EXISTENCE OF FRACTALS BY OPTIMAL POINTS FOR A CLASS OF DISCONTINUOUS MAPPINGS

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In this paper, we focus on the existence of the best proximity points in binormed linear spaces. We also provide some illustrations to support our claims. As consequences, we derive various fixed point r esults. We p resent an approach to the existence of fractals through best proximity points as applications.

Keywords: Best proximity point, fixed point, Banach space, contraction mappings.

1. Introduction and Preliminaries

The idea of a fixed point is not appropriate when the intersection of nonempty subsets $\mathscr{E}_1$ and $\mathscr{E}_2$ of a metric space (χ, d) is empty, because in this case FP equations $\Gamma u = u$ may not have a solution. If intersection of $\mathscr{E}_1$ and $\mathscr{E}_2$ is nonempty and FP equation $\Gamma u = u$ has solution, then Γ has a FP. Banach contraction theorem (BCT) plays an important role in nonlinear analysis. Due to its simplicity and applicability, it helps solve many kinds of nonlinear problems such as the existence of solutions of integral equation, differential equation and matrix equation so on.

Mandelbrot [15] gave a notion of fractals which describe a large family of irregular patterns in nature. Self similar sets are regarded as a valuable category of fractals due to their utility in mathematically modeling various physical phenomena. In 1981, Hutchinson [12] conducted an analysis of objects exhibiting self similarity, resulting in the establishment of an iterated function system which is one of the most common ways of building fractals. Barnsley [3] worked on this system to produce fractal sets within any given metric space by employing a finite collection of BCT. Garg and Chandok [11] also worked on this system and obtained some new results using contraction conditions.

In 1968, Maia established a very interesting and beautiful generalization of BCT using assumptions on two comparable metrics defined on the set χ. The beautiful idea of Maia's FP-theorem still attracts the interest of researchers working in FP-theory (see [1, 6] and references cited therein).

Consider the case when the FP equation $\Gamma u = u$ has no solution. In this case $d(\mathscr{E}_1, \mathscr{E}_2) > 0$. In this affairs it is interesting to find an approximate solution u such that the error $d(u, \Gamma u)$ is minimum in some sense. For a nonself mapping $\Gamma : \mathscr{E}_1 \to \mathscr{E}_2$ a point u, known as a best proximity point (BPP) if satisfies the following condition

$$d(u, \Gamma u) = d(\mathscr{E}_1, \mathscr{E}_2) = \inf\{d(u, v) : u \in \mathscr{E}_1, v \in \mathscr{E}_2\}.$$

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In 1969, Fan [10] gave the classical best approximation theorem in the context of a Hausdorff locally convex topological vector space χ. After that, many authors studied the best approximation problems in metric spaces and normed spaces (see [5, 18] and references cited therein). In 2006, Eldred and Veeramani [9] proved the existence of a best proximity point for cyclic contraction mappings. Thereafter, various authors obtained many best proximity point results using different types of contractions (see [8, 13, 17, 19] and references cited therein).

In this paper, we investigate the existence of best proximity points in the context of binormed linear spaces and derive various fixed point results as a result of our observations. Also, as an application, we give a method for the existence of fractals using the BPP. We provide some numerical examples to back up our findings.

Throughout this paper, we denote the set of natural numbers and real numbers by $\mathbb{N}$, $\mathbb{R}$ respectively, and the set of all nonempty compact subsets of χ by $C(\chi)$.

Now, we recall some definitions to be used in the sequel.

Definition 1.1. [3] *Let (χ,d) be a metric space. A mapping $h : C(\chi)\times C(\chi)\to\mathbb{R}$, defined as*

$$h(\mathscr{E}_1,\mathscr{E}_2)=\max\{D(\mathscr{E}_1,\mathscr{E}_2),D(\mathscr{E}_2,\mathscr{E}_1)\};\ \textit{where}\ \ D(\mathscr{E}_1,\mathscr{E}_2)=\sup_{x\in\mathscr{E}_1}\inf_{y\in\mathscr{E}_2}d(x,y),$$

is a metric on $C(\chi)$, called the Hausdorff metric h induced by metric d.

Here, it is interesting to note that if (χ,d) is complete and compact, then $(C(\chi),h)$ is also complete and compact.

Definition 1.2. [3] *Let $\Gamma_i : \chi\to\chi$ be self maps on a complete metric space (χ,d) such that $d(\Gamma_i x,\Gamma_i y)\le l_i d(x,y); l_i\in[0,1)$ for all $x,y\in\chi$. Then the system $\{\chi : \Gamma_i, i=1,2,\cdots,k\}$ is called iterated function system (IFS).*

Definition 1.3. [3] *Let (χ,d) be a complete metric space and $\{\chi : \Gamma_i, i=1,2,\cdots,k\}$ be IFS. Then Hutchinson mapping F on $C(\chi)$ is defined as $F(\mathscr{E}_1)=\cup_{i=1}^{k}\Gamma_i^*(\mathscr{E}_1)$, where $\Gamma_i^*(\mathscr{E}_1)=\{\Gamma_i(c) : c\in\mathscr{E}_1\}$. If $F(A)=A$, then $A\in C(\chi)$ is called an attractor or a fractal of IFS.*

Definition 1.4. [7] *A normed vector space χ is said to be a uniformly convex Banach space (UCBS) if for every $0<\varepsilon\le 2$ there is some $\delta>0$ such that for any two vectors with $||d||=1$ and $||e||=1$, the condition*

$$||d-e||\ge\varepsilon,$$

implies

$$\left|\left|\frac{d+e}{2}\right|\right|\le 1-\delta.$$

To prove the main result of the paper, we need the following interesting results of [3] and [9].

Lemma 1.1. [3] *If $\{A_i : 1\le i\le k\}$ and $\{B_i : 1\le i\le k\}$ are two finite collections of subsets of $C(\chi)$ for some $k\in\mathbb{N}$ then*

$$h(\cup_{i=1}^{k}A_i,\cup_{1=i}^{k}B_i)\le\max_{1\le i\le k}\{h(A_i,B_i)\}.\tag{1}$$

Lemma 1.2. [9] *Let $\mathscr{E}_1,\mathscr{E}_2$ be nonempty closed subsets of an UCBS χ with $\mathscr{E}_1$ convex. Let $\{d_n\}$ and $\{f_n\}$ be sequences in $\mathscr{E}_1$ and $\{e_n\}$ be a sequence in $\mathscr{E}_2$ satisfying:*

(i) $||f_n-e_n||\to d(\mathscr{E}_1,\mathscr{E}_2)$,

(ii) *for every* $\varepsilon > 0$ *there exists* N_0 *such that for all* $m > n \geq N_0$, $||d_n - e_n|| \leq d(\mathscr{E}_1, \mathscr{E}_2) + \varepsilon$.
Then, for every $\varepsilon > 0$ *there exists* N_1 *such that for all* $m > n \geq N_1$, $||d_n - f_n|| \leq \varepsilon$.

