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Scalable test functions for multidimensional continuous optimization

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Multidimensional scalable test functions are very important in testing the capabilities of new optimization methods, especially in evaluating their response with the increase of the search space dimension. The paper is proposing new sets of test functions for continuous optimization, both unconstrained (or only box constrained) and constrained.

Keywords: Optimization, Continuous Global Optimization Problem (CGOP), Constrained Optimization, Keane's Bump Function, Nesbitts Inequality

1. Introduction

The real world optimization problems that emerge from various scientific and engineering fields are characterized by complexity, non-linearity and increased numbers of decision variables and constraints. In order to be able to handle such difficult problems the researchers in the optimization field are continually proposing new improved optimization algorithms. Due to the intrinsic mathematical difficulty of the global optimization problem, in the last decades there is a trend in researching new nature inspired optimization algorithms, capable to provide acceptable solutions in convenient computing time, even though the global solution is not guaranteed. Such nature inspired algorithms, also named meta-heuristic, or population based algorithms, have some advantages over the traditional gradient based algorithms: they are able to handle more general classes o optimization problems, are derivative free (can be successfully applied when the derivatives are not available or do not exist) and can be easily parallelized on modern multiprocessor computers. Before the newly proposed optimization methods are applied to real world optimization problems, their properties are extensively evaluated by using known test functions from standard literature. In most of the cases the global solutions of the test functions are theoretically known, but sometimes only the best experimentally found solutions are available (for the so called open problems) and any improvement to the best known solutions provided by the tested optimization algorithm is considered as a competitive advantage.

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One important property that modern optimization methods need (especially in modern Big Data applications) is scalability, i.e. the ability to respond well when the dimension of the search space increases. In order to appropriately evaluate the scalability property of the optimization methods there is a need of multidimensional scalable test functions. Many sets of optimization test functions (benchmarks) are already known from the literature (see [1], [2], [3], [4], [5], [6], [7], [8], etc.), but there is still a need for multidimensional scalable test functions, and especially there is a lack of multidimensional scalable test functions for testing continuous constrained optimization methods. The goal of the present paper is to supplement the known collections of optimization test functions with some new proposals of multidimensional scalable problems, especially deceptive problems (for which the size of the basin of attraction of the global solution is small compared to the sizes of the basins of attractions of some local solutions), which can prove useful in further testing and comparing the capabilities of the numerous modern optimization methods.

The rest of the paper is organized as follows: Section 2 presents the general model of the Continuous Global Optimization Problem (*CGOP*); Section 3 presents the new proposed unconstrained (or box constrained) optimization test functions; Section 4 presents the new proposed constrained optimization test functions; and finally, Section 5 summarizes and draws some conclusions.

2. Continuous Global Optimization Problem (*CGOP*)

The Continuous Global Optimization Problem (*CGOP*) is generally formulated as ([9]):

$$\text{minimize} \qquad f(\mathbf{x}) \tag{1}$$

$$\text{subject to} \quad \mathbf{x} \in D$$

with

$$D = \{\mathbf{x} : \mathbf{l} \leq \mathbf{x} \leq \mathbf{u};\ \textit{and}\ \ g_i(\mathbf{x}) \leq 0,\ \ i = 1, \ldots, G; \\ \textit{and}\ \ h_j(\mathbf{x}) = 0,\ \ j = 1, \ldots, H\} \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^n$ is a real n-dimensional vector of decision variables ($\mathbf{x} = (x_1, x_2, \ldots, x_n)$), $f : \mathbb{R}^n \to \mathbb{R}$ is the continuous objective function, $D \subset \mathbb{R}^n$ is the non-empty set of feasible decisions (a proper subset of $\mathbb{R}^n$), $\mathbf{l}$ and $\mathbf{u}$ are explicit, finite (component-wise) lower and upper bounds on $\mathbf{x}$, $g_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, \ldots, G$ is a finite collection of continuous inequality constraint functions, and $h_j : \mathbb{R}^n \to \mathbb{R},\ j = 1, \ldots, H$ is a finite collection of continuous equality constraint functions. In the black box approach of the *CGOP* problem, which is specific for the derivative free meta-heuristic population based optimization methods, no other additional suppositions are made and it is assumed that no additional knowledge about the collections of real continuous functions can be obtained, i.e. for any point $\mathbf{x}$ in the boxed domain $\{\mathbf{x} : \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\}$ it is assumed the ability to calculate the values of the functions $f(\mathbf{x})$, $g_i(\mathbf{x})$, $i = 1, \ldots, G$, $h_j(\mathbf{x})$, $j = 1, \ldots, H$, but nothing more. However, in the gradient based optimization methods it is assumed that the methods have also access to the derivatives of the mentioned set of functions functions (if they exist).

