A MODIFIED ITERATIVE ALGORITHM FOR SOLVING SPLIT EQUALITY PROBLEMS

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In this paper, we investigate the split equality problems in Hilbert spaces. We propose an iterative algorithm with self-adaptive step-size for solving split equality problems. Convergent analysis of the suggested algorithm is proved under some suitable conditions.

Keywords: split equality problem, self-adaptive step-size, projection, subgradient, convergence.

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1. Introduction

Let H_1, H_2 and H_3 be three real Hilbert spaces. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two linear bounded operators. Let A^* and B^* be adjoint of A and B, respectively. The split equality problem can be mathematically formulated as the problem of finding $x \in C$ and $y \in Q$ such that

$$Ax = By. \tag{1}$$

When $H_2 = H_3$ and $B = I$, then the split problem (1) reduces to the split feasibility problem introduced by Censor and Elfving [?] which is to find $x \in C$ such that

$$Ax \in Q. \tag{2}$$

The split problems have a range of applications, for example, in image reconstruction, intensity modulated radiation therapy, signal processing and so on. The split problems have been studied extensively in the literature, please refer to [1]-[3], [7]-[13] and [[17, 18, 19, 24, 31]. Their theories are closely related to other nonlinear problems, such as fixed point problems ([4, 5], [20]-[29], [37, 39]), equilibrium problems ([27]) and variational inequality problems ([14], [30]-[43]). In order to solve the split equality problem (1), Moudafi [15] established an alternating CQ-algorithm:

$$\begin{cases} x_{n+1} = P_C(x_n - \rho_n A^*(Ax_n - By_n)), \\ y_{n+1} = P_Q(y_n + \rho_n B^*(Ax_{n+1} - By_n)), n \in N, \end{cases} \tag{3}$$

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where $\rho_n \in (\varepsilon, \min\{\frac{1}{\lambda_A}, \frac{1}{\lambda_B}\} - \varepsilon)$ with $\varepsilon > 0$, λ_A and λ_B are the spectral radii of of A^*A and B^*B, respectively.

Let $c : H_1 \to R$ and $q : H_2 \to R$ be convex and subdifferentiable functions. Set $C = \{x \in H_1 : c(x) \leq 0\}$ and $Q = \{y \in H_2 : q(y) \leq 0\}$. In [?], Moudafi established a relaxed alternating CQ-algorithm based on projections onto half-spaces as follows:

$$\begin{cases} x_{n+1} = P_{C_n}(x_n - \gamma A^*(Ax_n - By_n)), \\ y_{n+1} = P_{Q_n}(y_n + \beta B^*(Ax_{n+1} - By_n)), n \in N. \end{cases} \tag{4}$$

where $\gamma = \beta \in (0, \min\{\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}\})$ and

$$C_n = \{x \in H_1 : c(x_n) + \langle \xi_n, x - x_n \rangle \leq 0\},\ \xi_n \in \partial c(x_n),$$

and

$$Q_n = \{y \in H_2 : q(y_n) + \langle \eta_n, y - y_n \rangle \leq 0\},\ \eta_n \in \partial q(y_n).$$

Consequently, Moudafi and Byrne [?] suggested the following project Landweber algorithm for solving split equality problem (1)

$$\begin{cases} x_{n+1} = P_C(x_n - \rho_n A^*(Ax_n - By_n)), \\ y_{n+1} = P_Q(y_n + \rho_n B^*(Ax_n - By_n)), n \in N. \end{cases} \tag{5}$$

where $\rho_n \in (\varepsilon, \frac{2}{\lambda_A + \lambda_B})$.

Very recently, many scholars implement the algorithms by using the self-adaptive techniques ([10, 18, 23, 24, 28, 31, 39) for solving the split problems in which the information of the operator norms do not need to be known in advance. Inspired by the work in this direction, in this paper, we presented a modified iterative algorithm for solving split equality problem (1). The modofied algorithm uses a self-adaptive step-size and a new searching direction. We prove that the suggested algorithm converges to a solution of the split equality problem (1).

2. **Preliminaries**

In this part, we give some definitions, notions and lemmas, which will be used in the following parts.