Lemma 1.3. [9] *Let* $\mathscr{E}_1, \mathscr{E}_2$ *be nonempty closed subsets of an UCBS* χ *with* $\mathscr{E}_1$ *convex. Let* $\{d_n\}$ *and* $\{f_n\}$ *be sequences in* $\mathscr{E}_1$ *and* $\{e_n\}$ *be a sequence in* $\mathscr{E}_2$ *satisfying:*
(i) $||f_n - e_n|| \to d(\mathscr{E}_1, \mathscr{E}_2)$,
(ii) $||d_n - e_n|| \to d(\mathscr{E}_1, \mathscr{E}_2)$.
Then $||d_n - f_n||$ *converges to zero.*

2. **Main result**

First, we prove an approximation result.

Theorem 2.1. *Let* $\mathscr{E}_1$ *and* $\mathscr{E}_2$ *be nonempty closed subsets of a metric space* (χ, d). *Suppose that* $\Gamma : \mathscr{E}_1 \cup \mathscr{E}_2 \to \mathscr{E}_1 \cup \mathscr{E}_2$ *is an operator fulfilling the following hypotheses:*
$(\mathscr{T}_1)$ $\Gamma(\mathscr{E}_1) \subseteq \mathscr{E}_2$ *and* $\Gamma(\mathscr{E}_2) \subseteq \mathscr{E}_1$,
$(\mathscr{T}_2)$ $0 < a \leq d(u,v) \leq b < \infty$ *implies* $d(\Gamma u, \Gamma v) \leq \zeta_{a,b}(d(u,v)) + (1 - \zeta_{a,b})\, d(\mathscr{E}_1, \mathscr{E}_2)$,
for all $u \in \mathscr{E}_1$, $v \in \mathscr{E}_2$, $\zeta_{a,b} : [0,\infty] \to [0,\infty]$ *is a non-decreasing mapping such that* $\lim_{n\to\infty} \zeta^n_{a,b}(s) = 0$, $0 < \zeta_{a,b}(s) < s$ *for each* $s > 0$, *and* $\zeta^n_{a,b}(s)$ *is the nth iterate of* $\zeta_{a,b}$.
If $u_0 \in \mathscr{E}_1 \cup \mathscr{E}_2$ *and* $u_{n+1} = \Gamma u_n$ *where* $n \in \mathbb{N} \cup \{0\}$, *then* $d(u_n, \Gamma u_n) \to d(\mathscr{E}_1, \mathscr{E}_2)$.

Proof. Since $u_0 \in \mathscr{E}_1 \cup \mathscr{E}_2$, $u_0 \in \mathscr{E}_1$ or $u_0 \in \mathscr{E}_2$.
Case (i): Take $u_0 \in \mathscr{E}_1$. By $(\mathscr{T}_1)$, we have $u_1 = \Gamma u_0 \in \mathscr{E}_2$. Suppose that $d(\mathscr{E}_1, \mathscr{E}_2) > 0$, then there exist $u_0 \neq u_1$ and a large integer n_0, we get $\frac{1}{n_0} \leq d(u_0, u_1) \leq n_0$. Also $d(\mathscr{E}_1, \mathscr{E}_2) = \inf\{d(u,v) : u \in \mathscr{E}_1, v \in \mathscr{E}_2\}$, we obtain $d(\mathscr{E}_1, \mathscr{E}_2) \leq d(u_0, u_1)$. Using $(\mathscr{T}_2)$, we have

$$\begin{aligned} d(\Gamma u_0, \Gamma u_1) \leq& \zeta_{1/n_0,n_0}(d(u_0,u_1)) + (1 - \zeta_{1/n_0,n_0})\, d(\mathscr{E}_1, \mathscr{E}_2) \\ \leq& \zeta_{1/n_0,n_0}(d(u_0,u_1)) + (1 - \zeta_{1/n_0,n_0})\, d(u_0, u_1) \\ =& d(u_0, u_1). \end{aligned}$$

Since $u_1 \in \mathscr{E}_2$ by $(\mathscr{T}_1)$, $u_2 = \Gamma u_1 \in \mathscr{E}_1$. Again suppose that there exist $u_1 \neq u_2$ and a large integer n_1, we have $\frac{1}{n_1} \leq d(u_1, u_2) \leq n_1$, so by $(\mathscr{T}_2)$,

$$d(\Gamma u_1, \Gamma u_2) \leq \zeta_{1/n_0,n_0}(d(u_1,u_2)) + (1 - \zeta_{1/n_0,n_0}) d(u_1,u_2) \leq d(u_1,u_2).$$

Continuing this process, we construct the sequences $\{u_s\}$ for a large integer n_s such that

$$d(\Gamma u_s, \Gamma u_{s-1}) \leq \zeta_{1/n_s,n_s}(d(u_s,u_{s-1})) + (1 - \zeta_{1/n_s,n_s}) d(u_s,u_{s-1}) \leq d(u_s,u_{s-1}).$$

Therefore, $\{d(u_n, u_{n+1})\}$ is a bounded below and non-increasing sequence, so there exists $r \geq 0$ such that

$$r = \lim_{n\to\infty} d(u_n, u_{n+1}).$$

Now, we shall prove that $r = d(\mathscr{E}_1, \mathscr{E}_2)$. Thus, by contradiction, suppose that $r > d(\mathscr{E}_1, \mathscr{E}_2) > 0$. Then, for large N, we obtain

$$r \leq d(u_{N+s-1}, u_{N+s}) \leq r + 1, \text{ for all } s = 1, 2, \cdots,$$

which imply by $(\mathscr{T}_2)$ that

$$d(\Gamma u_{N+s-1}, \Gamma u_{N+s}) \leq \zeta_{r,r+1}(d(u_{N+s-1}, u_{N+s})) + (1 - \zeta_{r,r+1}) d(\mathscr{E}_1, \mathscr{E}_2), \text{ for all } s = 1, 2, \cdots. \tag{2}$$

