NECESSARY AND SUFFICIENT CONDITIONS FOR IDENTIFYING STRICTLY GEOMETRICALLY α-BIDIAGONALLY DOMINANT MATRICES

Lei-Lei Wang[1], Bo-Yan Xi[2], Feng Qi[3]

In the paper, the authors establish necessary and sufficient conditions for identifying strictly geometrically α-bidiagonally dominant matrices, present some new criteria of nonsingular H-matrix by using the theory of geometrically α-bidiagonally dominant matrices, and provide two numerical examples which illustrate the effectiveness and advantages of the new criteria.

Keywords: strictly geometrically α-bidiagonally dominant matrix, nonsingular H-matrix, chain of nonzero elements, irreducibility, diagonally dominant matrix
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1. Introduction and definitions

Let $\mathbb{C}^{n \times n}$ denote the set of all $n \times n$ complex matrices, $\mathbb{N}$ be the set of all positive integers, and $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. Further let

$$P_i(A) = \sum_{j \neq i} |a_{ij}|, \quad R_i(A) = \sum_{j \neq i} |a_{ji}|, \quad i, j \in \mathbb{N} \tag{1.1}$$

and

$$M = \{(i, j) : i, j \in \mathbb{N}, i \neq j\} = \mathbb{N}^2 \setminus \{(i, i) : i \in \mathbb{N}\}. \tag{1.2}$$

Definition 1.1 ([7, Definition 2.1] and [14, p. 427]). Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. If

$$|a_{ii}| \geq P_i(A) \tag{1.3}$$

for all $i \in \mathbb{N}$, then A is said to be diagonally dominant and denoted by $A \in D_n$. If the inequality (1.3) is strict for all $i \in \mathbb{N}$, then A is said to be strictly diagonally dominant and denoted by $A \in SD_n$.

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[1]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China; E-mail: `wangleilei501@qq.com`, `wangleilei501@163.com`.

[2]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China; E-mail: `baoyintu78@qq.com`, `baoyintu68@sohu.com`, `baoyintu78@imun.edu.cn`.

[3]Department of Mathematics, School of Science, Tianjin Polytechnic University, Tianjin City, 300387, China; E-mail: `qifeng618@gmail.com`, `qifeng618@hotmail.com`, `qifeng618@qq.com`; URL: `http://qifeng618.wordpress.com`.

If there exists a positive diagonal matrix

$$X = \text{diag}(x_1, x_2, \ldots, x_n) \tag{1.4}$$

such that AX is a strictly diagonally dominant matrix, then A is said to be a generalized strictly diagonally dominant matrix and denoted by $A \in GSD_n$.

It is said in [2, 12] and [14, p. 427] that A is a nonsingular H-matrix if and only if A is a generalized strictly diagonally dominant matrix. It was referenced in [7, Lemma 3.3] and [8, p. 241] that, if A is a nonsingular H-matrix, then there exists at least one strictly diagonally dominant row.

Definition 1.2 ([7, Definition 2.2] and [14, Definition 1]). Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. If there exists $\alpha \in [0, 1]$ such that $|a_{ii}| \geq \alpha P_i(A) + (1 - \alpha)R_i(A)$ holds for all $i \in \mathbb{N}$, then A is said to be an α-diagonally dominant matrix. If the inequality is strict, then A is said to be a strictly α-diagonally dominant matrix.

If there exists a positive diagonal matrix X such that AX is a strictly α-diagonally dominant matrix, then A is said to be a generalized strictly α-diagonally dominant matrix.

Definition 1.3 ([14, Definition 2]). Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. If there exists $\alpha \in [0, 1]$ such that $|a_{ii}a_{jj}| \geq \alpha P_i(A)P_j(A) + (1 - \alpha)R_i(A)R_j(A)$ holds for all $(i, j) \in M$, then A is said to be an α-bidiagonally dominant matrix. If the inequality is strict, then A is said to be a strictly α-bidiagonally dominant matrix.

Definition 1.4 ([15, 20]). Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. If there exists $\alpha \in [0, 1]$ such that

$$|a_{ii}| \geq [P_i(A)]^\alpha [R_i(A)]^{1-\alpha} \tag{1.5}$$

for all $(i, j) \in M$, then A is called a geometrically α-diagonally dominant matrix and denoted by $A \in PD_n^\alpha$; If the inequality (1.5) is strict, then A is said to be a strictly geometrically α-diagonally dominant matrix and denoted by $A \in SPD_n^\alpha$.

