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NUMERICAL RECKONING OF FIXED POINTS FOR GENERALIZED NONEXPANSIVE MAPPINGS IN CAT(0) SPACES WITH APPLICATIONS

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In this paper, we propose an iterative process for the reckoning of a fixed point of a mapping endowed with the (E) property in the setting of CAT(0) spaces. Results on strong and Δ-convergence for this algorithm are stated and proved. Numerical examples are provided, regarding the behavior of this method from different point of views. Several relevant theorems in the existing literature have been generalized and improved.

Keywords: Iterative algorithm; condition (E); fixed points; $\Delta-$convergence; strong convergence; CAT(0) space.

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1. Introduction

Various nonlinear equations can be transformed in fixed point problems, a fact which allows determining their solutions by means of iterative processes. After Picard [21] introduced his famous iterative algorithm, Mann [17] developed this idea further. Ishikawa [11] stated a two step algorithm for the determination of a fixed point for a suitable class of operators, by means of two auxiliary sequences of real numbers from $[0, 1]$. Agrawal *et al.* [2] introduced another two step method based on two sequences, which satisfy a condition defined by means of a divergent series, for nearly asymptotically nonexpansive mappings. Noor [19] developed a three step iterative scheme in order to solve a class of variational inequalities by means of a fixed point approach. Sintunavarat et al [27] introduced a new three step iteration scheme for approximating fixed points of the nonlinear self mappings on a normed linear spaces satisfying Berinde contractive condition. Sahu et al. [24] developed an S-iteration technique for finding common fixed points for nonself quasi-nonexpansive mappings in the framework of a uniformly convex Banach space. Suzuki [28] proved convergence theorems for an algorithm designed for mappings endowed with the property (C), which is obviously fulfilled by nonexpansive mappings. These results have been developed further by Pant and Shukla [20], to the class of generalized α-nonexpansive mappings. Extending more, the operators which fulfill the condition (E) were introduced by García-Falset *et al.* [9], and fixed point properties have been proved by means of almost fixed point sequences. Basarir and Sahin [3] performed a study of the S-iteration method in the framework of CAT(0) spaces, for a class of generalized nonexpansive mappings. The same geometric setting has been used by Dhompongsa and Panyanak [8] or by Khan and Abbas [12] in order to develop Δ-convergence theorems for various algorithms. Garodia and Uddin [10] stated the counterpart of the Thakur *et al.* [29] scheme in the setting of CAT(0) spaces, for Suzuki generalized nonexpansive mappings. Nanjaras *et al.* [18] developed a Mann type iterative

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process regarding the reckoning of a fixed point associated with operators which satisfy the (C) conditions, on CAT(0) spaces.

The paper is organized as follows. Section 2 contains some concepts and properties needed in the sequel. Section 3 refers to convergence properties regarding an iterative scheme introduced here in the framework of CAT(0) spaces and meaningful numerical examples.

2. **Preliminaries**

Throughout this paper $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, and $\mathcal{F}(\mathcal{T}) = \{x \in C : \mathcal{T}x = x\}$ is the set of all fixed points of a mapping $\mathcal{T} : C \to C$ where C is a convex subset of a linear space X.

Iterative procedures for the reckoning of a fixed point of a mapping endowed with suitable properties have been developed extensively.

In 1890, Picard [21] introduced his renowned iteration by $x_{n+1} = \mathcal{T}x_n$, $n \in \mathbb{N}$.

Sixty three years later, Mann [17] imposed his iteration method on Banach spaces, as follows

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \mathcal{T}x_n, \ n \in \mathbb{N}.$$

where $\{\alpha_n\} \subset (0, 1)$ and $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, (see also [23]).

In 1974, Ishikawa [11] made the debut of several step iteration processes, for the reckoning of fixed points associated with Lipschitzian pseudo-contractive mappings in the setting of Hilbert spaces. For $\{\alpha_n\}$, $\{\beta_n\}$ sequences of numbers from $(0, 1)$, he defined

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \mathcal{T}((1 - \beta_n)x_n + \beta_n \mathcal{T}x_n), \ n \in \mathbb{N}.$$

The Noor [19] iteration method appeared in 2000, related to a strongly monotone operator variational inequality on Hilbert spaces, and consists of

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \mathcal{T}((1 - \beta_n)x_n + \beta_n \mathcal{T}((1 - \gamma_n)x_n + \gamma_n \mathcal{T}x_n)), \ n \in \mathbb{N}. \tag{1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences of numbers from $(0, 1)$.

In 2014, Abbas and Nazir [1] defined their iteration method for nonexpansive operators on uniformly convex Banach spaces, as follows

$$x_{n+1} = \alpha_n \mathcal{T}((1-\beta_n)\mathcal{T}x_n + \beta_n \mathcal{T}((1-\gamma_n)x_n + \gamma_n \mathcal{T}x_n))_n + (1-\alpha_n)\mathcal{T}((1-\gamma_n)x_n + \gamma_n \mathcal{T}x_n), \ n \in \mathbb{N}. \tag{2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences of numbers from $(0, 1)$.

In 2014, Thakur *et al.* [30] stated their iteration method TTP14 for the class of nonexpansive mappings, on the framework of Banach spaces, by

$$x_{n+1} = (1-\alpha_n)\mathcal{T}x_n + \alpha_n \mathcal{T}((1-\beta_n)((1-\gamma_n)x_n + \gamma_n \mathcal{T}x_n) + \beta_n \mathcal{T}((1-\gamma_n)x_n + \gamma_n \mathcal{T}x_n)), \ n \in \mathbb{N}. \tag{3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences of numbers from $(0, 1)$.