Definition 2.1. *Let $F : X \subset R^N \to R^N$ be a mapping. F is said to be*

(i) *monotone, if*

$$\langle F(x) - F(y), x - y \rangle \geq 0, \quad \forall x, y \in X.$$

(ii) *co-coercive with modulus $\alpha > 0$, if*

$$\langle F(x) - F(y), x - y \rangle \geq \alpha \|F(x) - F(y)\|^2, \quad \forall x, y \in X.$$

(iii) *Lipschitz continuous with constant $\lambda > 0$, if*

$$\|F(x) - F(y)\| \leq \lambda \|x - y\|, \quad \forall x, y \in X.$$

(iv) *nonexpansive, if*

$$\|F(x) - F(y)\| \leq \|x - y\|, \quad \forall x, y \in X.$$

Definition 2.2. *An operator $f : X \to R$ is said to be lower semi-continuous at $x \in X$ if $\liminf_{y \to x} f(y) \geq f(x)$.*

Definition 2.3. *An operator $U : X \to X$ is called asymptotically regular if*

$$\lim_{k\to\infty} \|U^{k+1}x - U^k x\| = 0,$$

for all $x \in X$.

For the given nonempty closed convex set Ω in R^N, the orthogonal projection from R^N onto Ω is defined by

$$P_\Omega(x) = \arg\min\{\|x - y\| \mid y \in \Omega\}, x \in R^N.$$

Lemma 2.1 ([18])**.** *Let Ω be a nonempty closed convex subset in R^N. Then for any $x, y \in R^N$ and $z \in \Omega$, the following inequalities hold*

(i) $\langle P_\Omega(x) - x, z - P_\Omega(x)\rangle \geq 0$;
(ii) $\|P_\Omega(x) - P_\Omega(y)\|^2 \leq \langle P_\Omega(x) - P_\Omega(y), x - y\rangle$;
(iii) $\|P_\Omega(x) - z\|^2 \leq \|x - z\|^2 - \|P_\Omega(x) - x\|^2$.

Let $F : R^N \to R^N$ be a mapping. For any $x \in R^N$ and $\alpha > 0$, define

$$x(\alpha) = P_\Omega(x - \alpha F(x)), \tag{6}$$

and

$$e(x, \alpha) = x - x(\alpha). \tag{7}$$

It is known that $\|e(x, \alpha)\|/\alpha$, $\alpha > 0$ is nonincreasing ([**?**]) and $\|e(x, \alpha)\|$, $\alpha > 0$ is nondecreasing ([23]).

Lemma 2.2 ([18])**.** *For any $x \in R^N$ and $\alpha > 0$, we have*

$$min\{1, \alpha\}\|e(x, 1)\| \leq \|e(x, \alpha)\| \leq max\{1, \alpha\}\|e(x, 1)\|.$$

Lemma 2.3 ([10])**.** *Assume that $h : R^N \to R$ is a convex function, then it is subdifferentiable and it's subdifferentials are uniformly bounded on any bounded subset of R^N.*

3. Main results

Let C and Q be nonempty closed convex sets in R^N and R^M, respectively. Let A and B be J by N and J by M real matrices, respectively. Let $I = M + N$. Define

$$D = [A, \ -B], v^k = \begin{bmatrix} x^k, & y^k \end{bmatrix}^T,$$

$$D^T D = \begin{bmatrix} A^T A, & -A^T B \\ -B^T A, & B^T B \end{bmatrix}.$$

We give two assumptions as follows:
(B_1) The convex and nonempty sets C and Q are given by

$$C := \{x \in R^N | c(x) \leq 0\},$$

and

$$Q := \{y \in R^M | q(y) \leq 0\},$$

where $c : R^N \to R$ and $q : R^M \to R$ are convex functions.
(B_2) For any $x \in R^N, y \in R^M$, $\partial c(x)$ and $\partial q(y)$ denote the generalized gradient of $c(x)$ and $q(y)$, respectively. At least one subgradient $\xi \in \partial c(x)$ can be computed, so we define

$$\partial c(x) = \{\xi \in R^N \mid c(z) \geq c(x) + \langle \xi, z - x\rangle \ for \ all \ z \in R^N\}$$

and at least one subgradient $\eta \in \partial q(y)$ can be computed, so we defined

$$\partial q(y) = \{\eta \in R^M \mid q(u) \geq q(y) + \langle \eta, u - y\rangle \ for\ all\ u \in R^M\}.$$

Define

$$C_k := \{x | c(x^k) + \langle \xi^k, x - x^k\rangle \leq 0\},$$

and

$$Q_k := \{y | q(y^k) + \langle \eta^k, y - y^k\rangle \leq 0\},$$

where $\xi^k \in \partial c(x^k)$ and $\eta^k \in \partial q(y^k)$.