Definition 1.5. Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ and

$$J(A) = \{(i, j) \in M : |a_{ii}a_{jj}| > [P_i(A)P_j(A)]^\alpha [R_i(A)R_j(A)]^{1-\alpha}\}. \tag{1.6}$$

(1) If there exists an $\alpha \in [0, 1]$ such that

$$|a_{ii}a_{jj}| \geq [P_i(A)P_j(A)]^\alpha [R_i(A)R_j(A)]^{1-\alpha} \tag{1.7}$$

for all $(i, j) \in M$, then A is said to be a geometrically α-bidiagonally dominant matrix and denoted by $A \in DPD_n^\alpha$.

(2) If $J(A) = M$, then A is said to be a strictly geometrically α-bidiagonally dominant matrix and denoted by $A \in SDPD_n^\alpha$.

(3) If there exists a positive diagonal matrix X such that AX is strictly geometrically α-bidiagonally dominant, then A is said to be a generalized strictly geometrically α-bidiagonally dominant matrix and denoted by $A \in GSDPD_n^\alpha$.

Remark 1.1. If $\alpha = 1$, Definition 1.5 becomes [13, Definition 1.1] in which A is called a (strictly) doubly diagonally dominant matrix.

In [15, p. 1011] and [20, Definition 1], the matrix A defined in Definition 1.4 is respectively called a product α-diagonally dominant matrix and an α-diagonally dominant matrix. In [7, Definition 2.3], [9, p. 19], [11, p. 88], and [15, p. 1012], the matrix A defined by (1.7) was respectively called an α bi-diagonally dominant matrix, an α-connective diagonal dominant matrix, an α-bidiagonally dominant matrix, and a doubly product α-diagonally dominant matrix. It is clear that these four different notations of the same concept confuse us and make us be at loose ends.

It is common knowledge [3, 18] that the geometric mean $G(a, b; \alpha)$ of two positive numbers a and b with unit weight $(\alpha, 1-\alpha)$ for $\alpha \in [0, 1]$ is defined by $a^\alpha b^{1-\alpha}$. With the help of this notion, the term in the right hand side of (1.7) can be rewritten as $G(P_i(A), R_i(A); \alpha)G(P_j(A), R_j(A); \alpha)$ or $G(P_i(A), R_j(A); \alpha)G(P_j(A), R_i(A); \alpha)$. By this and considering Definitions 1.2, Definition 1.3, and the above mentioned confusion, for consistency and avoiding confusion, we would like to call the matrix defined by (1.5) a geometrically α-diagonally dominant matrix and to call the matrix A defined by (1.7) is a geometrically α-bidiagonally dominant matrix. In our opinion, this terminology is more meaningful and simple. This idea comes from the theory of means [3, 16, 17] and was ever used in generalizations of convex functions, see, for example, [19, 21, 22] and closely related references therein.

In what follows we will use the following notations:

$$M_1(A) = \{(i, j) \in M : P_i(A)P_j(A) < |a_{ii}a_{jj}| < R_i(A)R_j(A)\},$$

$$M_2(A) = \{(i, j) \in M : R_i(A)R_j(A) < |a_{ii}a_{jj}| < P_i(A)P_j(A)\},$$

$$M_3(A) = \{(i, j) \in M : |a_{ii}a_{jj}| \geq R_i(A)R_j(A) > P_i(A)P_j(A)\},$$

$$M_4(A) = \{(i, j) \in M : |a_{ii}a_{jj}| \geq P_i(A)P_j(A) > R_i(A)R_j(A)\},$$

$$M_5(A) = \{(i, j) \in M : |a_{ii}a_{jj}| > P_i(A)P_j(A) = R_i(A)R_j(A)\},$$

$$M_6(A) = \{(i, j) \in M : |a_{ii}a_{jj}| \leq R_i(A)R_j(A), |a_{ii}a_{jj}| \leq P_i(A)P_j(A)\},$$

which can be found in [14, p. 427]. It is obvious that

$$M = M_1(A) \cup M_2(A) \cup M_3(A) \cup M_4(A) \cup M_5(A) \cup M_6(A).$$

So far as we know, H-matrices have been playing an important role in computational mathematics, control theory, electric system theory, mathematics of economics, and many other fields. See [1, 2]. However, the practical discrimination of an H-matrix is very difficult. So it is meaningful to judge whether a matrix is an H-matrix or not. In recent years, many scholars have been working on its properties and criterion and have obtained lots of criteria for identifying nonsingular H-matrix by using iterative arithmetic and techniques in matrix theory and inequalities, and so on. In [14], some criteria for determining nonsingular H-matrices are obtained. For more information, please refer to [4, 5, 6, 8, 10, 12].