The general mathematical model presented in this section will be applied to the formal presentations of all the proposed optimization test functions in the next sections of the paper. All the proposed functions are multidimensional and scalable to the dimension of the search space n. All the other properties of the functions (such as, unimodality or multimodality) are described when they are known. The known global solutions (theoretically provable), or the best known global solutions (for open problems) are also specified. All the numerical results presented were obtained by applying metaheuristic optimization methods (see the methods presented in [10], [11] and [12]).

3. Unconstrained optimization test functions

In this section is described a new set of 13 unconstrained (or box constrained) optimization test functions.

- f_1 - unimodal, *global minimum* value (theoretical) $f_1^* = 0$ at $\mathbf{x}^* = (0, 0, \ldots, 0)$:

$$f_1(\mathbf{x}) = \sum_{j=1}^{n} (2x_{j-1} + x_j^2 x_{j+1} - x_{j+1})^2, \qquad (3)$$
$$n \geq 3,\ x_{n+1} = x_1,\ \ x_0 = x_n,\ \ -2 \leq x_j \leq 2,\ \ j = 1, \ldots, n$$

- f_2 - unimodal, *global minimum* value (theoretical) $f_2^* = 0$ at $\mathbf{x}^* = (2, 2, \ldots, 2)$:

$$f_2(\mathbf{x}) = \sum_{j=1}^{n} [\log_2(x_{j-1} x_j^2) - \log_3(x_{j+1}^5 - 5)]^2, \qquad (4)$$
$$n \geq 3,\ x_{n+1} = x_1,\ \ x_0 = x_n,\ \ 1.39 \leq x_j \leq 4,\ \ j = 1, \ldots, n$$

- f_3 - unimodal, *global minimum* value (theoretical) $f_3^* = 0$ at $\mathbf{x}^* = (2, 2, \ldots, 2)$:

$$f_3(\mathbf{x}) = \sum_{j=1}^{n} [x_j - |x_{j-1}^2 - 2x_j + 4|^{1/2} \log_2(4 - x_{j+1})]^2, \qquad (5)$$
$$n \geq 3,\ x_{n+1} = x_1,\ \ x_0 = x_n,\ \ -4 \leq x_j \leq 3.999,\ \ j = 1, \ldots, n$$

- f_4 - unimodal, *global minimum* value (theoretical) $f_4^* = 0$ at $\mathbf{x}^* = (1, 1, \ldots, 1)$:

$$f_4(\mathbf{x}) = \sum_{j=1}^{n} [2^{x_j} - \frac{1}{x_{j-1}x_{j+1}} - 1]^2, \quad n \geq 3,\ x_{n+1} = x_1,\ x_0 = x_n,\ 0.001 \leq x_j \leq 2,\ j = 1, \ldots, n \tag{6}$$

- f_5 - unimodal, *global minimum* value (theoretical) $f_5^* = \frac{2n}{n-1}$ at $\mathbf{x}^* = \left(\frac{1}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right)$:

$$f_5(\mathbf{x}) = \sum_{j=1}^{n} x_j + \sum_{j=1}^{n} \frac{x_j}{\left(-x_j + \sum_{j_1=1}^{n} x_{j_1}\right)^2}, \quad n \geq 3,\ 10^{-6} \leq x_j \leq 2,\ j = 1, \ldots, n \tag{7}$$

- f_6 - unimodal, *global minimum* value (theoretical) $f_6^* = \frac{2n}{(n-1)^2}$ at $\mathbf{x}^* = \left(\frac{1}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right)$:

$$f_6(\mathbf{x}) = \sum_{j=1}^{n} x_j^2 + \sum_{j=1}^{n} \frac{x_j^2}{\left(-x_j + \sum_{j_1=1}^{n} x_{j_1}\right)^4}, \quad n \geq 3,\ 10^{-6} \leq x_j \leq 2,\ j = 1, \ldots, n \tag{8}$$

- f_7 - unimodal, *global minimum* value (theoretical) $f_7^* = 0$ at $\mathbf{x}^* = (1, 1, \ldots, 1)$:

$$f_7(\mathbf{x}) = \frac{1}{n}(x_1 - 1)^2 + \sum_{j=1}^{n} (x_j^3 - 3x_{j+1}^2 + 3x_{j-1} - 1)^2, \quad n \geq 3,\ x_{n+1} = x_1,\ x_0 = x_n,\ -2 \leq x_j \leq 2,\ j = 1, \ldots, n \tag{9}$$