The numerical process TTP16, was introduced by Thakur *et al.* [29] for nonexpansive mappings, on on uniformly convex Banach spaces, as follows:

$$\begin{aligned} x_{n+1} = (1 - \alpha_n)\mathcal{T}((1 - \gamma_n)x_n + \gamma_n \mathcal{T}x_n) + \alpha_n \mathcal{T}((1 - \beta_n)((1 - \gamma_n)x_n + \gamma_n \mathcal{T}x_n) \\ + \beta_n \mathcal{T}((1 - \gamma_n)x_n + \gamma_n \mathcal{T}x_n)), \ n \in \mathbb{N}. \end{aligned} \tag{4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are in $(0, 1)$.

In 2019, Garodia et al. [10] studied convergence behaviour of this algorithm in the setting of CAT(0) spaces, for generalized nonexpansive mappings.

In 2019, Piri *et al.* [22] introduced a new iterative scheme to approximate a fixed point of generalized α-nonexpansive mappings in Banach spaces, as below.

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - \alpha_n)\mathcal{T}(\mathcal{T}((1 - \beta)x_n + \beta \mathcal{T}x_n)) + \alpha_n \mathcal{T}(\mathcal{T}(\mathcal{T}((1 - \beta)x_n + \beta \mathcal{T}x_n))) \ n \in \mathbb{N}. \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are in $(0,1)$.

In order to develop our new results, we need to recall some classes of mappings whose properties will be used in the sequel.

Definition 2.1. *Suppose K is a nonempty, closed, and convex subset of uniformly convex Banach space $(X, \|\cdot\|)$. A mapping $\mathcal{T}\colon K \to K$ is said to be*

- *nonexpansive, if $\|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|$, for all x, $y \in K$.*
- *quasi-nonexpansive, if $\mathcal{T}$ possesses fixed points and $\|\mathcal{T}x - p\| \leq \|x - p\|$, for all $x \in K$, and any $p \in \mathcal{F}(\mathcal{T})$.*

In 2011, García Falset *et. al.* [9] introduced the class of the mappings endowed with the (E) property, as follows.

Definition 2.2. Let $(X, \|\cdot\|)$ be a Banach space, and S a nonempty subset of X. A mapping $\mathcal{T}\colon S \to S$ satisfies the (E_μ) condition on the set S if there can be found a real number $\mu \geq 1$ so that

$$\|x - \mathcal{T}y\| \leq \mu\|x - \mathcal{T}x\| + \|x - y\|,$$

for all x, $y \in S$.

Moreover, it is said that $\mathcal{T}$ accomplishes the condition (E) if there exists $\mu \geq 1$ such that $\mathcal{T}$ fulfills the condition (E_μ).

This class of operators entails those endowed with the (C) property (so all nonexpansive mappings satisfy the condition (E)), but also other types of mappings, as proved in [9]. In the same paper, it is proved that a mapping endowed with the (E) property and has a fixed point is quasinonexpansive. Note that this condition can be easily formulated in the framework of metric spaces.

Motivated by above, in this paper we introduce a three-step iteration process with a single set of parameters,

$$\begin{cases} x_1 = x \in K \\ x_{n+1} = \mathcal{T}(\mathcal{T}(\mathcal{T}((1-\alpha_n)x_n + \alpha_n \mathcal{T}x_n))), \ n \in \mathbb{N}. \end{cases} \tag{5}$$

where $\{\alpha_n\}$ is in $(0,1)$.

The aim of this paper is to study the convergence of this iteration process (5) for mappings which fulfill the condition (E) in the framework of CAT(0) spaces. This setting has been considered here due to the fact that the non-positive curvature of Riemannian geometry, can be here presented in a wider sense, in this setting. Moreover, because of the absence of a natural linear and convex structure, many problems cannot be studied in usual metric spaces. Therefore we are aiming our study to those CAT(0) spaces, which are both Hilbert spaces as well as Banach spaces

Let us now recall some basics definitions, propositions and lemmas on CAT(0) spaces which shall be used in the next sections.

Let (X, d) be a metric space. A geodesic map is an isometric map $f\colon I \to X$ on a convex subset $I \subseteq \mathbb{R}$ to X, where the real line $\mathbb{R}$ is endowed with the Euclidean distance. The map f is called a geodesic segment (respectively ray, line) if I is a closed interval (respectively I is a half-line, $I = \mathbb{R}$).

A geodesic metric space is a metric space (X, d) in which any two points are joined by a geodesic segment.

Example 2.1. (i) The Euclidean space $(\mathbb{R}^n, d_{Eucl})$ is a geodesic metric space.
(ii) Any metric graph is a geodesic metric space.

Let (X, d) be a geodesic metric space. Given a triple $(x, y, z) \in X^3$, a Euclidean comparison triangle for (x, y, z) is a triple $(\bar{x}, \bar{y}, \bar{z})$ of points from the Euclidean plane $\mathbb{R}^2$

such that $d(x,y) = d_{Eucl}\ (\bar{x},\bar{y})$, $d(y,z) = d_{Eucl}(\bar{y},\bar{z})$ and $d(z,x) = d_{Eucl}(\bar{z},\bar{x})$. Notice that any triple in X admits some Euclidean comparison triangle.

Intuitively, a geodesic metric (X,d) is a CAT(0) space if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane.