In this article, basing on results in [14], according to the theory of geometrically α-bidiagonally dominant matrix, we will give several necessary and sufficient

conditions for specifying strictly geometrically α-bidiagonally dominant matrices and obtain several new practical criteria for nonsingular H-matrices. So the criteria for nonsingular H-matrices is expanded. These results improve and extend some existing ones. Finally, we will show the effectiveness and advantages of the proposed new criteria by two numerical examples.

In what follows, we always assume that $a_{ii}P_i(A)R_i(A) \neq 0$ for any $i \in \mathbb{N}$ and that both $M_1(A)$ and $M_2(A)$ are not empty, as done in [14, Corollary 1].

2. Lemmas

To attain our aim, we need the following lemmas.

Lemma 2.1 ([9, Theorem 1]). *Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. Then A is a nonsingular H-matrix if and only if $A \in GSDPD_n^\alpha$.*

Lemma 2.2 ([9, Theorem 3] and [14, Lemma 2]). *Let $\alpha \in [0, 1]$, $A = (a_{ij}) \in DPD_n^\alpha$, and $G(A) = \{i \in \mathbb{N} : |a_{ii}a_{jj}| > [P_i(A)P_j(A)]^\alpha[R_i(A)R_j(A)]^{1-\alpha}, j \neq i, j \in \mathbb{N}\}$. Then A is a nonsingular H-matrix if $G(A) \neq \emptyset$ and one of the following two statements is true:*

   (1) *when $\alpha \neq 0$, if $|a_{ii}a_{jj}| = [P_i(A)P_j(A)]^\alpha[R_i(A)R_j(A)]^{1-\alpha}$ for every i, j with $i \neq j$, there exists a nonzero elements chain $a_{i_0i_1}, a_{i_1i_2}, \ldots, a_{i_rj_0}$, where $i_0 \neq i_1$, $i_1 \neq i_2, \ldots, i_r \neq j_0$, such that $i_0 = i$ or $i_0 = j$ and $j_0 \in G(A)$;*

   (2) *when $\alpha = 0$, if $|a_{ii}a_{jj}| = R_i(A)R_j(A)$ for every i, j with $i \neq j$, there exists a nonzero elements chain $a_{j_0j_1}, a_{j_1j_2}, \ldots, a_{j_\ell i_0}$, where $j_0 \neq j_1$, $j_1 \neq j_2, \ldots$, $j_\ell \neq i_0$, such that $i_0 = i$ or $j_0 = j$ and $j_0 \in G(A)$.*

Lemma 2.3 ([20, Theorem 2]). *Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. Then A is a nonsingular H-matrix if it satisfies either of the following two conditions:*

   (1) *$A \in SPD_n^\alpha$;*

   (2) *$A \in PD_n^\alpha$ is irreducible and the strict inequality (1.5) holds for at least one.*

Lemma 2.4 ([13, Theorem 2.1 (iv)]). *Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ be an irreducible matrix. If $A \in DPD_n^1$ and if at least one of the inequalities in (1.7) strictly holds, then A is a nonsingular H-matrix.*

3. Main results

Now we start out to state and prove our main results.

Theorem 3.1. *Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. Then $A \in SDPD_n^\alpha$ if and only if $M_6(A) = \emptyset$ and*

$$\frac{\ln[R_s(A)R_t(A)] - \ln[|a_{ss}||a_{tt}|]}{\ln[R_s(A)R_t(A)] - \ln[P_s(A)P_t(A)]} < \frac{\ln[|a_{ii}a_{jj}|] - \ln[R_i(A)R_j(A)]}{\ln[P_i(A)P_j(A)] - \ln[R_i(A)R_j(A)]} \qquad (3.1)$$

for all $(s, t) \in M_1(A)$ and $(i, j) \in M_2(A)$.