- f_8 - multimodal, *global minimum* value (theoretical) $f_8^* = 0$ at $\mathbf{x}^* = (-1, -1, \ldots, -1)$, but there is another local minimum with a larger attraction basin close to $(-2, -2, \ldots, -2)$, which is frequently trapping the optimization methods:

$$f_8(\mathbf{x}) = \frac{1}{n}(x_1 + 1)^2 + \sum_{j=1}^{n} [x_j - 2(x_{j-1} + x_{j+1}) - x_{j-1}x_{j+1} - 2]^2, \quad n \geq 3,\ x_{n+1} = x_1,\ x_0 = x_n,\ -3 \leq x_j \leq 3,\ j = 1, \ldots, n \tag{10}$$

- f_9 - multimodal, *global minimum* value (theoretical) $f_9^* = 1$ at $\mathbf{x}^* = (1, 1, \ldots, 1)$, but the attraction basin of the global minimum is small and usually the optimization methods are trapped by a local minimum with a larger attraction basin located in the vicinity of the origin $(0, 0, \ldots, 0)$:

$$
\begin{aligned}
f_9(\mathbf{x}) = &\sum_{j=1}^{n}[x_j^2(2x_j^2 + x_{j+1} + 2) - x_j x_{j-1}(3x_j + 3x_{j-1} - x_{j+1})]^2 + \\
&+ e^{\frac{1}{n^2}\sum_{j=1}^{n}(x_j - 1)^2}, \\
&n \geq 3,\ \ x_{n+1} = x_1,\ \ x_0 = x_n,\ \ -1 \leq x_j \leq 2,\ \ j = 1, \ldots, n
\end{aligned}
\qquad (11)
$$

- f_{10} - multimodal, *global minimum* value (theoretical) $f_{10}^* = -1$ at $\mathbf{x}^* = (0.4, 0.4, \ldots, 0.4)$. This is a very difficult test function if approached as a black box model due to the small dimension of the attraction basin of the global minimum which is also masked by local maxima. Usually the optimization methods are trapped by one of the local minima in the corners of the limiting box. A graphical representation of this function is given in Fig. 1 for the 2-dimensional case:

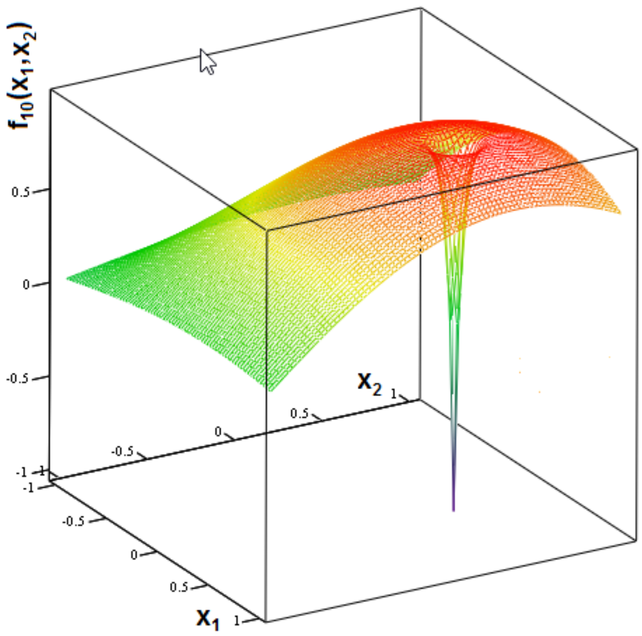

FIGURE 1. Difficult optimization problem (2-dimensional case)

$$f_{10}(\mathbf{x}) = -2e^{-20\sqrt{n}\left[\sum_{j=1}^{n}(x_j-0.4)^2\right]^{1/2}} + \prod_{j=1}^{n}\cos(x_j-0.4) \tag{12}$$
$$n \geq 1,\ \ -1 \leq x_j \leq 1,\ \ j = 1,\ldots,n$$

- f_{11} - multimodal, *global minimum* value (theor.) $f_{11}^* \approx -n(n-1)^{2/n-2}$ at $\mathbf{x}^* \approx \left((n-1)^{1/n-2}, (n-1)^{1/n-2}, \ldots, (n-1)^{1/n-2}\right)$, as an example for $n = 10$, $f_{11}^* = -17.323844081$ at $\mathbf{x}^* = (1.316326878, 1.316327304, 1.316327411, 1.316326966, 1.316327494, 1.316327450, 1.316327500, 1.316327146, 1.316327265, 1.316327356)$. The function has another local minimum with a larger attraction basin located in $(0, 0, \ldots, 0)$, which usually is trapping the optimization methods:

$$f_{11}(\mathbf{x}) = \sum_{j=1}^{n}\left(\sum_{j_1=1, j_1\neq j}^{n} x_{j_1} - \prod_{j_1=1, j_1\neq j+1}^{n} x_{j_1}\right)^2 - \sum_{j=1}^{n} x_j^2 \tag{13}$$
$$n \geq 3,\ \ x_{n+1} = x_1,\ \ 0 \leq x_j \leq 2,\ \ j = 1,\ldots,n$$