Definition 2.3. Let Δ be a geodesic triangle in the geodesic metric space (X,d) and let $\bar{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all x, $y \in \Delta$ and all comparison points $\bar{x},\bar{y} \in \bar{\Delta}$, the inequality $d(x,y) \leq d_{\mathbb{R}^2}(\bar{x},\bar{y})$ holds true. (X,d) is a CAT(0) space if the CAT(0) inequality is satisfied for any triangle from this space.

As examples of CAT(0) spaces, we enumerate the following.

Example 2.2. (i) The Euclidean space $(\mathbb{R}^n, d_{Eucl})$ is a CAT(0) space, and so is any pre-Hilbert space.
(ii) A metric graph X is a CAT(0) space if and only if X is a tree.

Assume now that x, y_1, y_2 are points in a CAT(0) space and y_0 is the midpoint of the segment $[y_1,y_2]$. Then the CAT(0) inequality implies

$$d(x,y_0)^2 \leq \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \qquad (CN)$$

This is the (CN) inequality of Bruhat and Tits [5]. In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

In the following, we mention some interesting and useful properties of CAT(0) spaces.

Lemma 2.1 ([8])**.** Let (X,d) be a CAT(0) space. Then

(1) (X,d) is uniquely geodesic.
(2) Let p, x, y be points of X, $\alpha \in [0,1]$, m_1 and m_2 denote, respectively, the points from $[p,x]$ and $[p,y]$ satisfying $d(p,m_1) = \alpha d(p,x)$ and $d(p,m_2) = \alpha d(p,y)$. Then the next statement is fulfilled
$$d(m_1,m_2) \leq \alpha d(x,y).$$
(3) Let x, $y \in X$, $x \neq y$ and z, $w \in [x,y]$ such that $d(x,z) = d(x,w)$. Then $z = w$.
(4) Let x, $y \in X$. For each $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that
$$d(x,z) = td(x,y) \quad \text{and} \quad d(y,z) = (1-t)d(x,y). \qquad (6)$$
(5) For x, y, $z \in X$ and $t \in [0,1]$, the next inequality holds true
$$d\big((1-t)x \oplus ty, z\big) \leq (1-t)d(x,z) + td(y,z).$$

For the sake of convenience, from now on the notation $(1-t)x \oplus ty$ will be used for the unique point z satisfying equalities (6).

Regarding the geometric properties, we recollect the ones which play a vital role in the development of our outcomes.

Let $\{s_n\}$ be a bounded sequence in a CAT(0) space (X,d). For $s \in X$, we set

$$r(s,\{s_n\}) = \limsup_{n\to\infty} d(s,s_n).$$

The asymptotic radius of $\{s_n\}$ is given by

$$r(\{s_n\}) = \inf\{r(s,\{s_n\}) : s \in X\}.$$

The asymptotic center of $\{s_n\}$ is the set

$$A(\{s_n\}) = \{s \in X : r(s,\{s_n\}) = r(\{s_n\})\}.$$

In 2006, Dhompongsa *et al.* [7] stated that, in the framework of CAT(0) spaces, the asymptotic center consists of exactly one point.

CAT(0) spaces feature an interesting type of convergence defined by means of asymptotic centers, namely the Δ-convergence.

Definition 2.4 ([14]). A sequence $\{s_n\}$ in a CAT(0) space X is said to be Δ-convergent to $s \in X$ if the unique asymptotic center of $\{u_n\}$ is s, for every subsequence $\{u_n\}$ of $\{s_n\}$.

Such kind of convergence will be represented by $\Delta - \lim_{n\to\infty} s_n = s$, and read as s is the Δ-limit of $\{s_n\}$.

We denote $W_\Delta(\{s_n\}) = \bigcup A(\{u_n\})$, where the union is considered over all subsequences $\{u_n\}$ of $\{s_n\}$.

The following lemmas have been proved by Dhompongsa and Panyanak [8].

Lemma 2.2. Suppose X is a CAT(0) space. Then, for all x, y, $z \in X$, and $t \in [0, 1]$, the next inequality is fulfilled

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

Lemma 2.3. Suppose (X, d) is a CAT(0) space. Then the next statements hold true.

1) Every bounded sequence in X has a $\Delta-$convergent subsequence.

2) If K is a closed, and convex subset of X and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is an element of the set K.

Lemma 2.4. Suppose that $\{s_n\}$ is a bounded sequence in a complete CAT(0) space so that $A(\{s_n\}) = \{s\}$, and $\{u_n\}$ is a subsequence of $\{s_n\}$, $A(\{u_n\}) = \{u\}$. If the sequence $\{d(s_n, u)\}$ converges, then $s = u$.

The next lemma proved by Laowang and Panyanak [15] regards the behaviour of some sequences with adequate properties in CAT(0) spaces.

Lemma 2.5. *Let (X, d) be a complete CAT(0) space and $x \in X$. Suppose $\{t_n\}$ is a sequence in $[b, c] \subset (0, 1)$ and $\{u_n\}$, $\{v_n\}$ are sequences in X such that* $\limsup_{n\to\infty} d(u_n, u) \leq r$, $\limsup_{n\to\infty} d(v_n, u) \leq r$ *and* $\lim_{n\to\infty} d(t_n v_n \oplus (1-t_n)u_n, x) = r$ *hold for some* $r \geq 0$. *Then* $\lim_{n\to\infty} d(u_n, v_n) = 0$.

Iteration (5) has its CAT(0) spaces version, as in the next lines.