Set $\zeta_k = [\xi^k, \eta^k]^T$. Let $l : R^N \times R^M$ and

$$l(v) = l(x, y) = c(x) + q(y), \quad \partial l(v^k) = \partial c(x^k) \times \partial q(y^k),$$

$$S := \{v | l(v) \leq 0\}, \quad S_k := \{v | \langle \zeta_k, v - v^k\rangle + l(v^k) \leq 0\}.$$

It is clear that $C \subset C_k$ and $Q \subset Q_k$ and $C \times Q \subset S$.

Note that the split equality problem (1) is equivalent to the following minimization problem

$$\min_{x \in C, y \in Q} \frac{1}{2}\|Ax - By\|^2.$$

Define

$$l(v) = \frac{1}{2}\|Ax - By\|^2,$$

i.e.

$$l(v) = \frac{1}{2}\|Dv\|^2.$$

For all k define $F_k : R^I \to R^I$ as follows:

$$F_k(v) = D^T D(v).$$

Next, we introduce our algorithm for solving (1).

Algorithm 3.1. *Let $\tau \in (0, 2)$, $\beta \in (0, 1)$ and $\mu \in (0, 1)$ be given constants and let $v^0 \in R^I$ be an initial guess. Let the sequence $\{v^k\}$ be defined by*

$$\begin{cases} \overline{v}^k = P_{S_k}(v^k - \lambda_k F_k(v^k)), \\ d_k = v^k - \overline{v}^k + \lambda_k F_k(\overline{v}^k) - \lambda_k F_k(v^k), \\ v^{k+1} = v^k - \lambda_k d_k, k \geq 0, \end{cases}$$

where $\lambda_k = \tau\beta^{m_k}$ and m_k is the smallest nonnegative integer m such that

$$\|F_k(v^k) - F_k(\overline{v}^k)\| \leq \mu \frac{\|v^k - \overline{v}^k\|}{\lambda_k}. \tag{8}$$

Lemma 3.1. *(i) F_k is Lipschitz continuous with constant L and co-coercive with modulus $\frac{1}{L}$, where L is the largest eigenvalue of the matrix $D^T D$; (ii) the search rule (8) is well defined and $\frac{\mu\beta}{L} < \lambda_k \leq \tau$.*

Proof. Take any $x, y \in R^I$. By the definition of F_k, we can get

$$\|F_k(x) - F_k(y)\|^2 = \|D^T Dx - D^T Dy\|^2 \leq L\|Dx - Dy\|^2,$$

and

$$\begin{aligned} \langle F_k(x) - F_k(y), x - y\rangle &= \langle D^T Dx - D^T Dy, x - y\rangle \\ &= \langle Dx - Dy, Dx - Dy\rangle \\ &= \|Dx - Dy\|^2. \end{aligned}$$

So

$$\|F_k(x) - F_k(y)\|^2 \leq L\|Dx - Dy\|^2 = L\langle F_k(x) - F_k(y), x - y\rangle,$$

that is

$$\langle F_k(x) - F_k(y), x - y\rangle \geq \frac{1}{L}\|F_k(x) - F_k(y)\|^2.$$

Obviously, $\lambda_k \leq \tau$ by the search rule (8). Note that λ_k/β violate the search rule (8), i.e.,

$$\|F_k(v^k) - F_k(P_{s_k}(v^k - \frac{\lambda_k}{\beta}F_k(v^k)))\| > \mu\frac{\|v^k - P_{S_k}(v^k - \frac{\lambda_k}{\beta}F_k(v^k))\|}{\frac{\lambda_k}{\beta}},$$

consequently, we can get

$$\lambda_k > \frac{\mu\beta}{L}.$$

The proof is completed. □

Theorem 3.1. *Assume that* (B_1) *and* (B_2) *are satisfied and* $\frac{\mu\beta}{L} < \lambda_k \leq \min\{L, \tau\}$. *Let* $\{v^k\}$ *be generated by Algorithm 3.1, if the solution set is nonempty, then* $\{v^k\}$ *converges to* $\overline{v} \in S$, $D\overline{v} = 0$ *and* $\overline{v}$ *is a solution of the split equality problem* (1).

