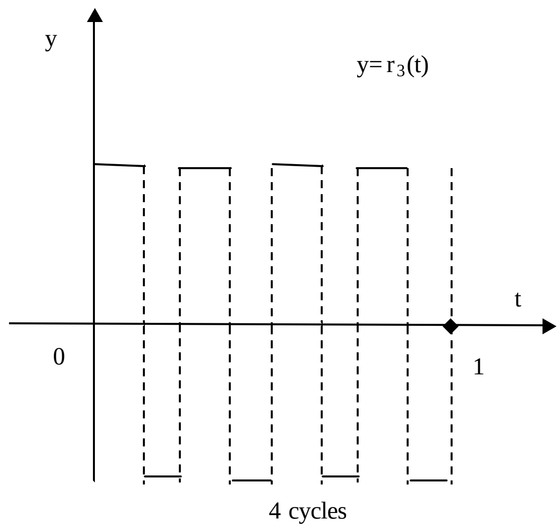


Fig. 1

It is easily proved that an orthogonal string is resultated in $L^2_{(0,1)}$ but not an orthonormal basis (because $\cos 2p t \perp r_n, \forall n$ and it is not satisfied condition b) from the previous lemma).

The functions r_n only take 1 and -1 values. The definition domain of the function r_n can be divided in 2^{n-1} cycles of length $\frac{1}{2^{n-1}}$, and on half of them r_n takes the value 1 and on the other half -1.

E. Borel considered, in 1910, the problem of finding a string of functions from $L^2_{(0,1)}$ which will only take 2 values, but which will form an orthonormal base.

The first example was made by Walsh in 1924 starting from the string (r_n), $n \geq 1$. He was the one to "completed" this string adding other functions. He considered the string (w_n) $n \geq 1$ of Walsh functions defined as the following: $w_1 = 1$; then for any integer $k \geq 1$ we have a unique form in basis 2:

$k = 2^{n_1} + 2^{n_2} + ...2^{n_p}$ with $n_1 > n_2 ...... > n_p \geq 0$ and we define $w_{k+1} = r_{n_1+1} \cdot r_{n_2+1} \cdot ...... \cdot r_{n_p+1}$.

In the case of $p = 1$, so $k = 2^{n_1} \left(n_1 \geq 0\right)$ we find again r_{n_1+1}. In this way the functions Rademacher $r_1, r_2, r_3......$ are among Walsh functions.

It is obvious that the Walsh functions take only the values 1 and -1.

The following theorem belongs to Walsh but we have another proof:

4. Theorem

The functions Walsh $\left(w_k\right), k \geq 1$ form an orthonormal basis in $L^2_{(0,1)}$.

Proof: The product of two functions Walsh will be of the following form $\left(r_{m_1}\right)^{a_1} \cdot \left(r_{m_2}\right)^{a_2} ......\left(r_{m_p}\right)^{a_p}$ where $m_1 > m_2 > ......m_p \geq 1$ are integers and a_k are equal to 1 or 2.

If two functions coincide then $a_k = 2$ and $r^2_{mk} \equiv 1$ a.e. so $\int_0^1 r^2_{m_k} \equiv 1$.

If f, g are distinct then in $\langle f, g \rangle$ we can renumber the indexes and we only remember the functions Rademacher at one squared with indexes $m_1, m_2......m_q$. But the product $r_{m_2}\left(x\right)......r_{m_q}\left(x\right)$ is constant with values 1 or -1 on each on the two semicycles 2^{m_2} or r_{m_2}. A typical semicycle I_{m_2} is divided in $2^{m_1 - m_2}$ semicycles of r_{m_1} in which r_{m_1} is alternative +1 and -1. So $\int_0^1 r_{m_1}......r_{m_q} = \sum_{I\,m_2} \int_{I\,m_2} (const) \cdot r_{m_1}$ so equal with 0, because $\int_{I\,m_2} r_{m_1} = 0$.

Let $f \in L^2_{(0,1)}$ and $F\left(x\right) = \int_0^x f\left(t\right) dt$ so $F\left(0\right) = 0$.