Let K be a nonempty, closed, and convex subset of a complete CAT(0) space X, and $\mathcal{T}\colon K \to K$ be a mapping. Let $x_1 \in K$ be arbitrary, and the sequence $\{x_n\}$ generated iteratively by

$$\begin{cases} x_1 = x \in K \\ x_{n+1} = \mathcal{T}\bar{x}_n, \\ \bar{x}_n = \mathcal{T}\tilde{x}_n, \\ \tilde{x}_n = \mathcal{T}((1-\alpha_n)x_n \oplus \alpha_n \mathcal{T}x_n), \ n \in \mathbb{N}. \end{cases} \tag{7}$$

where $\alpha_n \in (0, 1)$, for $n \in \mathbb{N}$.

Please note that Kirk [13] proved that any nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space has a fixed point.

3. Δ-Convergence and Strong Convergence Theorems

In the following, we will prove the strong and Δ-convergence of this iteration process (7). Our results will be generalization of some results of Chanchal Garodia *et al.* [10], Khan and Abbas [12], and Piri *et al.* [22].

The next theorem provides conditions for the boundedness of the sequence generated by Algorithm (7).

Theorem 3.1. *Let K be a nonempty, closed, convex subset of a complete CAT(0) space X, and $\mathcal{T}\colon K \to K$ be a mapping endowed with the property (E). Consider that $\{x_n\} \subset K$ is defined by (7), where $\{\alpha_n\}$ is in $(0,1)$ and $\mathcal{F}(\mathcal{T}) \neq \emptyset$. Then $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \mathcal{F}(\mathcal{T})$.*

Proof. Let $p \in \mathcal{F}(\mathcal{T})$ be a fixed point of $\mathcal{T}$, which is a quasinonexpansive mapping. From (7) and using Lemma 2.2, we have, for any $n \in \mathbb{N}$,

$$\begin{aligned} d(\tilde{x}_n, p) &= d(\mathcal{T}((1-\alpha_n)x_n \oplus \alpha_n \mathcal{T}x_n), p) \\ &\leq d((1-\alpha_n)x_n \oplus \alpha_n \mathcal{T}x_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(\mathcal{T}x_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &= d(x_n, p), \end{aligned} \tag{8}$$

and

$$d(\bar{x}_n, p) = d(\mathcal{T}\tilde{x}_n, p) \leq d(\tilde{x}_n, p) \leq d(x_n, p),\ n \in \mathbb{N}. \tag{9}$$

Inequalities (8) and (9) imply

$$d(x_{n+1}, p) = d(\mathcal{T}\bar{x}_n, p) \leq d(\bar{x}_n, p) \leq d(x_n, p),\ n \in \mathbb{N}.$$

Therefore, $d(x_n, p)$ is bounded below and nonincreasing. Hence $\lim_{n\to\infty} d(x_n, p)$ exists. The boundedness of the sequence $\{x_n\}$ follows then easily. □

Theorem 3.2. *Let K be a nonempty, closed, and convex subset of a complete CAT(0) space (X, d). Let $\mathcal{T}\colon K \to K$ be a mapping which satisfies the condition (E) on K, such that $\mathcal{F}(\mathcal{T}) \neq \phi$, and $\{x_n\}$ be defined by Algorithm (7), where $\{\alpha_n\}$ is in $(0,1)$. Then $\lim_{n\to\infty} d(x_n, \mathcal{T}x_n) = 0$.*

Proof. According to Theorem 3.1, the sequence $\{d(x_n, p)\}$ is convergent. Assume that $\lim_{n\to\infty} d(x_n, p) = l$. Inequality (9) from Theorem 3.1 compels $d(\bar{x}_n, p) \leq d(x_n, p)$, $n \in \mathbb{N}$, hence it follows that $\limsup d(\bar{x}_n, p) \leq \lim d(x_n, p) = l$. Therefore,

$$\limsup d(\bar{x}_n, p) \leq l. \tag{10}$$

Since T is a quasinonexpansive mapping, we have

$$\limsup d(\mathcal{T}x_n, p) \leq \lim d(x_n, p) = l. \tag{11}$$

Having in view inequality (9) from Theorem 3.1, we obtain $d(x_{n+1}, p) = d(\mathcal{T}\bar{x}_n, p) \leq d(\bar{x}_n, p)$, implying that $\lim d(x_{n+1}, p) \leq \liminf d(\bar{x}_n, p)$. Thus, we have

$$l \leq \liminf d(\bar{x}_n, p). \tag{12}$$

From relations (10) and (12), it follows that

$$\lim_{n\to\infty} d(\bar{x}_n, p) = l.$$

Moreover, the above mentioned inequality and relation (11) leads to

$$\begin{aligned} l &= \lim_{n\to\infty} d(\bar{x}_n, p) = \lim_{n\to\infty} d(\mathcal{T}\tilde{x}_n, p) \leq \liminf_{n\to\infty} d(\tilde{x}_n, p) \\ &\leq \liminf_{n\to\infty} d((\mathcal{T}((1-\alpha_n)x_n \oplus \alpha_n \mathcal{T}x_n)), p) \leq \liminf_{n\to\infty} d(((1-\alpha_n)x_n \oplus \alpha_n \mathcal{T}x_n), p) \\ &\leq \liminf_{n\to\infty}((1-\alpha_n)d(x_n, p) + \alpha_n d(\mathcal{T}x_n, p)) \leq \limsup_{n\to\infty}((1-\alpha_n)d(x_n, p) + \alpha_n d(\mathcal{T}x_n, p)) \\ &\leq \limsup_{n\to\infty}((1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p)) = l. \end{aligned}$$

This implies that

$$\lim_{n\to\infty}((1-\alpha_n)d(x_n,p)+\alpha_n d(\mathcal{T}x_n,p))=l.$$

Based on relation (11) and Lemma 2.5, we have drawn the conclusion that $\lim_{n\to\infty} d(x_n,\mathcal{T}x_n) = 0$, and the proof is completed. □

The next result refers to a Δ-convergence property associated with the iterative method (7).