Proof. Let v^* be a solution of the split equality problem (1). So $F_k(v^*) = 0, k = 0, 1, \cdots$. Using the monotonicity of F_k, we can get

$$\langle F_k(\overline{v}^k) - F_k(v^*), \overline{v}^k - v^*\rangle \geq 0,$$

i.e.,

$$\langle F_k(\overline{v}^k), \overline{v}^k - v^*\rangle \geq 0, \tag{9}$$

which is equivalent to

$$\langle F_k(\overline{v}^k), v^k - v^*\rangle \geq \langle F_k(\overline{v}^k), v^k - \overline{v}^k\rangle.$$

Thus, we obtain

$$\begin{aligned}\|v^{k+1} - v^*\|^2 &= \|v^k - \lambda_k d_k - v^*\|^2 \\ &= \|v^k - v^*\|^2 - 2\lambda_k\langle d_k, v^k - v^*\rangle + \lambda_k^2\|d_k\|^2 \\ &= \|v^k - v^*\|^2 - 2\lambda_k\langle d_k, \overline{v}^k - v^*\rangle - 2\lambda_k\langle d_k, v^k - \overline{v}^k\rangle + \lambda_k^2\|d_k\|^2.\end{aligned}$$

Note that

$$\begin{aligned}\langle d_k, \overline{v}^k - v^*\rangle &= \langle v^k - \overline{v}^k + \lambda_k F_k(\overline{v}^k) - \lambda_k F_k(v^k), \overline{v}^k - v^*\rangle \\ &= \langle v^k - \lambda_k F_k(v^k) - \overline{v}^k, \overline{v}^k - v^*\rangle + \lambda_k\langle F_k(\overline{v}^k), \overline{v}^k - v^*\rangle.\end{aligned} \tag{10}$$

Let $v^k - \lambda_k F_k(v^k) = w^k$. From the projection property (i) of Lemma 2.1, we have

$$\begin{aligned}\langle v^k - \lambda_k F_k(v^k) - \overline{v}^k, \overline{v}^k - v^*\rangle =&\langle w^k - \overline{v}^k, \overline{v}^k - v^*\rangle \\ =&\langle w^k - P_{S_k}(w^k), P_{S_k}(w^k) - v^*\rangle \\ \geq& 0,\end{aligned} \tag{11}$$

By (9), (10) and (11), we have

$$\langle d_k, \overline{v}^k - v^*\rangle \geq 0. \tag{12}$$

From the definition of $\overline{v}^k$ and Lemma 3.1, we can get

$$\begin{aligned}\langle d_k, v^k - \overline{v}^k\rangle &= \langle d_k, v^k - \overline{v}^k + \lambda_k F_k(\overline{v}^k) - \lambda_k F_k(v^k)\rangle + \lambda_k\langle d_k, F_k(v^k) - F_k(\overline{v}^k)\rangle \\ &= \|d_k\|^2 + \lambda_k\langle v^k - \overline{v}^k + \lambda_k F_k(\overline{v}^k) - \lambda_k F_k(v^k), F_k(v^k) - F_k(\overline{v}^k)\rangle \\ &= \|d_k\|^2 + \lambda_k\langle v^k - \overline{v}^k, F_k(v^k) - F_k(\overline{v}^k)\rangle - \lambda_k^2\|F_k(v^k) - F_k(\overline{v}^k)\|^2 \\ &\geq \|d_k\|^2 + (\lambda_k - \lambda_k^2\cdot\frac{1}{L})\langle v^k - \overline{v}^k, F_k(v^k) - F_k(\overline{v}^k)\rangle \\ &= \|d_k\|^2 + (\lambda_k - \lambda_k^2\cdot\frac{1}{L})\langle v^k - \overline{v}^k, D^TD(v^k) - D^TD(\overline{v}^k)\rangle \\ &= \|d_k\|^2 + (\lambda_k - \lambda_k^2\cdot\frac{1}{L})\langle D(v^k) - D(\overline{v}^k), D(v^k) - D(\overline{v}^k)\rangle \\ &= \|d_k\|^2 + (\lambda_k - \lambda_k^2\cdot\frac{1}{L})\|D(v^k) - D(\overline{v}^k)\|^2,\end{aligned} \tag{13}$$