- f_{12} - multimodal, *global minimum* value (theoretical) $f_{12}^* \approx -2.78n$ at $\mathbf{x}^* \approx (2.78, 2.78, \ldots, 2.78)$, as an example for $n = 10$, $f_{12}^* = -27.771045004$ at $\mathbf{x}^* = (2.777105469, 2.777105469, \ldots, 2.777105469)$. The function has another local minimum with a larger attraction basin located at $\mathbf{x}^* \approx (1.56, 1.56, \ldots, 1.56)$, which usually is trapping the optimization methods:

$$f_{12}(\mathbf{x}) = \sum_{j=1}^{n}\left(\log_{x_{j-1}} x_j + 2^{x_j x_{j+1}} - 7^{x_{j+1}\sin(x_{j-1}x_j)} - \cos x_j^2\right)^2 - \sum_{j=1}^{n} x_j \tag{14}$$
$$n \geq 3,\ \ x_{n+1} = x_1,\ \ x_0 = x_{n-1},\ \ 1.001 \leq x_j \leq \pi,\ \ j = 1,\ldots,n$$

- f_{13} - multimodal, *global minimum* value (theoretical) $f_{13}^* \approx -n$ at $\mathbf{x}^* \approx (-1, -1, \ldots, -1)$, as an example for $n = 10$, $f_{13}^* = -10.065145954$ at $\mathbf{x}^* = (-1.012645301, -1.012645301, \ldots, -1.012645301)$. The function has many local minima which have the potential of trapping the optimization methods, notably the one located in origin with $f_{13}(\mathbf{0}) = 0$:

$$f_{13}(\mathbf{x}) = \sum_{j=1}^{n}\left(x_j^4 x_{j-1} - x_j^3 x_{j+1} + x_{j-1}x_{j+1} - x_{j+1}\right)^2 + \sum_{j=1}^{n} x_j \tag{15}$$
$$n \geq 3,\ x_{n+1} = x_1,\ \ x_0 = x_{n-1},\ \ -2 \leq x_j \leq 2,\ \ j = 1,\ldots,n$$

4. Constrained optimization test functions

In this section is described a new set of 29 constrained optimization test functions.

- f_1 - multimodal, *global minimum* value (theoretical) $f_1^* = \sqrt{n+1}$ at $\mathbf{x}^* = \left(\sqrt{n+1}, 0, \ldots, 0\right)$ and cyclic permutations:

$$
\begin{gathered}
f_1(\mathbf{x}) = \sum_{j=1}^{n} x_j, \\
g(\mathbf{x}) = (n+1) - \sum_{j=1}^{n} x_j^2 - \prod_{j=1}^{n} x_j \leq 0, \\
n \geq 3,\ 0 \leq x_j \leq \sqrt{n+1} + 1,\ \ j \leq 1, \ldots, n
\end{gathered}
\tag{16}
$$

- f_2 - unimodal, *global maximum* value (theoretical) $f_2^* = n$ at $\mathbf{x}^* = (1, 1, \ldots, 1)$:

$$
\begin{gathered}
f_2(\mathbf{x}) = \sum_{j=1}^{n} x_j, \\
g(\mathbf{x}) = -(n+1) + \sum_{j=1}^{n} x_j^2 + \prod_{j=1}^{n} x_j \leq 0, \\
n \geq 3,\ \ 0 \leq x_j \leq 2,\ \ j = 1, \ldots, n
\end{gathered}
\tag{17}
$$

- f_3 - unimodal, *global maximum* value (theoretical) $f_3^* = n$ at $\mathbf{x}^* = (1, 1, \ldots, 1)$:

$$
\begin{gathered}
f_3(\mathbf{x}) = \sum_{j=1}^{n} x_j^2 x_{j+1}, \\
g(\mathbf{x}) = -n + \sum_{j=1}^{n} x_j^3 \leq 0, \\
n \geq 3,\ x_{n+1} = x_1,\ \ 0 \leq x_j \leq 2,\ \ j = 1, \ldots, n
\end{gathered}
\tag{18}
$$

- f_4 - unimodal, *global minimum* value (theoretical) $f_4^* = 0$ at $\mathbf{x}^* = (1, 1, \ldots, 1)$:

$$\begin{aligned} f_4(\mathbf{x}) &= \sum_{j=1}^{n} \frac{x_j}{x_{j+1}} - \sum_{j=1}^{n} x_j, \\ g(\mathbf{x}) &= -(n+1) + \sum_{j=1}^{n} x_j x_{j+1} + \prod_{j=1}^{n} x_j \leq 0, \\ n \geq 3,\ & x_{n+1} = x_1,\ \ 0.001\ \leq x_j \leq 2,\ \ j = 1, \ldots, n \end{aligned} \tag{19}$$