Theorem 3.3. *Let $\mathcal{T}\colon K\to K$ be a mapping which fulfills the condition (E) on a nonempty, closed, and convex subset K of a complete CAT(0) space (X,d) such that the set of the fixed points of T is not empty. If $\{x_n\}$ is a sequence defined by the iteration process (7), then $\{x_n\}$ is Δ-convergent to a fixed point of $\mathcal{T}$.*

Proof. From Theorem 3.1 and Theorem 3.2, it is clear that $\lim_{n\to\infty} d(x_n,p)$ exists for each $p\in\mathcal{F}(\mathcal{T})$, the sequence $\{x_n\}$ is bounded, and $\lim_{n\to\infty} d(x_n,Tx_n)=0$. Let $W_\Delta(\{x_n\})=\bigcup A(\{u_n\})$, where the reunion is taken over all subsequences $\{u_n\}$ of $\{x_n\}$.

First we will show that $W_\Delta(\{x_n\})\subseteq\mathcal{F}(\mathcal{T})$. Let $u\in W_\Delta(\{x_n\})$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\})=u$. By Lemma 2.3 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta-\lim_{n\to\infty} v_n=v$ and $v\in K$. Since $\lim_{n\to\infty} d(\mathcal{T}x_n,x_n)=0$ and $\{v_n\}$ is a subsequence of $\{x_n\}$, $\lim_{n\to\infty} d(v_n,Tv_n)=0$. Since $\mathcal{T}$ satisfies the condition (E), there exists $\mu\geq 1$, so that for all x, $y\in K$, $d(x,\mathcal{T}y)\leq \mu d(x,\mathcal{T}x)+d(x,y)$. This inequality compels that

$$d(v_n,\mathcal{T}v)\leq \mu d(v_n,\mathcal{T}v_n)+d(v_n,v).$$

Taking lim sup in both sides of this relation, it follows that

$$\begin{aligned}\limsup_{n\to\infty} d(v_n,\mathcal{T}v) &\leq \limsup_{n\to\infty}\big(\mu d(v_n,\mathcal{T}v_n)+d(v_n,v)\big)\\ &\leq \mu\limsup_{n\to\infty} d(v_n,\mathcal{T}v_n)+\limsup_{n\to\infty} d(v_n,v)=\limsup_{n\to\infty} d(v_n,v).\end{aligned}$$

As $\Delta-\lim_{n\to\infty} v_n=v$, we get $\limsup_{n\to\infty} d(v_n,v)\leq\limsup_{n\to\infty} d(v_n,\mathcal{T}v)$, and hence

$$\limsup_{n\to\infty} d(v_n,v)=\limsup_{n\to\infty} d(v_n,\mathcal{T}v).$$

It follows that $\mathcal{T}v=v$ i.e. $v\in\mathcal{F}(\mathcal{T})$.

Presume that $u\neq v$. By Theorem 3.1, $\lim_{n\to\infty} d(x_n,v)$ exists as $v\in\mathcal{F}(\mathcal{T})$. We now claim that $v=u$. Then by the uniqueness property regarding the asymptotic centers, we have

$$\begin{aligned}\limsup_{n\to\infty} d(v_n,v)&<\limsup_{n\to\infty} d(v_n,u)\leq\limsup_{n\to\infty} d(u_n,u)\\ &<\limsup_{n\to\infty} d(u_n,v)=\limsup_{n\to\infty} d(x_n,v)=\limsup_{n\to\infty} d(v_n,v)\end{aligned}$$

which is a contradiction. Thus $u=v$ and hence $W_\Delta(\{x_n\})\subseteq\mathcal{F}(\mathcal{T})$.

To show that the sequence $\{x_n\}$ is $\Delta-$convergent to a fixed point of T, we show that $W_\Delta(\{x_n\})$ consists of exactly one point. In this respect, consider $\{u_n\}$ a subsequence of $\{x_n\}$. By using Lemma 2.3, there can be found a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta-\lim_{n\to\infty} v_n=v$ and $v\in K$. Let $A(\{u_n\})=\{u\}$ and $A(\{x_n\})=\{x\}$. It has already been proved that $u=v$ and $v\in\mathcal{F}(\mathcal{T})$. Since $v\in\mathcal{F}(\mathcal{T})$, by Theorem 3.1, $\{d(x_n,v)\}$ is convergent. Lemma 2.4 leads to $v=x$. Therefore $W_\Delta(\{x_n\})=\{x\}$. This completes the proof. □

Using Theorem 3.1 and Theorem 3.2, now we are in a position to prove a strong convergence result.

Theorem 3.4. *Let $\mathcal{T}\colon K \to K$ be a mapping endowed with the property (E), defined on a nonempty, closed, and convex subset K of a complete CAT(0) space (X, d), which possesses at least one fixed point. Denote by $\{x_n\}$ the sequence defined by the iteration process* (7)*. Then $\{x_n\}$ converges to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}(\mathcal{T})) = 0$.*

Proof. Presume first that the sequence $\{x_n\}$ converges to a point $p \in \mathcal{F}(\mathcal{T})$. Then $\lim_{n\to\infty} d(x_n, p) = 0$, so $\liminf_{n\to\infty} d(x_n, \mathcal{F}(\mathcal{T})) = 0$, and the conclusion has been proved.