Again by $(\mathscr{T}_2)$, we have $d(u_{N+s-1}, u_{N+s}) = d(\Gamma u_{N+s-2}, \Gamma u_{N+s-1}) \leq \zeta_{r,r+1}(d(u_{N+s-2}, u_{N+s-1})) + (1-\zeta_{r,r+1})d(\mathscr{E}_1, \mathscr{E}_2)$. Since $\zeta_{r,r+1}$, is non-decreasing mapping, we obtain

$$\zeta_{r,r+1}(d(u_{N+s-1}, u_{N+s})) \leq \zeta_{r,r+1}(d(u_{N+s-2}, u_{N+s-1}) + (1-\zeta_{r,r+1})d(\mathscr{E}_1, \mathscr{E}_2)). \tag{3}$$

Put (3) in (2), we have

$$d(\Gamma u_{N+s-1}, \Gamma u_{N+s}) \leq \zeta^2_{r,r+1}(d(u_{N+s-2}, u_{N+s-1})) + (1-\zeta^2_{r,r+1})d(\mathscr{E}_1, \mathscr{E}_2).$$

Thus, by induction, we get

$$d(u_{N+s}, u_{N+s+1}) \leq \zeta^s_{r,r+1}(d(u_N, u_{N+1})) + (1-\zeta^s_{r,r+1})d(\mathscr{E}_1, \mathscr{E}_2), \text{ for all } s = 1, 2, \cdots.$$

Since $\zeta^s_{r,r+1}(r+1) \to 0$ as $s \to \infty$, $d(u_{N+s}, u_{N+s+1}) \to d(\mathscr{E}_1, \mathscr{E}_2)$ as $s \to \infty$, which implies a contradiction. This shows that $d(u_n, u_{n+1}) \to d(\mathscr{E}_1, \mathscr{E}_2)$.
Case (ii) If $u_0 \in \mathscr{E}_2$. In Similar way, we obtain

$$d(u_n, u_{n+1}) \to d(\mathscr{E}_1, \mathscr{E}_2).$$

□

Next, we prove an existence result for a BPP.

Theorem 2.2. *Suppose that all the assumptions of Theorem 2.1 hold. Additionally if $u_0 \in \mathscr{E}_1$, χ is complete and $\{u_{2n}\}$ has a convergent subsequence in $\mathscr{E}_1$ then Γ has a BPP.*

Proof. Let $\{u_{2n(s)}\}$ be a subsequence of $\{u_{2n}\}$ which converges to a point $u \in \mathscr{E}_1$. Now

$$d(u, u_{2n(s)-1}) \leq d(u, u_{2n(s)}) + d(u_{2n(s)}, u_{2n(s)-1}). \tag{4}$$

Taking $n \to \infty$ in (4), we get

$$d(u, u_{2n(s)-1}) \to d(\mathscr{E}_1, \mathscr{E}_2).$$

Since $d(\mathscr{E}_1, \mathscr{E}_2) \leq d(u_{2n(s)}, \Gamma u) \leq d(u_{2n(s)-1}, u)$. Then Γ has a BPP. □

Next, we are ready to prove the main result, which gives existence, uniqueness and convergence for best proximity points in binormed linear space.

Theorem 2.3. *Let $\mathscr{E}_1, \mathscr{E}_2$ be nonempty closed subsets of a uniformly convex binormed linear space $(\chi, ||.||_1, ||.||_2)$ with $\mathscr{E}_1$ convex and $||.||_2 \leq ||.||_1$. Suppose that χ is complete with respect to $||.||_2$ and $\Gamma : \mathscr{E}_1 \cup \mathscr{E}_2 \to \mathscr{E}_1 \cup \mathscr{E}_2$ is an operator fulfilling the following hypotheses:*
$(\mathscr{T}_1)$ *$\Gamma(\mathscr{E}_1) \subseteq \mathscr{E}_2$ and $\Gamma(\mathscr{E}_2) \subseteq \mathscr{E}_1$,*
$(\mathscr{T}_2)$ *$0 < a \leq ||u-v||_1 \leq b < \infty$ implies $||\Gamma u - \Gamma v||_1 \leq \zeta_{a,b}(||u-v||_1) + (1-\zeta_{a,b})\, ||\mathscr{E}_1 - \mathscr{E}_2||_1$, for all $u \in \mathscr{E}_1$, $v \in \mathscr{E}_2$, $\zeta_{a,b} : [0,\infty] \to [0,\infty]$ is a non-decreasing mapping such that $\lim_{n\to\infty} \zeta^n_{a,b}(s) = 0$, $0 < \zeta_{a,b}(s) < s$ for each $s > 0$, and $\zeta^n_{a,b}(s)$ is the nth iterate of $\zeta_{a,b}$.*

If $u_0 \in \mathscr{E}_1$ and $u_{n+1} = \Gamma u_n$ where $n \in \mathbb{N}$, then Γ has a unique BPP in $\mathscr{E}_1$ where $||\mathscr{E}_1 - \mathscr{E}_2||_1 = \inf\{||u-v||_1 : u \in \mathscr{E}_1, v \in \mathscr{E}_2\}$.

Proof. By Theorem 2.1, we have

$$||u_{2n} - u_{2n+1}||_1 \to ||\mathscr{E}_1 - \mathscr{E}_2||_1 \text{ and } ||u_{2n+2} - u_{2n+1}||_1 \to ||\mathscr{E}_1 - \mathscr{E}_2||_1. \tag{5}$$

Since χ is a uniformly convex binormed linear space by Lemma 1.3, we get

$$||u_{2n} - u_{2(n+1)}||_1 \to 0. \tag{6}$$

We now show that for every $\varepsilon > 0$ there exists N_0 such that for all $M > N \geq N_0$, $||u_{2M} - \Gamma u_{2N}||_1 < ||\mathscr{E}_1 - \mathscr{E}_2||_1 + \varepsilon = r_1$. Suppose not, then, for sufficiently large N and M, we have

$$r_1 \leq ||u_{2(M+s)} - \Gamma u_{2(N+s)}||_1 \leq r_1 + 1, \text{ for all } s = 1, 2, \cdots.$$

This M can be chosen such that it is the least integer greater than N to satisfy the above inequality. Now

$$\begin{aligned} ||\mathscr{E}_1 - \mathscr{E}_2||_1 + \varepsilon &\leq ||u_{2(M+s)} - \Gamma u_{2(N+s)}||_1 \\ &\leq ||u_{2(M+s)} - u_{2(M-1+s)}||_1 + ||u_{2(M-1+s)} - \Gamma u_{2(N+s)}||_1 \\ &< ||u_{2(M+s)} - u_{2(M-1+s)}||_1 + ||\mathscr{E}_1 - \mathscr{E}_2||_1 + \varepsilon. \end{aligned}$$