Proof. We first prove the necessity. Since $A \in SDPD_n^\alpha$ and $M_6(A) = \emptyset$, there exists $\alpha \in [0,1]$ such that $|a_{ss}||a_{tt}| > [P_s(A)P_t(A)]^\alpha[R_s(A)R_t(A)]^{1-\alpha}$ for all $(s,t) \in M_1(A)$. So, we have $\ln[|a_{ss}||a_{tt}|] - \ln[R_s(A)R_t(A)] > \alpha\{\ln[P_s(A)P_t(A)] - \ln[R_s(A)R_t(A)]\}$, that is,

$$\frac{\ln[R_s(A)R_t(A)] - \ln[|a_{ss}||a_{tt}|]}{\ln[R_s(A)R_t(A)] - \ln[P_s(A)P_t(A)]} < \alpha. \tag{3.2}$$

On the other hand, we have $|a_{ii}a_{jj}| > [P_i(A)P_j(A)]^\alpha[R_i(A)R_j(A)]^{1-\alpha}$ for every $(i,j) \in M_2(A)$, which can be reformulated as

$$\ln[P_i(A)P_j(A)] - \ln[|a_{ii}a_{jj}|] < (1-\alpha)\{\ln[P_i(A)P_j(A)] - \ln[R_i(A)R_j(A)]\},$$

that is,

$$\alpha < \frac{\ln[|a_{ii}a_{jj}|] - \ln[R_i(A)R_j(A)]}{\ln[P_i(A)P_j(A)] - \ln[R_i(A)R_j(A)]}. \tag{3.3}$$

Therefore, from (3.2) and (3.3), we obtain (3.1).

Now we begin to prove the sufficiency. Obviously, for all $(s,t) \in M_1(A)$ and $(i,j) \in M_2(A)$, from (3.1), it follows that

$$0 < \frac{\ln[R_s(A)R_t(A)] - \ln[|a_{ss}||a_{tt}|]}{\ln[R_s(A)R_t(A)] - \ln[P_s(A)P_t(A)]} < \frac{\ln[|a_{ii}a_{jj}|] - \ln[R_i(A)R_j(A)]}{\ln[P_i(A)P_j(A)] - \ln[R_i(A)R_j(A)]} < 1.$$

Therefore, there must exist $\alpha \in (0,1)$ such that

$$\frac{\ln[R_s(A)R_t(A)] - \ln[|a_{ss}||a_{tt}|]}{\ln[R_s(A)R_t(A)] - \ln[P_s(A)P_t(A)]} < \alpha$$
$$< \frac{\ln[|a_{ii}a_{jj}|] - \ln[R_i(A)R_j(A)]}{\ln[P_i(A)P_j(A)] - \ln[R_i(A)R_j(A)]}. \tag{3.4}$$

The left hand side of (3.4) implies that

$$\ln[|a_{ss}||a_{tt}|] - \ln[R_s(A)R_t(A)] > \alpha\{\ln[P_s(A)P_t(A)] - \ln[R_s(A)R_t(A)]\},$$

i.e., $|a_{ss}||a_{tt}| > [P_s(A)P_t(A)]^\alpha[R_s(A)R_t(A)]^{1-\alpha}$ for $(s,t) \in M_1(A)$. The right hand side of (3.4) means that

$$\frac{\ln[P_i(A)P_j(A)] - \ln[|a_{ii}a_{jj}|]}{\ln[P_i(A)P_j(A)] - \ln[R_i(A)R_j(A)]} < 1 - \alpha,$$

i.e., $|a_{ii}a_{jj}| > [P_i(A)P_j(A)]^\alpha[R_i(A)R_j(A)]^{1-\alpha}$ for $(i,j) \in M_2(A)$. For all $(i,j) \in M_3(A) \cup M_4(A) \cup M_5(A)$ and $\alpha \in (0,1)$, it is easy to see that $|a_{ii}a_{jj}| > [P_i(A)P_j(A)]^\alpha[R_i(A)R_j(A)]^{1-\alpha}$. Since $M_6(A) = \emptyset$, there exists $\alpha \in [0,1]$ such that $|a_{ii}a_{jj}| > [P_i(A)P_j(A)]^\alpha[R_i(A)R_j(A)]^{1-\alpha}$ for $(i,j) \in M$. Consequently, by Definition 1.5, we obtain $A \in SDPD_n^\alpha$. $\square$