According to (12) and (13), we obtain

$$\begin{aligned}\|v^{k+1} - v^*\|^2 &= \|v^k - v^*\|^2 - 2\lambda_k\langle d_k, \overline{v}^k - v^*\rangle - 2\lambda_k\langle d_k, v^k - \overline{v}^k\rangle + \lambda_k^2\|d_k\|^2 \\ &\leq \|v^k - v^*\|^2 - 2\lambda_k\langle d_k, \overline{v}^k - v^*\rangle + \lambda_k^2\|d_k\|^2 \\ &- 2\lambda_k[\|d_k\|^2 + (\lambda_k - \lambda_k^2\cdot\frac{1}{L})\|D(v^k) - D(\overline{v}^k)\|^2] \\ &= \|v^k - v^*\|^2 - \lambda_k(2-\lambda_k)\|d_k\|^2 - 2\lambda_k\langle d_k, \overline{v}^k - v^*\rangle \\ &- 2\lambda_k^2(1 - \lambda_k\cdot\frac{1}{L})\|D(v^k) - D(\overline{v}^k)\|^2.\end{aligned} \tag{14}$$

Since $\frac{\mu\beta}{L} < \lambda_k \leq \min\{L, \tau\}$, we deduce that $\{\|v^k - v^*\|\}$ is monotone. Hence, $\{v^k\}$ is bounded and convergent. Furthermore,

$$\lim_{k\to\infty}\|D(v^k) - D(\overline{v}^k)\| = \lim_{k\to\infty}\|D(v^k - \overline{v}^k)\| = 0.$$

Therefore

$$\lim_{k\to\infty}\|v^k - \overline{v}^k\| = 0. \tag{15}$$

By (8), we have

$$\begin{aligned}\|v^{k+1} - v^k\| &\leq \|v^{k+1} - \overline{v}^k\| + \|v^k - \overline{v}^k\| \\ &= \|v^k - \lambda_k d_k - \overline{v}^k\| + \|v^k - \overline{v}^k\| \\ &= \|v^k - \overline{v}^k - \lambda_k(v^k - \overline{v}^k + \lambda_k F_k(\overline{v}^k) - \lambda_k F_k(v^k))\| + \|v^k - \overline{v}^k\| \\ &= \|v^k - \overline{v}^k - \lambda_k(v^k - \overline{v}^k) - \lambda_k^2(F_k(\overline{v}^k) - F_k(v^k))\| + \|v^k - \overline{v}^k\| \\ &\leq \|v^k - \overline{v}^k - \lambda_k(v^k - \overline{v}^k)\| + \lambda_k^2 \parallel F_k(\overline{v}^k) - F_k(v^k) \parallel + \parallel v^k - \overline{v}^k \parallel \\ &\leq \|v^k - \overline{v}^k\| + \lambda_k\|v^k - \overline{v}^k\| + \lambda_k^2\frac{\mu\|v^k - \overline{v}^k\|}{\lambda_k} + \parallel v^k - \overline{v}^k \parallel \\ &= (2 + (1+\mu)\lambda_k)\|v^k - \overline{v}^k\|,\end{aligned}$$

which together with (15) implies that

$$\lim_{k\to\infty}\|v^{k+1} - v^k\| = 0. \tag{16}$$

Suppose that $\overline{v}$ is a cluster point of $\{v^k\}$ and $\{v^{k_i}\}$ is a convergent subsequence of $\{v^k\}$. Next, we will prove that $\overline{v}$ is a solution of the split equality problem (1) and $D\overline{v} = 0$. Set

$$e_k(v, \lambda) = v - P_{S_k}(v - \lambda F_k(v)),\ \lambda > 0,\ k = 0, 1, \cdots.$$

From Lemma 2.2, Lemma 3.1 and equation (15), we have

$$\begin{aligned}\lim_{i\to\infty}\|e_{k_i}(v^{k_i},1)\| &\le \lim_{i\to\infty}\frac{\|v^{k_i}-\overline{v}^{k_i}\|}{\min\{1,\lambda_{k_i}\}}\\ &\le \lim_{i\to\infty}\frac{\|v^{k_i}-\overline{v}^{k_i}\|}{\min\{1,\underline{\lambda}\}}\\ &=0,\end{aligned} \tag{17}$$

where $\underline{\lambda}=\frac{\mu\beta}{L}$.