Using (6) and taking $s \to \infty$ in above inequality we have, $||u_{2(M+s)} - \Gamma u_{2(N+s)}||_1 = ||\mathscr{E}_1 - \mathscr{E}_2||_1 + \varepsilon$. Consider,

$$\begin{aligned} ||u_{2(M+s)} - \Gamma u_{2(N+s)}||_1 \leq& ||u_{2(M+s)} - u_{2(M+1+s)}||_1 + ||u_{2(M+1+s)} - \Gamma u_{2(N+1+s)}||_1 \\ &+ ||\Gamma u_{2(N+1+s)} - \Gamma u_{2(N+s)}||_1 \\ \leq& ||u_{2(M+s)} - u_{2(M+1+s)}||_1 + ||\Gamma u_{2(N+1+s)} - \Gamma u_{2(N+s)}||_1 \\ &+ \zeta^2_{r_1, r_1+1} ||u_{2(M+s)} - \Gamma u_{2(N+s)}||_1 + (1 - \zeta^2_{r_1, r_1+1}) ||\mathscr{E}_1 - \mathscr{E}_2||_1. \end{aligned} \quad (7)$$

Taking $s \to \infty$ in (7) and using (6) we get,

$$\begin{aligned} ||\mathscr{E}_1 - \mathscr{E}_2||_1 + \varepsilon &\leq \zeta^2_{r_1, r_1+1} (||\mathscr{E}_1 - \mathscr{E}_2||_1 + \varepsilon) + (1 - \zeta^2_{r_1, r_1+1}) ||\mathscr{E}_1 - \mathscr{E}_2||_1 \\ &= ||\mathscr{E}_1 - \mathscr{E}_2||_1 + \zeta^2_{r_1, r_1+1} \varepsilon, \end{aligned}$$

which is a contradiction because $\zeta^2_{r_1, r_1+1}(\varepsilon) = \zeta_{r_1, r_1+1}(\zeta_{r_1, r_1+1}(\varepsilon)) < \zeta_{r_1, r_1+1}(\varepsilon) < \varepsilon$. Therefore, $\{u_{2n}\}$ is a Cauchy sequence in $\mathscr{E}_1$ with respect $||.||_1$. Since $||.||_2 \leq ||.||_1$, $\{u_{2n}\}$ is a Cauchy sequence in $\mathscr{E}_1$ with respect $||.||_2$. As $\mathscr{E}_1$ is a closed subset of χ, it is a complete subspace. By the completeness of $\mathscr{E}_1$, $\{u_{2n}\}$ converges to a point u in $\mathscr{E}_1$, then by Theorem 2.2, we get Γ has a BPP, $||u - \Gamma u||_2 = ||\mathscr{E}_1 - \mathscr{E}_2||_2$ in $\mathscr{E}_1$.

Next, we have to prove that Γ has a unique BPP. Suppose that $u, v \in \mathscr{E}$ and $u \neq v$ such that $||u - \Gamma u||_2 = ||\mathscr{E}_1 - \mathscr{E}_2||_2$ and $||v - \Gamma v||_2 = ||\mathscr{E}_1 - \mathscr{E}_2||_2$ where necessarily, $\Gamma^2 u = u$ and $\Gamma^2 v = v$. Also, $||u - \Gamma v||_2 \leq ||u - v|| + ||\mathscr{E}_1 - \mathscr{E}_2||_2$. This shows $0 < ||\mathscr{E}_1 - \mathscr{E}_2||_2 < ||u - \Gamma v||_2 \leq ||u - v|| + ||\mathscr{E}_1 - \mathscr{E}_2||_2$. Therefore,

$$||\Gamma u - v||_2 = ||\Gamma u - \Gamma^2 v||_2 \leq ||u - \Gamma v||_2.$$

Similarly, $||\Gamma v - u||_2 \leq ||v - \Gamma u||_2$, which implies $||\Gamma v - u||_2 = ||v - \Gamma u||_2$. Since $||v - \Gamma u||_2 > ||\mathscr{E}_1 - \mathscr{E}_2||_2$, this shows $||\Gamma v - u||_2 < ||v - \Gamma u||_2$, which is a contradiction. Therefore, Γ has a unique BPP. □

If $\mathscr{E}_1 = \mathscr{E}_2 = \chi$ in Theorem 2.3, we get following FP result:

Corollary 2.1. [6] *Assume that Γ is fulfilling the following assumption:*

- $0 < a \leq ||u - v||_1 \leq b < \infty$ *implies* $||\Gamma u - \Gamma v||_1 \leq \zeta_{a,b}(||u - v||_1)$,

for all $u, v \in \chi$, $\zeta_{a,b} : [0, \infty] \to [0, \infty]$ is non-decreasing a mapping such that $\lim_{n \to \infty} \zeta^n_{a,b}(s) = 0$, $0 < \zeta_{a,b}(s) < s$ for each $s > 0$, and $\zeta^n_{a,b}(s)$ is the nth iterate of $\zeta_{a,b}$.
If $u_0 \in \chi$ and $u_{n+1} = \Gamma u_n$ where $n \in \mathbb{N}$, then Γ has a FP.

If $\mathscr{E}_1 = \mathscr{E}_2 = \chi$ and $\zeta_{a,b}(s) = rs; r \in (0,1), s \in [0, \infty)$ in Theorem 2.3, we get following FP result:

Corollary 2.2. [14] *Assume that Γ is fulfilling the following condition:*

- $||\Gamma u-\Gamma v||_1 \leq r||u-v||_1$,

for all $u,v\in\chi$, $r\in(0,1)$ and $u_{n+1}=\Gamma u_n$ where $n\in\mathbb{N}$. Then Γ has a FP.

3. Numerical Illustrations

We give several illustrations that support our findings in this section.