Theorem 3.2. *Let* $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ *and* $M_6(A) = \emptyset$. *If* $(s,t) \in M_1(A)$ *and* $(i,j) \in M_2(A)$ *satisfy*

$$\frac{\ln[R_s(A)R_t(A)] - \ln[|a_{ss}||a_{tt}|]}{\ln[R_s(A)R_t(A)] - \ln[P_s(A)P_t(A)]} < \frac{\ln[|a_{ii}a_{jj}|] - \ln[R_i(A)R_j(A)]}{\ln[P_i(A)P_j(A)] - \ln[R_i(A)R_j(A)]}, \tag{3.5}$$

then A *is a nonsingular H-matrix.*

Proof. From Theorem 3.1, it is clear that $A \in SDPD_n^\alpha$. Hence, by Lemma 2.1, we obtain that A is a nonsingular H-matrix. $\qquad\square$

Remark 3.1. Since

$$[P_i(A)P_j(A)]^\alpha[R_i(A)R_j(A)]^{1-\alpha} \le \alpha P_i(A)P_j(A) + (1-\alpha)R_i(A)R_j(A)$$

for $(i,j) \in M(A)$, then $DD_\alpha \subset SDPD_n^\alpha$, where DD_α is defined in [14].

Corollary 3.2.1. *Let* $A = (a_{ij}) \in C^{n\times n}$. *If at least one of inequalities*

$$\frac{\ln[R_s(A)R_t(A)] - \ln[|a_{ss}||a_{tt}|]}{\ln[R_s(A)R_t(A)] - \ln[P_s(A)P_t(A)]} \le \frac{\ln[|a_{ii}a_{jj}|] - \ln[R_i(A)R_j(A)]}{\ln[P_i(A)P_j(A)] - \ln[R_i(A)R_j(A)]}$$

for $(s,t) \in M_1(A)$ *and* $(i,j) \in M_2(A)$ *hold and* $|a_{rr}||a_{qq}| = P_r(A)P_q(A) = R_r(A)R_q(A)$ *for* $(r,q) \in M_6(A) \ne \emptyset$, *and if there are nonzero elements* $a_{li_1}, a_{i_1 i_2}, \ldots, a_{i_h k}$ *such that* $l = r$ *or* $l = q$, $k = k_1$ *or* $k = k_2$, *and* $(k_1, k_2) \in M - M_6(A)$, *then* A *is a nonsingular H-matrix.*

Proof. By Theorem 3.2, we see that $|a_{ii}a_{jj}| \ge [P_i(A)P_j(A)]^\alpha[R_i(A)R_j(A)]^{1-\alpha}$ for all $(i,j) \in M_1(A) \cup M_2(A) \cup M_3(A) \cup M_4(A) \cup M_5(A)$. On the other hand, for all $(r,q) \in M_6(A) \ne \emptyset$, we obtain $|a_{rr}||a_{qq}| = [P_r(A)P_q(A)]^\alpha[R_r(A)R_q(A)]^{1-\alpha}$ and by the assumption, there exists a nonzero elements chain $a_{li_1}, a_{i_1 i_2}, \ldots, a_{i_h k}$ such that $l = r$ or $l = q$; $k = k_1$ or $k = k_2$ and $(k_1, k_2) \in M - M_6(A)$, on the base of the Lemma 2.2, we can acquire that A is a nonsingular H-matrix. $\qquad\square$

Corollary 3.2.2. *Let* $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ *be an irreducible matrix. If* $|a_{rr}||a_{qq}| = P_r(A)P_q(A) = R_r(A)R_q(A)$ *for all* $(r,q) \in M_6(A)$ *and*

$$\frac{\ln[R_s(A)R_t(A)] - \ln[|a_{ss}||a_{tt}|]}{\ln[R_s(A)R_t(A)] - \ln[P_s(A)P_t(A)]} \le \frac{\ln[|a_{ii}a_{jj}|] - \ln[R_i(A)R_j(A)]}{\ln[P_i(A)P_j(A)] - \ln[R_i(A)R_j(A)]} \qquad (3.6)$$

for $(s,t) \in M_1(A)$ *and* $(i,j) \in M_2(A)$, *where at least one of the inequalities in (3.6) strictly holds, then* A *is a nonsingular H-matrix.*

Proof. Obviously, according to the assumption, by the similar method in the proof of Corollary 3.2.1, we can obtain $A \in DPD_n^a$.

For $\alpha = 0$ or $\alpha = 1$, since A is irreducible, from Lemma 2.4, it follows that A is a nonsingular H-matrix.