Conversely, suppose now that $\liminf_{n\to\infty} d(x_n, \mathcal{F}(\mathcal{T})) = 0$. According to Theorem 3.1,

$$d(x_{n+1}, p) \leq d(x_n, p), \qquad \text{for all } p \in \mathcal{F}(\mathcal{T}).$$

Because $d(x_{n+1}, \mathcal{F}(\mathcal{T})) = \inf_{q\in\mathcal{F}(\mathcal{T})} d(x_{n+1}, q) \leq d(x_{n+1}, p)$, for all fixed points p of $\mathcal{T}$, it follows that $d(x_{n+1}, \mathcal{F}(\mathcal{T})) \leq d(x_n, \mathcal{F}(\mathcal{T}))$, and, as a consequence, $\lim_{n\to\infty} d(x_n, \mathcal{F}(\mathcal{T}))$ exists. Having in view the hypothesis of the theorem, we get $\lim_{n\to\infty} d(x_n, \mathcal{F}(\mathcal{T})) = 0$.

Let us prove now that $\{x_n\}$ is a Cauchy sequence in K. Consider $\epsilon > 0$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}(\mathcal{T})) = 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, \mathcal{F}(\mathcal{T})) < \frac{\epsilon}{4}$, for all $n \geq n_0$. In particular, $\inf\{d(x_{n_0}, p) : p \in \mathcal{F}(\mathcal{T})\} < \frac{\epsilon}{4}$. Therefore, there exists $p^* \in \mathcal{F}(\mathcal{T})$ such that $d(x_{n_0}, p^*) < \frac{\epsilon}{2}$. If m, $n \geq n_0$, it can be noticed that

$$d(x_{n+m}, x_n) < d(x_{n+m}, p^*) + d(p^*, x_n) \leq 2d(x_{n_0}, p^*) < 2\frac{\epsilon}{2} = \epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in the closed subset K of a complete CAT(0) space. Let $x \in K$ be its limit. As $\lim_{n\to\infty} d(x_n, \mathcal{F}(\mathcal{T})) = 0$, it follows that $d(x, \mathcal{F}(\mathcal{T})) = 0$. According to Chidume and Maruster [6], the set $\mathcal{F}(\mathcal{T})$ is closed, which allows us to conclude that $x \in \mathcal{F}(\mathcal{T})$. □

Senter *et al.* [26] introduced the condition (A) as follows.

Let $(B, \|\cdot\|)$ be a Banach space, and $K \subseteq B$. A mapping $\mathcal{T}\colon K \to K$ is said to satisfy the condition (A) if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$, for all $r \in (0,\infty)$ such that $d(x, \mathcal{T}x) \geq f(d(x, \mathcal{F}(\mathcal{T})))$ for all $x \in K$.

A similar definition can be easily formulated in the framework of CAT(0) spaces.

Theorem 3.5. *Let $\mathcal{T}\colon K \to K$ be a mapping defined on a nonempty, closed, and convex subset K of a complete CAT(0) space X, endowed with the property (E), which satisfies the condition (A), such that $\mathcal{F}(\mathcal{T})$ is not empty. If $\{x_n\}$ is a sequence defined by Algorithm* (7)*, then $\{x_n\}$ converges strongly to a fixed point of $\mathcal{T}$.*

Proof. By Theorem 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \mathcal{F}(\mathcal{T})$. Since $d(x_{n+1}, p) \leq d(x_n, p)$, $n \in \mathbb{N}$, it follows that

$$\inf_{q\in\mathcal{F}(\mathcal{T})} d(x_{n+1}, q) \leq d(x_n, p), \text{ for any } p \in \mathcal{F}(\mathcal{T}),$$

which yields $d(x_{n+1}, \mathcal{F}(\mathcal{T})) \leq d(x_n, \mathcal{F}(\mathcal{T}))$. This compels that the sequence $\{d(x_n, \mathcal{F}(\mathcal{T}))\}$ is nonincreasing and bounded from below. It follows that the limit $\lim_{n\to\infty} d(x_n, \mathcal{F}(\mathcal{T}))$ exists. Also, by Theorem 3.2, $\lim_{n\to\infty} d(x_n, \mathcal{T}x_n) = 0$.

Since the condition (A) is fulfilled, $\lim_{n\to\infty} f(d(x_n, \mathcal{F}(\mathcal{T}))) \leq \lim_{n\to\infty} d(x_n, \mathcal{T}x_n) = 0$. It follows that $\lim_{n\to\infty} f(d(x_n, \mathcal{F}(\mathcal{T}))) = 0$. Keeping in mind that f is a nondecreasing function satisfying $f(0) = 0$, and $f(r) > 0$, for all points $r \in (0,\infty)$, we obtain that $\lim_{n\to\infty} d(x_n, \mathcal{F}(\mathcal{T})) = 0$. Since all the conditions in Theorem 3.4 are satisfied, the sequence $\{x_n\}$ converges strongly to a fixed point of $\mathcal{T}$. □

Recall that a complete simply connected Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold. For some fundamental definitions, properties and notations of Riemannian manifolds, one can refer to [4, 25]. We now continue our discussion with an example which regards a Hadamard manifold (all Hadamard manifolds are CAT(0) spaces), inspired by [16].