Example 3.1. *Consider $\chi=\mathbb{R}$, define $||.||_1,||.||_2:\chi\to\mathbb{R}_+$ by*

$$||u||_1=|u| \text{ and } ||u||_2=\frac{|u|}{2}$$

for all $u\in\chi$. It is easy to see that $||u||_2<||u||_1$, for all $u\in\chi$. Suppose $\mathscr{E}_1=[-3,-1]$ and $\mathscr{E}_2=[1,3]$ are two subsets of χ, then $||\mathscr{E}_1-\mathscr{E}_2||_1=2$ and $||\mathscr{E}_1-\mathscr{E}_2||_2=1$. Define a mapping $\Gamma:\mathscr{E}_1\cup\mathscr{E}_2\to\mathscr{E}_1\cup\mathscr{E}_2$ by

$$\Gamma(u)=\begin{cases}\frac{-u+3}{4}, & u\in[-3,-1]\\ \frac{-u-3}{4}, & u\in[1,3]\end{cases}$$

for all $u\in\mathscr{E}_1\cup\mathscr{E}_2$. Since $||u-v||_1\in[2,6]$ that is $2=a\leq||u-v||_1\leq b=6$. Next, we prove that Γ satisfies the following inequality,

$$||\Gamma u-\Gamma v||_1\leq\zeta_{a,b}(||u-v||_1)+(1-\zeta_{a,b})\ ||\mathscr{E}_1-\mathscr{E}_2||_1$$

for all $u\in\mathscr{E}_1$ and $v\in\mathscr{E}_2$. Let $\zeta_{a,b}(s)=\frac{s}{b-a};s>0$.

$$\begin{aligned}||\Gamma u-\Gamma v||_1&=\left|\left|\frac{-u+3}{4}-\frac{-v-3}{4}\right|\right|_1\\&\leq\left|\frac{v-u}{4}\right|+\frac{3}{2}=\zeta_{a,b}(||u-v||_1)+(1-\zeta_{a,b})\ ||\mathscr{E}_1-\mathscr{E}_2||_1.\end{aligned}$$

It implies

$$||\Gamma u-\Gamma v||_1\leq\zeta_{a,b}(||u-v||_1)+(1-\zeta_{a,b})\ ||\mathscr{E}_1-\mathscr{E}_2||_1, \tag{8}$$

for all $u\in\mathscr{E}_1$, $v\in\mathscr{E}_2$, and $\Gamma(\mathscr{E}_1)\subseteq\mathscr{E}_2$, $\Gamma(\mathscr{E}_2)\subseteq\mathscr{E}_1$ see Figure 1. Since $||u||_2<||u||_1$, for all $u\in\chi$, we have

$$||\Gamma u-\Gamma v||_2\leq\zeta_{a,b}(||u-v||_2)+(1-\zeta_{a,b})\ ||\mathscr{E}_1-\mathscr{E}_2||_2.$$

Starting with point $u_0=0\in\mathscr{E}_1$, we construct a sequence as

u_{n+1}	u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	$\cdots$
Γu_n	-1	1	-1	1	-1	1	-1	1	$\cdots$

We found that $\{u_{2n}\}$ has a subsequence $(-1,-1,-1,-1,\ldots)$, which converges to -1. All the conditions of Theorem 2.3 satisfied, Γ has a BPP -1.

Example 3.2. *Suppose a two-dimensional real sequence space $\chi=\ell_2$ induced with norms*

$$||z_1||_1=\sqrt{u_1^2+v_1^2} \text{ and } ||z_1||_2=\frac{1}{\sqrt{2}}(|u_1|+|v_1|)$$

for all $z_1=(u_1,v_1)\in\chi$ and

$$\mathscr{E}_1=\left\{(0,u_1):-2\leq u_1\leq-\frac{1}{2}\right\} \text{ and } \mathscr{E}_2=\left\{(0,u_1):\frac{1}{2}\leq u_1\leq 2\right\},$$

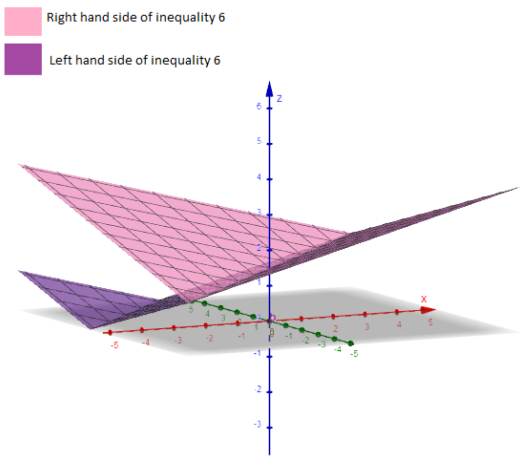

FIGURE 1. This graph shows the left and right side of inequality 8

are two subsets of χ*. Here,* $||\mathscr{E}_1 - \mathscr{E}_2||_1 = 1$*. Define a mapping* $\Gamma : \mathscr{E}_1 \cup \mathscr{E}_2 \to \mathscr{E}_1 \cup \mathscr{E}_2$ *by*

$$\Gamma(u) = \begin{cases} \frac{-u+1}{3}, & u \in [-2, -\frac{1}{2}] \\ \frac{-u-1}{3}, & u \in [\frac{1}{2}, 2] \end{cases}$$

If $z_1, z_2 \in \mathscr{E}_1$*,* $1 = a \leq ||z_1 - z_2|| \leq b = 4$*. Next, we show that* Γ *satisfies the following inequality,*

$$||\Gamma z_1 - \Gamma z_2||_1 \leq \zeta_{a,b}(||z_1 - z_2||_1) + (1 - \zeta_{a,b})\ ||\mathscr{E}_1 - \mathscr{E}_2||_1$$

for all $z_1 \in \mathscr{E}_1$ *and* $z_2 \in \mathscr{E}_2$*. Let* $\zeta_{a,b}(s) = \frac{s}{(b-a)}; s > 0$*. Consider*

$$\begin{aligned} ||\Gamma z_2 - \Gamma z_1||_1 &= ||(0, v_1) - (0, v_2)||_1 \\ &= \left| \frac{v_2 - v_1 + 2}{3} \right| \\ &\leq \left| \frac{v_2 - v_1}{3} \right| + \frac{2}{3} = \zeta_{a,b}(||z_1 - z_2||_1) + (1 - \zeta_{a,b})\ ||\mathscr{E}_1 - \mathscr{E}_2||_1. \end{aligned}$$

It shows that Γ *satisfies the following inequality,*

$$||\Gamma z_1 - \Gamma z_2||_1 \leq \zeta_{a,b}(||z_1 - z_2||_1) + (1 - \zeta_{a,b})\ ||\mathscr{E}_1 - \mathscr{E}_2||_1 \tag{9}$$

for all $z_1 \in \mathscr{E}_1$*,* $z_2 \in \mathscr{E}_2$ *and* $\Gamma(\mathscr{E}_1) \subseteq \mathscr{E}_2$*,* $\Gamma(\mathscr{E}_2) \subseteq \mathscr{E}_1$ *see Figure 2. Since* $||u||_2 \leq ||u||_1$*, for all* $u \in \chi$*, we have*

$$||\Gamma u - \Gamma v||_2 \leq \zeta_{a,b}(||u - v||_2) + (1 - \zeta_{a,b})\ ||\mathscr{E}_1 - \mathscr{E}_2||_2.$$