- f_5 - multimodal, *global maximum* value (theoretical) $f_5^* = n - 1$ at $\mathbf{x}^* = (n-1, 1, \ldots, 1)$ and cyclic permutations:

$$\begin{aligned} f_5(\mathbf{x}) &= \prod_{j=1}^{n} x_j, \\ g_1(\mathbf{x}) &= -2(n-1) + \sum_{j=1}^{n} x_j \leq 0, \\ g_2(\mathbf{x}) &= n(n-1) - \sum_{j=1}^{n} x_j^2 \leq 0, \\ n \geq 3,\ & ,0 \leq x_j \leq n,\ \ j = 1, \ldots, n \end{aligned} \tag{20}$$

- f_6 - multimodal, *global minimum* value (theoretical) $f_6^* = 2^{n-1}$ at $\mathbf{x}^* = (1, 2, \ldots, 2)$ and cyclic permutations:

$$\begin{aligned} f_6(\mathbf{x}) &= \prod_{j=1}^{n} x_j, \\ h_1(\mathbf{x}) &= -(2n-1) + \sum_{j=1}^{n} x_j = 0, \\ h_2(\mathbf{x}) &= -(4n-3) + \sum_{j=1}^{n} x_j^2 = 0, \\ n \geq 3,\ & 0 \leq x_j \leq 3,\ \ j = 1, \ldots, n \end{aligned} \tag{21}$$

- f_7 - unimodal, *global minimum* value (theoretical) $f_7^* = 1$ at $\mathbf{x}^* = (1, 1, \ldots, 1)$:

$$
\begin{aligned}
f_7(\mathbf{x}) &= \sum_{j=1}^{n} \frac{1}{(1-x_j)+\sum\limits_{j_1=1}^{n} x_{j_1}}, \\
g(\mathbf{x}) &= -1+\sum_{j=1}^{n} \frac{x_j}{(1-x_j)+\sum\limits_{j_1=1}^{n} x_{j_1}} \leq 0, \\
& n \geq 3,\ \ 0 \leq x_j \leq 2,\ \ j=1,\dots,n
\end{aligned} \tag{22}
$$

- f_8 - unimodal, *global minimum* value (theoretical) $f_8^* = n - \frac{2n}{n+1}$ at $\mathbf{x}^* = (1,1,\dots,1)$:

$$
\begin{aligned}
f_8(\mathbf{x}) &= \sum_{j=1}^{n} \frac{-x_j+\sum\limits_{j_1=1}^{n} x_{j_1}}{x_j+\sum\limits_{j_1=1}^{n} x_{j_1}}, \\
h(\mathbf{x}) &= -n+\sum_{j=1}^{n} x_j = 0, \\
& n \geq 3,\ 0.001 \leq x_j \leq 2,\ \ j=1,\dots,n
\end{aligned} \tag{23}
$$

- f_9 - unimodal, *global maximum* value (theoretical) $f_9^* = \frac{n}{2}$ at $\mathbf{x}^* = (1,1,2,\dots,2^{n-2})$:

$$
\begin{aligned}
f_9(\mathbf{x}) &= \sum_{j=1}^{n-1} \frac{x_j}{x_{j+1}}, \\
h(\mathbf{x}) &= -2^{n-1}+\sum_{j=1}^{n} x_j = 0, \\
g_k(\mathbf{x}) &= -x_{k+1}+\sum_{j=1}^{k} x_j \leq 0,\ \ k=1,\dots,n-1, \\
& n \geq 3,\ 0.001 \leq x_j \leq 2^{n-1},\ \ j=1,\dots,n
\end{aligned} \tag{24}
$$

- f_{10} - unimodal, *global minimum* value (theoretical) $f_{10}^* = n$ at $\mathbf{x}^* = (1,1,\dots,1)$:

$$
\begin{aligned}
f_{10}(\mathbf{x}) &= \sum_{j=1}^{n} \frac{x_j^3 + x_j^2}{1 + \prod\limits_{j_1=1, j_1 \neq j}^{n} x_{j_1}}, \\
h(\mathbf{x}) &= -1 + \prod_{j=1}^{n} x_j = 0, \\
n \geq 3,\ & 0 \leq x_j \leq 2,\ \ j = 1, \ldots, n
\end{aligned} \tag{25}
$$