Because after defining Walsh's function we have

$w_1=1; w_2=r_1; w_3=r_2; w_4=r_1\cdot r_2, w_5=r_3$ we get:

$$f\perp w_1 \Rightarrow \int_0^1 f w_1 = 0 \text{ which is } F(1)=0 \tag{1}$$

$$f\perp w_2 \Rightarrow \int_0^1 f w_2 = 0 \Leftrightarrow \int_0^1 f r_1 = 0 \Leftrightarrow \int_0^{1/2} f - \int_{1/2}^1 f = 0 \Leftrightarrow$$

$$F\left(\frac{1}{2}\right)-\left(F(1)-F\left(\frac{1}{2}\right)\right)=0 \Leftrightarrow F\left(\frac{1}{2}\right)=0 \tag{2}$$

$$f\perp w_3 \Rightarrow \int_0^1 f w_3 = 0 \Leftrightarrow \int_0^1 f r_2 = 0 \Leftrightarrow \int_0^{1/4} f - \int_{1/4}^{1/2} f + \int_{1/2}^{3/4} f - \int_{3/4}^1 f = 0 \Leftrightarrow$$

$$F\left(\frac{1}{4}\right)-F\left(\frac{1}{2}\right)+F\left(\frac{1}{4}\right)+F\left(\frac{3}{4}\right)-F\left(\frac{1}{2}\right)-F(1)+F\left(\frac{3}{4}\right)=0$$

$$\Leftrightarrow F\left(\frac{1}{4}\right)+F\left(\frac{3}{4}\right)=0 \tag{3}$$

$$f\perp w_4 \Rightarrow \int_0^1 f w_4 = 0 \Leftrightarrow \int_0^1 f r_1 r_2 = 0 \Leftrightarrow \int_0^{1/4} f - \int_{1/4}^{3/4} f + \int_{3/4}^1 f =$$

$$=F\left(\frac{1}{4}\right)-F\left(\frac{3}{4}\right)+F\left(\frac{1}{4}\right)+F(1)-F\left(\frac{3}{4}\right)=0$$

$$\Leftrightarrow F\left(\frac{1}{4}\right)-F\left(\frac{3}{4}\right)=0 \tag{4}$$

From relation (3) and (4) we have

$$F\left(\frac{1}{4}\right)=0, F\left(\frac{3}{4}\right)=0 \tag{5}$$

$$f\perp w_5 \Rightarrow \int_0^1 f w_5 = 0 \Leftrightarrow \int_0^1 f r_3 = 0 \Leftrightarrow$$

$$\int_0^{1/8} f - \int_{1/8}^{1/4} f + \int_{1/4}^{3/8} f - \int_{3/8}^{1/2} f + \int_{1/2}^{5/8} f - \int_{5/8}^{3/4} f + \int_{3/4}^{7/8} f - \int_{7/8}^{1} f = 0$$

$$\Leftrightarrow F\left(\frac{1}{8}\right) - F\left(\frac{1}{4}\right) + F\left(\frac{1}{8}\right) + F\left(\frac{3}{8}\right) - F\left(\frac{1}{4}\right) - F\left(\frac{1}{2}\right) + F\left(\frac{3}{8}\right) + F\left(\frac{5}{8}\right) - F\left(\frac{1}{2}\right) - F\left(\frac{3}{4}\right) +$$

$$+F\left(\frac{5}{8}\right) + F\left(\frac{7}{8}\right) - F\left(\frac{3}{4}\right) - F(1) + F\left(\frac{7}{8}\right) = 0 \Leftrightarrow$$

$$\Leftrightarrow F\left(\frac{1}{8}\right) + F\left(\frac{3}{8}\right) + F\left(\frac{5}{8}\right) + F\left(\frac{7}{8}\right) = 0 \tag{6}$$

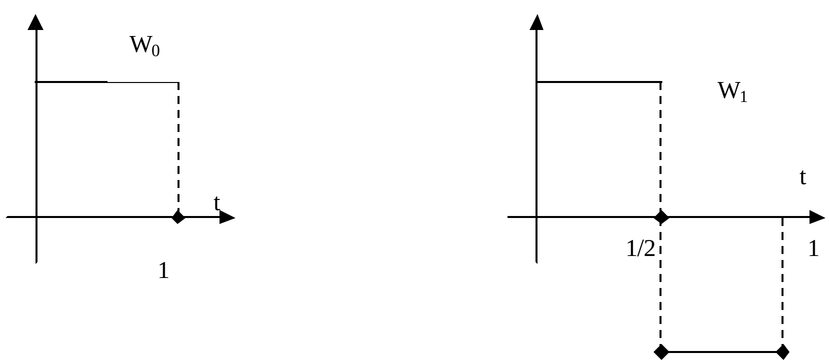
Than from $f \perp w_6, w_7, w_8$ it results $F\left(\frac{1}{8}\right) = F\left(\frac{3}{8}\right) = F\left(\frac{5}{8}\right) = F\left(\frac{7}{8}\right) = 0$.