Starting with point $u_0 = (0, 1) \in \mathscr{E}_1$*, we construct a sequence as*

u_{n+1}	u_0	u_1	u_2	u_3	u_4	u_5	u_6	$\cdots$
Γu_n	$(0, -\frac{1}{2})$	$(0, \frac{1}{2})$	$(0, -\frac{1}{2})$	$(0, \frac{1}{2})$	$(0, -\frac{1}{2})$	$(0, \frac{1}{2})$	$(0, -\frac{1}{2})$	$\cdots$

We found that $\{u_{2n}\}$ *has a subsequence* $((0, -\frac{1}{2}), (0, -\frac{1}{2}), (0, -\frac{1}{2}), (0, -\frac{1}{2}), \ldots)$*, which converges to* $(0, -\frac{1}{2})$*. All the conditions of Theorem 2.3 satisfied,* Γ *has a (BPP)* $(0, -\frac{1}{2})$*.*

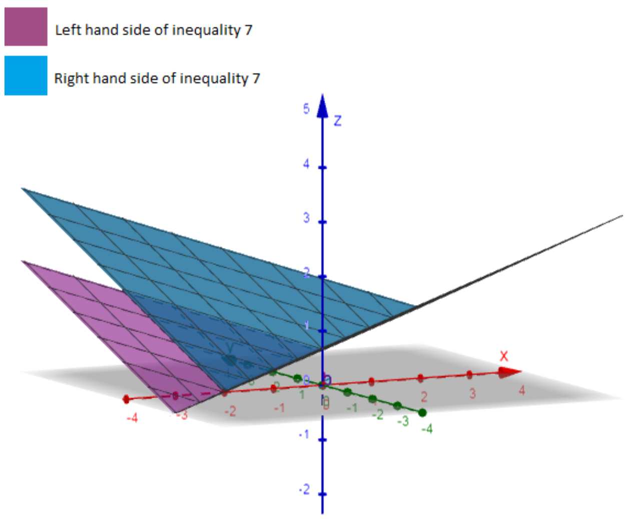

FIGURE 2. This graph shows the left and right side of inequality 9

4. Application to Fractals

In this section, we present a approach to fractals through (BPP) using contraction condition. Now, we establish a lemma which is useful to prove our application part.

Lemma 4.1. [2] *Let (χ,d) be a metric space and $\mathscr{E}_1,\mathscr{E}_2 \subseteq \chi$ with $\mathscr{E}_{1_0} \neq \emptyset$. Then, we have $H(C(\mathscr{E}_1),C(\mathscr{E}_2)) = d(\mathscr{E}_1,\mathscr{E}_2)$, where*

$$\mathscr{E}_{1_0} = \{x \in \mathscr{E}_1 : \text{ there exists some } y \in \mathscr{E}_2 \text{ such that } d(x,y) = d(\mathscr{E}_1,\mathscr{E}_2)\} \text{ and}$$
$$H(C(\mathscr{E}_1),C(\mathscr{E}_2)) = \inf\left\{h(U',V'); U' \in C(\mathscr{E}_1) \text{ and } V' \in C(\mathscr{E}_2)\right\}.$$

Lemma 4.2. *Let $\mathscr{E}_1,\mathscr{E}_2 \in C(\chi)$ be two subsets of a metric space (χ,d). Then*

$$\sup_{x\in\mathscr{E}_1}\inf_{y\in\mathscr{E}_2} \Im(d(x,y)) \leq \Im(h(\mathscr{E}_1,\mathscr{E}_2)),$$

where $\Im : [0,\infty) \to [0,\infty)$ is a non-decreasing mapping.

Proof. Since $\mathscr{E}_1$ and $\mathscr{E}_2$ are compact, there exists $y_1 \in \mathscr{E}_2$ such that

$$\inf_{y_1\in\mathscr{E}_2} d(x,y) = d(x,y_1).$$

Then

$$\begin{aligned}\Im(d(x,y_1)) =& \Im(\inf_{y\in\mathscr{E}_2} d(x,y))\\ \leq& \Im(\sup_{x\in\mathscr{E}_1}\inf_{y\in\mathscr{E}_2} d(x,y))\\ =& \Im(D(\mathscr{E}_1,\mathscr{E}_2))\\ \leq& \Im(h(\mathscr{E}_1,\mathscr{E}_2)).\end{aligned}$$

□

Theorem 4.1. *Let $\mathscr{E}_1, \mathscr{E}_2$ be subsets of metric space (χ, d) with $\mathscr{E}_{1_0} \neq \emptyset$, $\Gamma : \mathscr{E}_1 \cup \mathscr{E}_2 \to \mathscr{E}_1 \cup \mathscr{E}_2$ be a map such that $\Gamma\mathscr{E}_1 \subseteq \mathscr{E}_2$, $\Gamma\mathscr{E}_2 \subseteq \mathscr{E}_1$ and satisfying*

$$0 < a \leq d(u,v) \leq b < \infty \text{ implies } d(\Gamma u, \Gamma v) \leq \zeta_{a,b}(d(u,v)) + (1-\zeta_{a,b})\, d(\mathscr{E}_1, \mathscr{E}_2), \tag{10}$$

with respect to $\mathscr{E}_1$ and $\mathscr{E}_2$, $\zeta : [0,\infty] \to [0,\infty]$ is a mapping such that $\lim_{n\to\infty} \zeta^n(s) = 0$, $0 < \zeta(s) < s$ for each $s > 0$ and $\zeta^n(s)$ is the nth iterate of ζ.
Then $\Gamma^* : C(\mathscr{E}_1) \cup C(\mathscr{E}_2) \to C(\mathscr{E}_1) \cup C(\mathscr{E}_2)$ is a map such that $\Gamma^*(C(\mathscr{E}_1)) \subseteq C(\mathscr{E}_2), \Gamma^*(C(\mathscr{E}_2)) \subseteq C(\mathscr{E}_1)$ and satisfying

$$0 < a \leq h(U,V) \leq b < \infty \text{ implies } d(\Gamma^* U, \Gamma^* V) \leq \zeta_{a,b}(h(U,V)) + (1-\zeta_{a,b})\, H(C(\mathscr{E}_1), C(\mathscr{E}_2)), \tag{11}$$

between $C(\mathscr{E}_1)$ and $C(\mathscr{E}_2)$ with respect to the Hausdorff metric h.