- f_{11} - unimodal, *global maximum* value (theoretical) $f_{11}^* = \frac{n}{2}$ at $\mathbf{x}^* = (1, 1, \ldots, 1)$:

$$
\begin{aligned}
f_{11}(\mathbf{x}) &= \sum_{j=1}^{n} \frac{x_j}{(1 + x_j^3 x_{j+1})}, \\
g(\mathbf{x}) &= 1 - \prod_{j=1}^{n} x_j \leq 0, \\
n \geq 3,\ & x_{n+1} = x_1,\ \ 0 \leq x_j \leq 2,\ \ j = 1, \ldots, n
\end{aligned} \tag{26}
$$

- f_{12} - unimodal, *global maximum* value (theoretical) $f_{12}^* = \frac{1}{2}$ at $\mathbf{x}^* = (1, 1, \ldots, 1)$:

$$
\begin{aligned}
f_{12}(\mathbf{x}) &= \sum_{j=1}^{n} \frac{1}{(x_j + 2n - 1)}, \\
g(\mathbf{x}) &= 1 - \prod_{j=1}^{n} x_j \leq 0, \\
n \geq 3,\ & 0 \leq x_j \leq 2,\ \ j = 1, \ldots, n
\end{aligned} \tag{27}
$$

- f_{13} (Generalized Nesbitt Inequality, see [13]) - unimodal, *global minimum* value (theoretical) $f_{13}^* = \frac{n}{n-1}$ at $\mathbf{x}^* = (1, 1, \ldots, 1)$:

$$
\begin{aligned}
f_{13}(\mathbf{x}) &= \sum_{j=1}^{n} \frac{x_j}{\left(-x_j + \sum\limits_{j_1=1}^{n} x_{j_1}\right)}, \\
h(\mathbf{x}) &= -n + \sum_{j=1}^{n} x_j = 0, \\
n \geq 3,\ & 0.001 \leq x_j \leq 2,\ \ j = 1, \ldots, n
\end{aligned} \tag{28}
$$

- f_{14} - unimodal, *global minimum* value (theoretical) $f_{14}^* = 0$ at $\mathbf{x}^* = (1, 1, \dots, 1)$:

$$
\begin{aligned}
f_{14}(\mathbf{x}) &= (n-1)\sum_{j=1}^{n} x_j - 2\sum_{1\le j_1<j_2\le n}^{n} x_{j_1}x_{j_2},\\
g(\mathbf{x}) &= 1 - \sum_{j=1}^{n} \frac{1}{\left(1 + \sum\limits_{j_1=1, j_1\neq j}^{n} x_{j_1}\right)} \le 0,\\
n &\ge 3,\ \ 0 \le x_j \le 2,\ \ j = 1, \dots, n
\end{aligned}
\tag{29}
$$

- f_{15} - unimodal, *global minimum* value (theoretical) $f_{15}^* = 1$ at $\mathbf{x}^* = (1, 1, \dots, 1)$:

$$
\begin{aligned}
f_{15}(\mathbf{x}) &= \frac{1}{\left(\sum\limits_{j=1}^{n} \frac{1}{x_j^2 + n}\right)} - \frac{1}{\left(\sum\limits_{j=1}^{n} \frac{1}{x_j}\right)},\\
g(\mathbf{x}) &= n - \sum_{j=1}^{n} x_j \le 0,\\
n &\ge 2,\ 0.001 \le x_j \le 2,\ \ j = 1, \dots, n
\end{aligned}
\tag{30}
$$

- f_{16} - unimodal, *global maximum* value (theoretical) $f_{16}^* = 1$ at $\mathbf{x}^* = (1, 1, \dots, 1)$:

$$
\begin{aligned}
f_{16}(\mathbf{x}) &= \sum_{j=1}^{2n+1} \frac{x_j}{n(x_j^2 + 1) + 1},\\
g(\mathbf{x}) &= -(2n+1) + \sum_{j=1}^{2n+1} x_j \le 0,\\
n &\ge 1,\ 0 \le x_j \le 2,\ \ j = 1, \dots, 2n+1
\end{aligned}
\tag{31}
$$

- f_{17} - unimodal, *global minimum* value (theoretical) $f_{17}^* = 2n + 1$ at $\mathbf{x}^* = (0, 0, \dots, 0)$:

$$
\begin{aligned}
f_{17}(\mathbf{x}) &= \sum_{j=1}^{2n+1} \log_2 \left(1 + 3^{x_j + x_{j+1}}\right), \\
h(\mathbf{x}) &= \sum_{j=1}^{2n+1} x_j = 0, \\
n \geq 1,\ & x_{2n+2} = x_1,\ \ -2 \leq x_j \leq 2,\ \ j = 1, \ldots, 2n+1
\end{aligned} \tag{32}
$$