Example 3.1. Let $\mathbb{E}^{3,1}$ be the Minkowski space $\mathbb{R}^{3+1}$ endowed with the Lorentz inner product

$$\langle x, y \rangle = \sum_{k=1}^{3} x^k y^k - x^4 y^4, \qquad x = (x^k),\, y = (y^k) \in \mathbb{R}^{3+1}.$$

According to [4], p. 93, the set $\mathbb{H}^3 = \{x \in \mathbb{E}^{3,1} : \langle x, x \rangle = -1,\, x^4 > 0\}$ can be organized as a Riemannian manifold. The corresponding distance is $d \colon \mathbb{H}^3 \times \mathbb{H}^3 \to \mathbb{R}$, where $d(x, y)$ is the unique non-negative value for which $\cosh d(x, y) = -\langle x, y \rangle$.

Let $x \in \mathbb{H}^3$ and a unit vector v from the tangent space $\mathcal{T}_x\mathbb{H}^3$. The corresponding geodesic is

$$\gamma \colon \mathbb{R} \to \mathbb{H}^3 \qquad \gamma(t) = (\cosh t)x + (\sinh t)y,$$

while the exponential map is

$$\exp_x \colon \mathcal{T}_x\mathbb{H}^3 \to \mathbb{H}^3,\ \exp_x(rv) = (\cosh r)x + (\sinh r)v,\ r \in \mathbb{R}^+,\, x \in \mathbb{H}^3,\, v \in \mathcal{T}_x\mathbb{H}^3,$$

while its inverse is given by

$$\exp_x^{-1} y = r(x, y)V(x, y),\ y \in \mathbb{H}^3,$$

where $r(x, y) = \operatorname{arccosh}(-\langle x, y \rangle)$ and $V(x, y) = \dfrac{y + \langle x, y \rangle x}{\sqrt{\langle x, y \rangle^2 - 1}}$.

In the following, consider the nonexpansive mapping

$$\mathcal{T} \colon \mathbb{H}^3 \to \mathbb{H}^3,\ \mathcal{T}x = (-x^1, -x^2, -x^3, x^4),\ x = (x^k) \in \mathbb{H}^3,$$

with the unique fixed point $(0, 0, 0, 1)$.

As an initial value we considered $x_0 = (1, 1, 1, 2)$. We have considered $\alpha_n = \frac{3}{5}$ in the scheme introduced here. Comparisons made with respect to the algorithms introduced by Abbas and Nazir [1], Noor [19], Thakur *et al.* [30] (TTP14), Thakur *et al.* [29] (TTP16), for the choice of all parameter sequences equal to $\frac{3}{5}$, are presented below. In the second column we have indicated the number of iteration at which an error smaller than 10^{-9} is obtained.

Process	No. of iteration
TTP14	iteration#29
Noor	iteration #24
TTP16	iteration#10
Abbas	iteration # 9
Algorithm (7)	iteration #8

Now, we present an example of a mapping which fulfills the condition (E) and illustrates the convergence of the iteration process (7) with respect to different initial values.

Example 3.2. Let the set $K = [0, \infty)$ be equipped with the usual norm $|\cdot|$ and let

$$\mathcal{T} \colon K \to K,\ \mathcal{T}(x) = \begin{cases} \frac{x}{2}, & \text{if } x > 2, \\ 0, & \text{otherwise.} \end{cases}$$

Piri [22] proved that the mapping $\mathcal{T}$ does not satisfy the condition (C), but it is a generalized α-nonexpansive mapping, so it fulfills the condition (E).

For $\alpha_n = \dfrac{n}{n^2 + 1}$, we obtain Table 1 and Figure 1 for different initial values.

TABLE 1. Comparison Table for Example 3.2

Steps	$x_1 = 10^1$	$x_1 = 10^2$	$x_1 = 10^3$	$x_1 = 1500$	$x_1 = 10^5$
0	10	100.0000	1000.0000	1500.0000	10000.0000
1	2.5000	25.0000	250.0000	375.0000	2500.0000
2	0.4688	4.6875	46.8750	70.3125	468.7500
3	0.0000	0.9375	9.3750	14.0625	93.7500
4	0.0000	0.0000	1.9922	2.9883	19.9219
5	0.0000	0.0000	0.0000	0.6592	4.3945
6	0.0000	0.0000	0.0000	0.0000	0.9929
7	0.0000	0.0000	0.0000	0.0000	0.0000

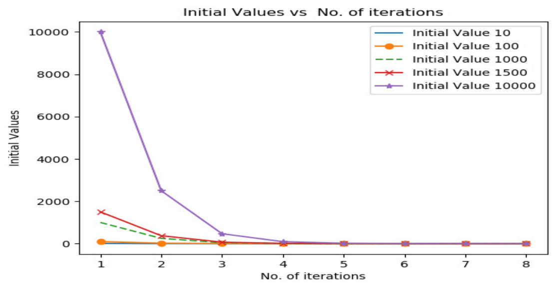

FIGURE 1. Convergence behavior of process (7) for Example 3.2 for various initial values

Example 3.3. *Let $K = [0, 1]$ which is a closed, and convex subset of the CAT(0) space $X = \mathbb{R}$, endowed with the usual metric. Define a mapping*

$$\mathfrak{T}\colon K \to K,\ \mathfrak{T}x = \begin{cases} \frac{x}{2}, & \text{if } x \neq 1, \\ \frac{7}{11}, & \text{if } x = 1. \end{cases}$$