Based on Lemma 2.1 and $v^*\in S_{k_i}$, we obtain

$$\langle v^{k_i}-F_{k_i}(v^{k_i})-P_{S_{k_i}}(v^{k_i}-\lambda_{k_i}F_{k_i}(v^{k_i})),v^*-P_{S_{k_i}}(v^{k_i}-\lambda_{k_i}F_{k_i}(v^{k_i}))\rangle\le 0,$$

i.e.,

$$\langle e_{k_i}(v^{k_i},1)-F_{k_i}(v^{k_i}),v^{k_i}-v^*-e_{k_i}(v^{k_i},1)\rangle\ge 0.$$

Hence

$$\langle e_{k_i}(v^{k_i},1),v^{k_i}-v^*\rangle-\|e_{k_i}(v^{k_i},1)\|^2-\langle F_{k_i}(v^{k_i}),v^{k_i}-v^*\rangle+\langle F_{k_i}(v^{k_i}),e_{k_i}(v^{k_i},1)\rangle\ge 0.$$

Consequently, we obtain

$$\begin{aligned}&\langle v^{k_i}-v^*,e_{k_i}(v^{k_i},1)\rangle\\ \ge&\|e_{k_i}(v^{k_i},1)\|^2-\langle F_{k_i}(v^{k_i}),e_{k_i}(v^{k_i},1)\rangle+\langle F_{k_i}(v^{k_i}),v^{k_i}-v^*\rangle\\ =&\|e_{k_i}(v^{k_i},1)\|^2-\langle F_{k_i}(v^{k_i}),e_{k_i}(v^{k_i},1)\rangle+\langle F_{k_i}(v^{k_i})-F_{k_i}(v^*),v^{k_i}-v^*\rangle\\ =&\|e_{k_i}(v^{k_i},1)\|^2-\langle F_{k_i}(v^{k_i}),e_{k_i}(v^{k_i},1)\rangle+\langle D^TDv^{k_i}-D^TDv^*,v^{k_i}-v^*\rangle\\ =&\|e_{k_i}(v^{k_i},1)\|^2-\langle F_{k_i}(v^{k_i}),e_{k_i}(v^{k_i},1)\rangle+\langle Dv^{k_i}-Dv^*,Dv^{k_i}-Dv^*\rangle\\ =&\|e_{k_i}(v^{k_i},1)\|^2-\langle F_{k_i}(v^{k_i}),e_{k_i}(v^{k_i},1)\rangle+\|Dv^{k_i}-Dv^*\|^2\\ =&\|e_{k_i}(v^{k_i},1)\|^2-\langle F_{k_i}(v^{k_i}),e_{k_i}(v^{k_i},1)\rangle+\|Dv^{k_i}\|^2,\end{aligned} \tag{18}$$

Owing to

$$\|F_{k_i}(v^{k_i})\|=\|F_{k_i}(v^{k_i})-F_{k_i}(v^*)\|\le L\|v^{k_i}-v^*\|,\quad \forall i=0,1,\cdots.$$

and $\{v^{k_i}\}$ is bounded, the sequence $\{F_{k_i}(v^{k_i})\}$ is bounded, too. Hence, from (17) and (18), we can deduce

$$\lim_{i\to\infty}\|Dv^{k_i}\|=0,$$

that is

$$\lim_{i\to\infty}Dv^{k_i}=0,$$

so we get $D\overline{v}=0$.

Assume that the $\{v^{k_n}\}$ converges to $\overline{v}$ and $D\overline{v}=0$. Since $\|v^k-v^{k+1}\|\to 0$, we know that v^k is asymptotically regular. Owing to $v^{k_n+1}\in S_{k_n}$, we obtain

$$\langle \zeta_{k_n},v^{k_n+1}-v^{k_n}\rangle+l(v^{k_n})\le 0,$$

thus

$$l(v^{k_n})\le-\langle\zeta_{k_n},v^{k_n+1}-v^{k_n}\rangle\le\zeta\|v^{k_n+1}-v^{k_n}\|.$$

From the lower semi-continuity of $l(v)$ and the asymptotic regularity of $\{v^k\}$, we obtain

$$l(\overline{v})\le\liminf_{n\to\infty}l(v^{k_n})\le 0.$$

Hence, $\overline{v} \in S$ and the sequence $\{\|\overline{v} - v^k\|\}$ converges to 0. Thus, we use $\overline{v}$ to replace v^* in $\|v^{k+1} - v^*\|$ and $\{\| v^k - \overline{v} \|\}$ is convergent. Since the subsequent $\{\| v^{k_i} - \overline{v} \|\}$ converges to 0, $v^k \to \overline{v}$ as $k \to \infty$. This completes the proof. □

4. Conclusion

In this paper, we proposed a new iterative algorithm with Armijo-like search for solving the split equality problem. The algorithm doesn't require computing the matrix inverse and the largest eigenvalue of the matrix $A^T A$. Convergent analysis of the suggested algorithm is proved under some suitable conditions.

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