Proof. Suppose that $\mathscr{E}_1$ and $\mathscr{E}_2$ are closed, so infimum and supremum exist in $\mathscr{E}_1 \cup \mathscr{E}_2$. Therefore, $\inf_{u\in\mathscr{E}_1, v\in\mathscr{E}_2} d(u,v) \leq d(u,v) \leq \sup_{u\in\mathscr{E}_1, v\in\mathscr{E}_2} d(u,v)$. Therefore, $d(u,v)$ is bounded, that is, there exist two real numbers a,b such that $0 < a \leq d(u,v) \leq b < \infty$. We know that a finite union of compact sets is a compact set. Therefore, $C(\mathscr{E}_1) \cup C(\mathscr{E}_2) = C(\mathscr{E}_1 \cup \mathscr{E}_2)$ is compact. It is a trivial observation that for all $U \in C(\mathscr{E}_1)$ and $V \in C(\mathscr{E}_2)$, then $\Gamma^*(U) \subseteq (V)$, $\Gamma^*(V) \subseteq (U)$. Next we prove that Γ^* satisfying (11) between $C(\mathscr{E}_1)$ and $C(\mathscr{E}_2)$ with respect to the Hausdorff metric h. Since $C(\mathscr{E}_1)$ and $C(\mathscr{E}_2)$ are compact then $\max\{D(U,V), D(V,U)\}$ exist in $C(\mathscr{E}_1) \cup C(\mathscr{E}_2)$. Therefore, $h(U,V)$ is bounded that is there exist two real numbers a,b such that $0 < a \leq h(U,V) \leq b < \infty$. Consider

$$\begin{aligned}
D(\Gamma^* U, \Gamma^* V) =& D(\{\Gamma u : u \in U\}, \{\Gamma v : v \in V\}) \\
=& \sup_{u\in U} \inf_{v\in V} d(\Gamma u, \Gamma v) \\
\leq& \sup_{u\in U} \inf_{v\in V} (\zeta_{a,b}(d(u,v)) + (1-\zeta_{a,b})\, d(\mathscr{E}_1, \mathscr{E}_2)) \\
=& \sup_{u\in U} \inf_{v\in V} (\zeta_{a,b}(d(u,v))) + (1-\zeta_{a,b})\, H(C(\mathscr{E}_1), C(\mathscr{E}_2)), \text{ by Lemma 4.1} \\
\leq& \zeta_{a,b}(h(U,V)) + (1-\zeta_{a,b})\, H(C(\mathscr{E}_1), C(\mathscr{E}_2)), \text{ by Lemma 4.2.}
\end{aligned}$$

Similarly,

$$D(\Gamma^* V, \Gamma^* U) \leq \zeta_{a,b}(h(V,U)) + (1-\zeta_{a,b})\, H(C(\mathscr{E}_1), C(\mathscr{E}_2)).$$

This shows that

$$\begin{aligned}
h(\Gamma^* U, \Gamma^* V) =& \max\{D(\Gamma^* U, \Gamma^* V), D(\Gamma^* V, \Gamma^* U)\} \\
\leq& \zeta_{a,b}(h(U,V)) + (1-\zeta_{a,b})\, H(C(\mathscr{E}_1), C(\mathscr{E}_2)).
\end{aligned}$$

Hence Γ^* satisfying (11) between $C(\mathscr{E}_1)$ and $C(\mathscr{E}_2)$ with respect to the Hausdorff metric h. □

Theorem 4.2. *Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ be a finite family of mappings such that $\Gamma_i(\mathscr{E}_1) \subseteq \mathscr{E}_2, \Gamma_i(\mathscr{E}_2) \subseteq \mathscr{E}_1; 1 \leq i \leq k : k \in \mathbb{N}$, satisfying (10), for some $\zeta_{1_{(a,b)}}, \cdots, \zeta_{k_{(a,b)}}$ respectively with respect to $\mathscr{E}_1$ and $\mathscr{E}_2$ with $\mathscr{E}_{1_0} \neq \emptyset$, where $\zeta_{i_{(a,b)}} : [0,\infty] \to [0,\infty]$ is a non-decreasing mapping such that $\lim_{n\to\infty} \zeta^n_{i_{(a,b)}}(s) = 0$, $0 < \zeta_{i_{(a,b)}}(s) < s$ for each $s > 0$ and $\zeta^n_{i_{(a,b)}}(s)$ is the nth iterate of $\zeta_{i_{(a,b)}}$.*
Then $F = \cup_{i=1}^k \Gamma_i^*$ is a mapping such that $F(C(\mathscr{E}_1)) \subseteq (C(\mathscr{E}_2))$, $F(C(\mathscr{E}_2)) \subseteq (C(\mathscr{E}_1))$, satisfying (11) between $C(\mathscr{E}_1)$ and $C(\mathscr{E}_2)$ with respect to the Hausdorff metric h induced by d.

Proof. From the construction of F it is immediate that for all $U \in C(\mathscr{E}_1)$ and $V \in C(\mathscr{E}_2)$, $F(U) \subseteq V, F(V) \subseteq U$. Consider

$$\begin{aligned} h(FU,FV) =& h(\cup_{i=1}^{k}\Gamma_i^* U, \cup_{i=1}^{k}\Gamma_i^* V) \\ &\leq \max_{1\leq i\leq k} h(\Gamma^* U, \Gamma^* V) \\ &\leq \max_{1\leq i\leq k} \zeta_{i_{(a,b)}}(h(U,V)) + (1-\zeta_{a,b})\, H(C(\mathscr{E}_1), C(\mathscr{E}_2)) \\ &\leq \zeta_{a,b}(h(U,V)) + (1-\zeta_{a,b})\, H(C(\mathscr{E}_1), C(\mathscr{E}_2)), \end{aligned}$$

where $\zeta_{i_{(a,b)}}(s) = \max_{1\leq i\leq k} \zeta_{i_{(a,b)}}(s)$. □

Now, we are ready to present our result for the existence of best proximity point.