For $\alpha \in (0,1)$, from $A \in DPD_n^\alpha$, it follows that

$$|a_{ii}a_{jj}| \ge [P_i(A)P_j(A)]^\alpha[R_i(A)R_j(A)]^{1-\alpha} \qquad (3.7)$$

for all $(i,j) \in M$. Thus, from $J(A) \ne \emptyset$, we can easily obtain that there exists at most a number $i_0 \in \mathbb{N}$ such that $|a_{i_0 i_0}| \le [P_{i_0}(A)]^\alpha[R_{i_0}(A)]^{1-\alpha}$. Without loss of generality, we assume that $|a_{11}| \le [P_1(A)]^\alpha[R_1(A)]^{1-\alpha}$ and $|a_{jj}| \ge [P_j(A)]^\alpha[R_j(A)]^{1-\alpha}$ for $2 \le j \le n$. Putting $d^\alpha = |a_{11}|^{-1}[P_1(A)]^\alpha[R_1(A)]^{1-\alpha}$, by (3.7), we have $d^\alpha[P_j(A)]^\alpha[R_j(A)]^{1-\alpha} \le |a_{jj}|$ for $2 \le j \le n$. Let $X = \text{diag}(d, 1, \ldots, 1)$ and $B = AX = (b_{ij})$. Then

$$[P_1(B)]^\alpha[R_1(B)]^{1-\alpha} = [P_1(A)]^\alpha[R_1(dA)]^{1-\alpha} = d^\alpha d^{1-\alpha}|a_{11}| = b_{11}$$

and

$$[P_j(B)]^\alpha [R_j(B)]^{1-\alpha} = [P_1(dA)]^\alpha [R_1(A)]^{1-\alpha}$$
$$= d^\alpha [P_1(A)]^\alpha [R_1(A)]^{1-\alpha} \leq |a_{jj}| = b_{jj}, \quad 2 \leq j \leq n.$$

Thus, we obtain $B \in PD_n^\alpha$. Since B is irreducible, from Lemma 2.3, it follows that B is a nonsingular H-matrix, and so is A. Corollary 3.2.2 is proved. $\qquad\square$

Theorem 3.3. *Let* $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. *Then* $A \in GSDPD_n^\alpha$ *if and only if there exists a positive diagonal matrix* (1.4) *such that* $M_6(AX) = \emptyset$ *and*

$$\frac{\ln[R_s(AX)R_t(AX)] - \ln[|a_{ss}x_s||a_{tt}x_t|]}{\ln[R_s(AX)R_t(AX)] - \ln[P_s(AX)P_t(AX)]}$$
$$< \frac{\ln[|a_{ii}x_i||a_{jj}x_j|] - \ln[R_i(AX)R_j(AX)]}{\ln[P_i(AX)P_j(AX)] - \ln[R_i(AX)R_j(AX)]}$$

for all $(s,t) \in M_1(AX)$ *and* $(i,j) \in M_2(AX)$.

Proof. By the similar method as in the proof of Theorem 3.1, we can easily prove this theorem. $\qquad\square$

Theorem 3.4. *Let* $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. *If there exists a positive diagonal matrix* $X = \mathrm{diag}(x_1, x_2, \ldots, x_n)$ *such that* $M_6(AX) = \emptyset$ *and*

$$\frac{\ln[R_s(AX)R_t(AX)] - \ln[|a_{ss}x_s||a_{tt}x_t|]}{\ln[R_s(AX)R_t(AX)] - \ln[P_s(AX)P_t(AX)]}$$
$$< \frac{\ln[|a_{ii}x_i||a_{jj}x_j|] - \ln[R_i(AX)R_j(AX)]}{\ln[P_i(AX)P_j(AX)] - \ln[R_i(AX)R_j(AX)]}$$

for all $(s,t) \in M_1(AX)$ *and* $(i,j) \in M_2(AX)$, *then* A *is a nonsingular H-matrix.*

Proof. From Theorem 3.3, it is easy to obtain that $A \in GSDPD_n^\alpha$. As a result, by Lemma 2.1, it follows that A is a nonsingular H-matrix. $\qquad\square$

Remark 3.2. According to Theorem 3.4, we can obtain similar results to Corollaries 3.2.1 and 3.2.2.

4. Two numerical examples

Finally we provide two numerical examples which illustrate the effectiveness and advantages of the new criteria.