Than through incomplete induction it results that F is cancelled on $S = \left\{ \frac{2k+1}{2^n}, k, n \in \mathbb{N}, 2k+1 \leq 2^n - 1 \right\}$. Because S is dense in $(0,1)$ and F is continuous $\Rightarrow F = 0$ on $(0,1)$ so $f = 0$ a.e. in $(0,1)$.

5. Observation

In many papers the Walsh functions $(w_k), k \geq 1$ are ordered differently so that each w_k could have exactly $k+1$ "crossing in 0" in the interval $(0,1)$.

The graphs of the first 3 Walsh functions are shown in fig. 2.



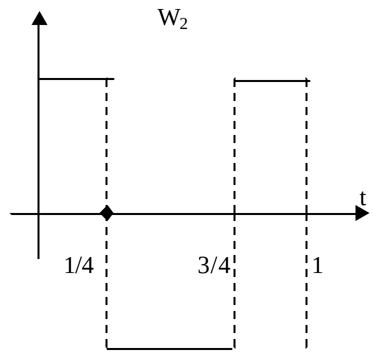


Fig.2

For the sinc signals it is fundamental that notion of frequency.

Let the family of functions $(\sin 2\pi wt), w \in \mathbb{N}^*$ for any w the function $t \to \sin 2\pi wt$, has the period $\frac{1}{w}$ and in any semi-open interval of length $\frac{1}{w}$, $\sin 2\pi wt$ has $2w$ zeroes.

The index w appears as being equal with half the number of zeroes of the signal $\sin 2\pi wt$ in a time unit.

For the Walsh functions, the notion of sequence is defined as being half the number of changes of signal in time unit.

Haar build an orthonormal base for $L^2_{(0,1)}$, formed with functions having at most three values, which approximates uniformly any continuous function $f:[0,1] \to \mathbb{R}$.

Functions Haar have a systematic application in the wavelets theory.

The definition of functions $h_n : (0,1) \to \mathbb{R}$, $n \geq 1$ is the following:

$$h_1 = 1\ ;\ \ h_{2^k+l}(x) = \begin{cases} 2^{k/2},\ if\ x \in \left[\dfrac{l-1}{2^k}, \dfrac{l-\frac{1}{2}}{2^k}\right] \\ -2^{k/2},\ if\ x \in \left[\dfrac{l-\frac{1}{2}}{2^k}, \dfrac{l}{2^k}\right] \\ 0, \text{to the rest of the interval} \left(0,1\right) \end{cases}$$

for $l = 1,2,\ldots\ldots\ldots 2^k;\ k \geq 0$

The definition domain of each function Haar divided in 2^k cycles of length $\frac{1}{2^k}$.

The graph of a Haar function is shown in figure 3.

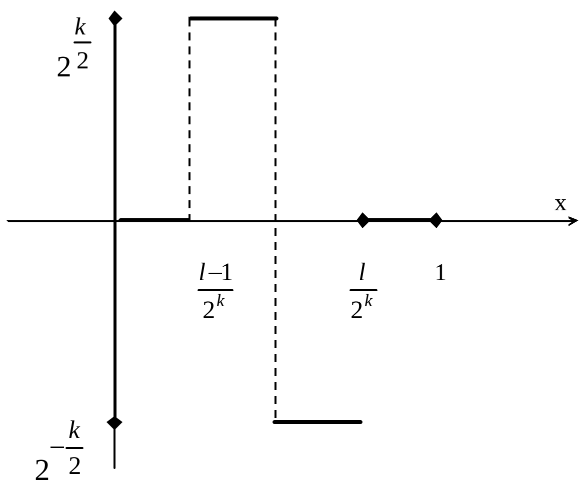


Fig. 3

It can be proven that the string $\left(h_n\right)_{n\geq 1}$ forms an orthonormal basis in $L^2_{(0,1)}$.

Conclusions

I have studied a series of functions (f_n) from $L^2(a,b)$ which will form an orthonormal basis and, moreover, each function f_n will have at most two values. The results have applications in wavelet's theory.

I have given a new proof for the fact that the string $(w_k)_{k\geq 1}$ of Walsh's functions is a Hillbert base in $L^2(0,1)$.

In this article I offered a rather more simple proof than the one in paper [1].

R E F E R E N C E S

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