- f_{18} - unimodal, *global minimum* value (theoretical) $f_{18}^* = 16$ at $\mathbf{x}^* = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$:

$$
\begin{aligned}
f_{18}(\mathbf{x}) &= \sum_{j=1}^{n} \frac{(x_j + x_{j+1})^4}{x_j x_{j+1}}, \\
g(\mathbf{x}) &= 1 - \sum_{j=1}^{n} x_j^2 \leq 0, \\
n \geq 3,\ & x_{n+1} = x_1,\ \ 10^{-6} \leq x_j \leq 1,\ \ j = 1, \ldots, n
\end{aligned} \tag{33}
$$

- f_{19} - multimodal, *global minimum* value (theoretical) $f_{19}^* = 1.8$ at $\mathbf{x}^* = (0.5, 0.5, 0, \ldots, 0)$ and cyclic permutations:

$$
\begin{aligned}
f_{19}(\mathbf{x}) &= \sum_{j=1}^{n} \frac{x_j + x_{j+1}}{1 + x_j x_{j+1}}, \\
g(\mathbf{x}) &= 1 - \sum_{j=1}^{n} x_j \leq 0, \\
n \geq 3,\ & x_{n+1} = x_1,\ \ 0 \leq x_j \leq 1,\ \ j = 1, \ldots, n
\end{aligned} \tag{34}
$$

- f_{20} - unimodal, *global maximum* value (theoretical) $f_{20}^* = \frac{2n\sqrt{n}}{n+1}$ at $\mathbf{x}^* = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$:

$$
\begin{aligned}
f_{20}(\mathbf{x}) &= \sum_{j=1}^{n} \frac{x_j + x_{j+1}}{1 + x_j x_{j+1}}, \\
g(\mathbf{x}) &= -1 + \sum_{j=1}^{n} x_j^2 \leq 0, \\
n \geq 3,\ & x_{n+1} = x_1,\ \ 0 \leq x_j \leq 1,\ \ j = 1, \ldots, n
\end{aligned} \tag{35}
$$

- f_{21} - unimodal, *global maximum* value (theoretical) $f_{21}^* = 2^n$ at $\mathbf{x}^* = (1, 1, \ldots, 1)$:

$$
\begin{aligned}
f_{21}(\mathbf{x}) &= 2\prod_{j=1}^{n}(x_j^2+1) - \prod_{j=1}^{n}(x_j^3+1), \\
g(\mathbf{x}) &= -1 + \prod_{j=1}^{n} x_j \le 0, \\
n \ge 3,\ & x_{n+1} = x_1,\ \ 0 \le x_j \le 2,\ \ j = 1, \dots, n
\end{aligned}
\tag{36}
$$

- f_{22} - unimodal, *global minimum* value (open problem) for $n = 10$, $f_{22}^* = 0.885506173$ at $\mathbf{x}^* = (1.808714501, 0.611851955, 0.711171539, 0.811288889, 0.903346486, 0.984843787, 1.059007430, 1.125675681, 1.187060342, 1.243895193)$:

$$
\begin{aligned}
f_{22}(\mathbf{x}) &= \sum_{j=1}^{n} \frac{x_j^j}{\prod_{j_1=1}^{j}(x_{j_1}+1)}, \\
g(\mathbf{x}) &= 1 - \prod_{j=1}^{n} x_j \le 0, \\
n \ge 2,\ & 0 \le x_j \le 3,\ \ j = 1, \dots, n
\end{aligned}
\tag{37}
$$

- f_{23} - unimodal, *global maximum* value (theoretical) $f_{23}^* = 1$ at $\mathbf{x}^* = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$:

$$
\begin{aligned}
f_{23}(\mathbf{x}) &= \frac{\displaystyle\sum_{j=1}^{n} \frac{1}{1-x_j^2}}{\displaystyle\sum_{j=1}^{n} \frac{1}{1-x_j x_{j+1}}}, \\
h(\mathbf{x}) &= -1 + \sum_{j=1}^{n} x_j = 0, \\
n \ge 3,\ & x_{n+1} = x_1,\ \ 0 \le x_j \le 0.999,\ \ j = 1, \dots, n
\end{aligned}
\tag{38}
$$

- f_{24} - unimodal, *global minimum* value (theoretical) $f_{24}^* = 1$ at $\mathbf{x}^* = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$:

$$
\begin{aligned}
f_{24}(\mathbf{x}) &= \sum_{j=1}^{n} \frac{\log_{x_j}^2 x_{j+1}}{n x_{j-1} + n - 1}, \\
g(\mathbf{x}) &= -1 + \sum_{j=1}^{n} x_j \le 0, \\
n \ge 3,\ & x_{n+1} = x_1,\ \ x_0 = x_n,\ \ 10^{-6} \le x_j \le 1,\ \ j = 1, \ldots, n
\end{aligned} \tag{39}
$$