Example 1. Let

$$A = \begin{bmatrix} 2.3 & 1 & 2 \\ 1 & 3 & 3 \\ 0.5 & 1.5 & 3.5 \end{bmatrix}.$$

An easy computation yields

$$\frac{R_2(A)R_3(A) - |a_{22}||a_{33}|}{R_2(A)R_3(A) - P_2(A)P_3(A)} = \frac{4}{9} > \frac{21}{55} = \frac{|a_{11}||a_{22}| - R_1(A)R_2(A)}{P_1(A)P_2(A) - R_1(A)R_2(A)}.$$

Thus A does not satisfy conditions of [14, Theorem 2].

By the criteria presented in Theorem 3.2, we obtain

$$
0.3906\cdots = \frac{\ln[|R_2(A)R_3(A)|] - \ln[|a_{22}||a_{33}|]}{\ln[|R_2(A)R_3(A)|] - \ln[P_2(A)P_3(A)]}
$$
$$
< \frac{\ln[|a_{11}||a_{22}|] - \ln[R_1(A)R_2(A)]}{\ln[P_1(A)P_2(A)] - \ln[R_1(A)R_2(A)]} = 0.5242\cdots.
$$

Since $M_6(A) = \emptyset$, the matrix A satisfies conditions of Theorem 3.2, then A is a nonsingular H-matrix. Let $X = \mathrm{diag}(1, 1.04, 0.6)$, then $AX \in SD_n$. $\qquad\square$

Example 2. Let

$$
A = \begin{bmatrix} 0.3 & 1 & 2 \\ 0.125 & 3 & 3 \\ 0.0625 & 1.5 & 3.5 \end{bmatrix}.
$$

Since

$$
\frac{R_2(A)R_3(A) - |a_{22}||a_{33}|}{R_2(A)R_3(A) - P_2(A)P_3(A)} = 0.2625\cdots > \frac{|a_{11}||a_{22}| - R_1(A)R_2(A)}{P_1(A)P_2(A) - R_2(A)R_3(A)}
$$
$$
= 0.0484\cdots > \frac{|a_{11}||a_{33}| - R_1(A)R_3(A)}{P_1(A)P_3(A) - R_1(A)R_3(A)} = 0.0300\cdots,
$$

and

$$
\frac{\ln[|a_{11}||a_{22}|] - \ln[R_1(A)R_2(A)]}{\ln[P_1(A)P_2(A)] - \ln[R_1(A)R_2(A)]} = 0.2177\cdots
$$
$$
> \frac{\ln[R_2(A)R_3(A)] - \ln[|a_{22}||a_{33}|]}{\ln[R_2(A)R_3(A)] - \ln[P_2(A)P_3(A)]} = 0.1854\cdots
$$
$$
> \frac{\ln[|a_{11}||a_{33}|] - \ln[R_1(A)R_3(A)]}{\ln[P_1(A)P_3(A)] - \ln[R_1(A)R_3(A)]} = 0.0704\cdots,
$$

and since $M_6(A) = \emptyset$, the matrix A does not satisfy the conditions in [14, Theorem 2] and Theorem 3.2 in this paper. However, if we choose the positive diagonal matrix $X = \mathrm{diag}(8, 1, 1)$, then

$$
AX = \begin{bmatrix} 2.4 & 1 & 2 \\ 1 & 3 & 3 \\ 0.5 & 1.5 & 3.5 \end{bmatrix}.
$$

Since

$$
\frac{\ln[R_2(AX)R_3(AX)] - \ln[|a_{22}x_2||a_{33}x_3|]}{\ln[R_2(AX)R_3(AX)] - \ln[P_2(AX)P_3(AX)]} = 0.3906\cdots
$$
$$
< \frac{\ln[|a_{11}x_1||a_{22}x_2|] - \ln[R_1(AX)R_2(AX)]}{\ln[P_1(AX)P_2(AX)] - \ln[R_1(AX)R_2(AX)]} = 0.5608\cdots,
$$

the matrix A satisfies conditions of Theorem 3.4 in this paper, and then A is a nonsingular H-matrix. In fact, if we take the positive diagonal matrix $Y = \mathrm{diag}(1, 0.13, 0.08)$, then $AY \in SD_n$. $\qquad\square$

5. **Conclusions**

In conclusion, we unify the notion "the geometrically α-bidiagonally dominant matrix", establish necessary and sufficient conditions for identifying strictly geometrically α-bidiagonally dominant matrices, and present some new criteria of nonsingular H-matrix.

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