Theorem 4.3. *Let $\mathscr{E}_1$, $\mathscr{E}_2$ be two closed subsets of complete metric space (χ, d) with $\mathscr{E}_{1_0} \neq \emptyset$ and $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ be a finite family of mappings such that $\Gamma_i(\mathscr{E}_1) \subseteq \mathscr{E}_2, \Gamma_i(\mathscr{E}_2) \subseteq \mathscr{E}_1; 1 \leq i \leq k : k \in \mathbb{N}$, satisfying (10), for some $\zeta_{1_{(a,b)}}, \cdots, \zeta_{k_{(a,b)}}$ respectively with respect to $\mathscr{E}_1$ and $\mathscr{E}_2$ where $\zeta_{i_{(a,b)}} : [0,\infty] \to [0,\infty]$ is a non-decreasing mapping such that $\lim_{n\to\infty} \zeta_{i_{(a,b)}}(s) = 0$, $0 < \zeta_{i_{(a,b)}}(s) < s$ for each $s > 0$ and $\zeta^n_{i_{(a,b)}}(t)$ is the nth iterate of $\zeta_{i_{(a,b)}}$. Then $F = \cup_{i=1}^{k}\Gamma_i^*$ defined in Theorem 4.2, has a (BPP).*

Proof. By Theorem 4.2, F is a mapping such that $F(U) \subseteq V, F(V) \subseteq U$ and satisfying (11), for all $U \in C(\mathscr{E}_1)$ and $V \in C(\mathscr{E}_2)$. Again, since (χ, d) is complete metric space $(C(\chi), h)$ is also complete metric space. On the other hand, $\mathscr{E}_1, \mathscr{E}_2$ are closed subsets of χ, they are also complete. Also $C(\mathscr{E}_1) \cup C(\mathscr{E}_2) = C(\mathscr{E}_1 \cup \mathscr{E}_2)$ is compact subset of $C(\chi)$ thus closed, then it is complete subspace. By Theorem 2.2, F has a (BPP). □

If we take $\mathscr{E}_1 = \mathscr{E}_2$ in Theorem 4.3 then we get following (FP) result:

Corollary 4.1. *Let (χ, d) be a complete metric space and $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ be a finite family of mappings satisfying (10), for some $\zeta_{1_{(a,b)}}, \cdots, \zeta_{k_{(a,b)}}$ respectively with respect to χ, where $\zeta_{i_{(a,b)}} : [0,\infty] \to [0,\infty]$ is a non-decreasing mapping such that $\lim_{n\to\infty} \zeta^n_{i_{(a,b)}}(s) = 0$ and $0 < \zeta_{i_{(a,b)}}(s) < s$ for each $s > 0$ and $\zeta^n_{i_{(a,b)}}(s)$ is the nth iterate of $\zeta_{i_{(a,b)}}$. Then $F = \cup_{i=1}^{k}\Gamma_i^*$ defined in Theorem 4.2, has a attractor.*

If we take $\zeta_{i_{(a,b)}}(s) = r_i s$ where $1 \leq i \leq k$, for some $r_i \in (0,1), s \in [0,\infty)$, then we get following result:

Corollary 4.2. *Let (χ, d) be a complete metric space and $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ be a finite family of mappings satisfying*

$$d(\Gamma u, \Gamma v) \leq r' d(u,v) + (1-r')\, d(\mathscr{E}_1, \mathscr{E}_2). \tag{12}$$

Then $F = \cup_{i=1}^{k}\Gamma_i^*$ defined in Theorem 4.2, has an attractor where $r'(s) = \max_{1\leq i\leq k} r_i(s)$.

Example 4.1. *Let $\chi = \mathbb{R}$, be endowed with the metric $d : \chi \times \chi \to \mathbb{R}$ defined by*

$$d(u,v) = |u - v|.$$

Consider

$$\mathscr{E}_1 = [-0.5, 1.3] \text{ and } \mathscr{E}_2 = [0, \frac{2}{3}],$$

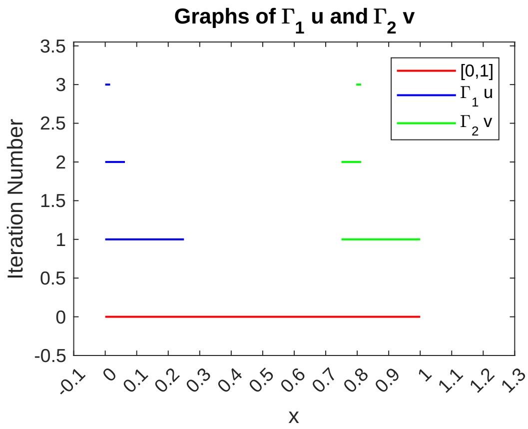

FIGURE 3.

the two compact subsets of χ. *Also* $d(\mathscr{E}_1,\mathscr{E}_2)=0$. *Take* $r_1(s)=\frac{s}{2}$ *and* $r_2(s)=\frac{s}{3}$ *for all* $s>0$. *Define* $\Gamma_1,\Gamma_2:\mathscr{E}_1\cup\mathscr{E}_2\to\mathscr{E}_1\cup\mathscr{E}_2$ *by*

$$\Gamma_1(u)=\frac{u}{4} \text{ and } \Gamma_2(u)=1-\frac{u}{4},$$

for all $u\in\mathscr{E}_1\cup\mathscr{E}_2$ *with* $\Gamma_1(\mathscr{E}_1)\subset\mathscr{E}_2,\Gamma_1(\mathscr{E}_2)\subset\mathscr{E}_1$ *and* $\Gamma_2(\mathscr{E}_1)\subset\mathscr{E}_2,\Gamma_2(\mathscr{E}_2)\subset\mathscr{E}_1$. *Next we prove that* Γ_1,Γ_2 *satisfies the* (12). *If* $u\in\mathscr{E}_1$ *and* $v\in\mathscr{E}_2$ *then*

$$\begin{aligned} d(\Gamma_1 u,\Gamma_1 v) &= d(\frac{u}{4},\frac{v}{4})\\ &= \frac{|u-v|}{4}\le\frac{1}{2}d(u,v)=r_1(d(u,v)). \end{aligned}$$

This shows that Γ_1 *satisfies the* (12). *Similarly* Γ_2 *satisfies the* (12). *By Corollary 4.2 mapping* $F=\cup_{i=1}^{2}\Gamma_i^*$ *has a unique fractal. If* $A_1=[0,1]$ *, then :*

$$\begin{aligned} A_2 &= [0,\frac{1}{4}]\cup[\frac{3}{4},1],\\ A_3 &= [0,\frac{1}{16}]\cup[\frac{3}{4},\frac{13}{16}],\\ A_3 &= [0,\frac{1}{64}]\cup[\frac{51}{64},\frac{13}{16}],\\ &\vdots \end{aligned}$$

The first few iterations are shown in Figure 3.

5. Conclusion

We find some novel best proximity point results in binormed linear spaces. Many known results in the literature are also generalized by our findings. We also discuss an approach to the existence of fractals through best proximity points as applications.

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