- f_{25} - multimodal, *global minimum* value (open problem), for $n = 10$, $f_{25}^* = 1.874972874$ at $\mathbf{x}^* = (0.413805624, 4.548875314, 4.546871049, 0.411744067, 0.015928136, 0.011641123, 0.0117279236, 0.011646751, 0.011759360, 0.016000783)$ and cyclic permutations:

$$
\begin{aligned}
f_{25}(\mathbf{x}) &= \sum_{j=1}^{n} \frac{x_j^2}{1 + x_j(x_{j-1} + x_{j+1})}, \\
g(\mathbf{x}) &= n - \sum_{j=1}^{n} x_j \le 0, \\
n \ge 3,\ & x_{n+1} = x_1,\ \ x_0 = x_n,\ \ 0 \le x_j \le n,\ \ j = 1, \ldots, n
\end{aligned} \tag{40}
$$

- f_{26} - unimodal, *global minimum* value (theoretical) $f_{26}^* = n$ at $\mathbf{x}^* = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$:

$$
\begin{aligned}
f_{26}(\mathbf{x}) &= e^{-\frac{1}{(n^2-n)^{1/2}}} \sum_{j=1}^{n} e^{\frac{x_j}{\left(-x_j + \sum_{j_1=1}^{n} x_{j_1}\right)^{1/2}}}, \\
g(\mathbf{x}) &= 1 - \sum_{j=1}^{n} x_j \le 0, \\
n \ge 2,\ & 10^{-6} \le x_j \le 1,\ \ j = 1, \ldots, n
\end{aligned} \tag{41}
$$

- f_{27} - multimodal, *global minimum* value (theoretical) $f_{27}^* = n$ with many solutions, as an example for $n = 9$, $\mathbf{x}^* = (2.062398515 \times 10^{-5}, 7.099654204 \times 10^{-5}, 1.475907850, 5.591264529, 4.733189813, 4.490091124, 1.610322128, 1.246368226, 3.477101693)$:

$$
\begin{aligned}
f_{27}(\mathbf{x}) &= \sum_{j=1}^{n} [x_{n+1}^2 + x_{n+2}^2 - 2(x_{n+1}\cos x_j + x_{n+2}\sin x_j) + 1]^{1/2}, \\
h_1(\mathbf{x}) &= \sum_{j=1}^{n} \cos x_j = 0, \\
h_2(\mathbf{x}) &= \sum_{j=1}^{n} \sin x_j = 0, \\
n \geq 2,\ \ &-2 \leq x_{n+1}, x_{n+2} \leq 2,\ \ 0 \leq x_j < 2\pi,\ \ j = 1,\ldots,n
\end{aligned}
\tag{42}
$$

- f_{28} - unimodal, *global minimum* value (theoretical) $f_{28}^* = \frac{n}{2}$ at $\mathbf{x}^* = (1, 1, \ldots, 1)$:

$$
\begin{aligned}
f_{28}(\mathbf{x}) &= \sum_{j=1}^{n} \frac{x_{j-1}^{x_j}}{x_j^2(x_{j+1}+1)}, \\
g(\mathbf{x}) &= -1 + \prod_{j=1}^{n} x_j \leq 0, \\
n \geq 3,\ x_{n+1} &= x_1,\ \ x_0 = x_{n-1},\ \ 0.001 \leq x_j < 2,\ \ j = 1,\ldots,n
\end{aligned}
\tag{43}
$$

- f_{29} - unimodal, *global maximum* value (theoretical) $f_{29}^* = \frac{1}{2}$ at $\mathbf{x}^* = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$:

$$
\begin{aligned}
f_{29}(\mathbf{x}) &= \sum_{j=1}^{n} \frac{x_j}{1 + x_{j-1} + \log_{x_j} x_{j+1} - x_{j+1}}, \\
g(\mathbf{x}) &= -1 + \sum_{j=1}^{n} x_j \leq 0, \\
3 \leq n \leq 10000,\ \ x_{n+1} &= x_1,\ \ x_0 = x_{n-1},\ \ 1.0e-4 \leq x_j < 0.999, \\
j &= 1,\ldots,n
\end{aligned}
\tag{44}
$$

5. **Conclusions**

The paper proposed two new sets of optimization test functions: a set of 13 continuous unconstrained (or box constrained) test functions, and a set of 29 continuous constrained test functions. All the proposed functions are multidimensional and scalable to the dimension of the search space, n, which is a useful property when the response in performance (efficiency and success rate) of an optimization method is investigated with the increase of the dimension of the search space. It is the hope of the author that the proposed new sets of optimization test functions will represent a valuable addition to the known collections of optimization test functions and will prove useful in investigating the properties of new or existing optimization methods.

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