It is obvious that $0 \in F(\mathfrak{T})$, and that $\mathfrak{T}$ fulfills the condition (E) for $\mu = \frac{11}{8} > 1$. The operator T does not satisfy the condition (C) of Suzuki. Indeed, if we consider $x = \frac{4}{5}$ and $y = 1$, then

$$\frac{1}{2}|x - \mathfrak{T}x| = \frac{1}{2}\left|\frac{4}{5} - \frac{2}{5}\right| = \frac{1}{5} = |x - y|.$$

On the other hand,

$$|\mathfrak{T}x - \mathfrak{T}y| = \left|\mathfrak{T}\frac{4}{5} - \mathfrak{T}1\right| = \left|\frac{2}{5} - \frac{7}{11}\right| = \frac{13}{55} > \left|\frac{4}{5} - 1\right| = |x - y|.$$

Thus, $\mathfrak{T}$ fails to satisfy condition (C). Furthermore, we have examined the influence of parameters α_n, β_n and γ_n. For this we have considered various sets of parameters and present a study regarding the number of iterations required. Each iteration starts with a particular initial value and the respective number of iterations, average of the number of iterations for different initial points are given in Figure 2. We have examined the fastness and stability of different iterations relative to above mentioned set of parameters. The observations are given in Figure 2 and Figure 3. We have concluded that the new iteration process (7) not only converges faster than the known iterations but also is stable with respect to the parameters α_n, β_n and γ_n. From Figure 2, we also observe that the average number of iterations of the new iteration process (7) is the smallest with respect to other processes.

We now discuss the influence of parameters $\alpha_n, \beta_n, \gamma_n$ by considering the following fives sets of parameters:

Case 1. $\alpha_n = \sqrt{\frac{2n}{3n+5}}$, $\beta_n = \frac{1}{\sqrt{2n+9}}$, $\gamma_n = \frac{2n}{7n+9}$
Case 2. $\alpha_n = \frac{n}{n+2}$, $\beta_n = \frac{1}{\sqrt{n+5}}$, $\gamma_n = \frac{2n}{5n+3}$
Case 3. $\alpha_n = \frac{3n}{8n+4}$, $\beta_n = \frac{1}{n+4}$, $\gamma_n = \frac{n}{(5n+2)^2}$
Case 4. $\alpha_n = \frac{2n}{3n+2}$, $\beta_n = \frac{n}{\sqrt{49n^2+1}}$, $\gamma_n = \sqrt{\frac{2n}{(3n+5)}}$
Case 5. $\alpha_n = \frac{n}{n+1}$, $\beta_n = \frac{n}{n+5}$, $\gamma_n = \frac{n}{\sqrt{2n^2+9}}$.

Comparison of various iteration processes for Example 3.3

Iterations																									
Init. Value	0.25					0.5					0.75					1					Iterations Average ↓				
Case	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
Mann	70	59	167	88	55	71	60	171	90	55	72	60	171	91	56	73	61	174	92	57	71.5	60	171.2	90.2	56.2
Ishikawa	65	52	165	82	36	66	53	169	83	36	67	54	171	84	36	68	55	172	85	37	66.5	53.5	169.2	83.5	36.5
Noor	64	51	165	80	31	65	52	169	82	31	66	53	171	83	32	67	53	172	84	32	65.5	52.2	169.2	82.2	31.7
Abbas	29	27	39	30	23	29	27	39	30	23	29	28	40	50	23	30	28	41	31	24	29.5	27.5	39.7	30.5	23.5
TTP16	39	35	49	37	24	39	35	50	38	24	40	36	50	58	25	41	36	51	39	25	39.7	35.5	50	38	24.7
Piri	18	17	21	19	16	18	18	22	20	16	19	18	22	20	16	19	19	23	21	17	18.5	18	22	20	16.2
Alg. (7)	14	14	16	15	14	14	14	16	15	14	15	14	16	15	14	15	15	17	16	15	14.5	14.2	16.2	15.2	14.2

FIGURE 2. Table depicting Comparision of various iterations process under distinct parameters for Example 3.3

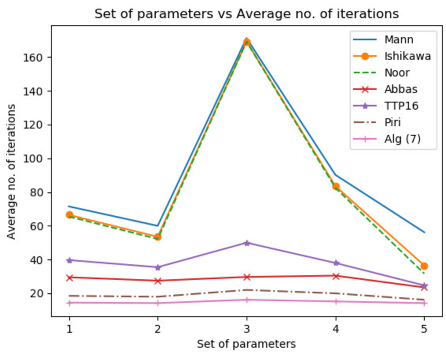

FIGURE 3. Average no. of iterations under distinct parameters for Example 3.3

4. **Conclusions**

In this paper, we obtained some strong and Δ-convergence results in CAT(0) space for a new iterative scheme for operators endowed with the (E) property. Our results extend and generalize many results in the literature. More precisely, Theorem 3.3, Theorem 3.4 and Theorem 3.5 extend Theorem 1, Theorem 2 and Theorem 3 of Khan and Abbas [12] in the sense that it provides a convergent scheme for approximating fixed points a class of mappings more general than that of nonexpansive mappings. Theorem 3.3, Theorem 3.4 and Theorem 3.5 generalize Theorem 3.1, Theorem 3.2 and Theorem 3.3 of Garodia and Uddin [10] proved for the TTP 14 [30] iteration scheme for generalized nonexpansive mappings.

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