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EXISTENCE AND MULTIPLICITY RESULTS FOR A MIXED STURM-LIOUVILLE TYPE BOUNDARY VALUE PROBLEM

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In this paper, existence results of positive solutions for a mixed boundary value problem with Sturm-Liouville equation are established. Multiplicity results are also pointed out. The approach is based on variational methods.

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1. Introduction

In 1836-1837, the French mathematicians Jacques Charles François Sturm (1803-1855) and Joseph Liouville (1809-1882) published several papers that initiated a new subtopic of mathematical analysis: the Sturm-Liouville theory. Sturm and Liouville were concerned with the general *linear*, homogeneous second-order differential equation of the form

$$(p(x)u')' + q(x)u = \lambda w(x)u \quad \text{if } x \in [a, b], \tag{1}$$

where the potentials are given functions. Under various boundary conditions, Sturm and Liouville established that solutions of problem (1) can exist only for particular values of the real parameter λ, which is called an *eigenvalue*. Relevant examples of linear Sturm-Liouville problems are the Bessel equation and the Legendre equation.

The classical Sturm-Liouville theory does not depend upon the calculus of variations, but stems from the theory of ordinary linear or nonlinear differential equations. Linear Sturm-Liouville equations can be also studied in the context of functional analysis by means of self-adjoint operators or integral operators with a continuous symmetric kernel (the Green's function of the problem). Certain applications involving linear partial differential equations can be treated with the help of the Sturm-Liouville theory, for instance the normal modes of vibration of a thin membrane. We also refer to [20], where it is studied a perturbed *nonlinear* Sturm-Liouville problem with superlinear convex nonlinearity. In the recent paper [16], the authors study a class of discrete anisotropic Sturm-Liouville problems.

In the present paper, we are concerned with a class of *nonlinear* Sturm-Liouville problems and we establish some qualitative properties of the eigenvalues

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by using variational principles. A feature of our work is the presence of a Lipschitz continuous function, which creates further technical constraints.

Consider the following Sturm-Liouville problem with mixed conditions on a bounded interval $[a,b]$ in $\mathbb{R}$:

$$\begin{cases} -(pu')' + qu = \lambda f(x,u) + g(u) & \text{in } (a,b), \\ u(a) = u'(b) = 0. \end{cases} \tag{2}$$

We assume that $p, q \in L^\infty([a,b])$ are such that

$$p_0 := \operatorname{ess\,inf}_{x\in[a,b]} p(x) > 0 \qquad \text{and} \qquad q_0 := \operatorname{ess\,inf}_{x\in[a,b]} q(x) \geq 0,$$

λ is a positive parameter, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function and $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|g(t_1) - g(t_2)| \leq L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$, and $g(0) = 0$.

Mixed boundary value problems, as well as Dirichlet or Neumann problems, have been widely studied because of their applications to various fields of applied sciences, as mechanical engineering, control systems, computer science, economics, artificial or biological neural networks and many others.

In this connection, several existence and multiplicity results for solutions to second order ordinary differential nonlinear equations, with mixed conditions at the ends, have been investigated making use of fixed point theorems, lower and upper solutions and variational methods. We refer the reader to the papers [1, 2, 9, 22, 25] and references therein.

In the present paper, first we obtain the existence of at least one solution for problem (2). It is worth noticing that, usually, to obtain the existence of one solution, asymptotic conditions both at zero and at infinity on the nonlinear term are requested (see, for instance, [26, Theorem 1]), while, here, it is required only a unique algebraic condition (see (A_6) in Theorem 3.3). As a consequence, by combining with the classical Ambrosetti-Rabinowitz condition, the existence of two solutions is obtained (see Theorem 4.1). Subsequently, an existence result of three nonnegative solutions is obtained combining two algebraic conditions which guarantee the existence of two local minima for the Euler-Lagrange functional and applying the mountain pass theorem as given by Pucci and Serrin (see [17]) to ensure the existence of the third critical point (see Theorem 4.3).

Our approach is variational and the main tool is a local minimum theorem established in [3], of whose two its consequences are here applied (see Theorems 2.1 and 2.2). These Theorems have been successfully employed in several works in order to obtain existence results for different kinds of problems (see, for instance, [4, 5, 6, 7, 8, 11, 12]).

As an example, we state here the following special case of Theorem 4.3.

Theorem 1.1. *Let $h : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that*

$$\lim_{t\to 0^+} \frac{h(t)}{t} = +\infty, \qquad \lim_{t\to +\infty} \frac{h(t)}{t} = 0,$$

and

$$48\int_0^1 h(x)dx < \int_0^2 h(x)dx.$$

Then, for each

$$\lambda \in \left]\frac{32}{\int_0^2 h(x)dx}, \frac{2}{3\int_0^1 h(x)dx}\right[,$$

the problem

$$\begin{cases} -3u'' = \lambda h(u) + u & \text{in } (0,1), \\ u(0) = u'(1) = 0 \end{cases}$$

admits at least three nonnegative classical solutions.

2. **Preliminaries**

Our main tools are Theorems 2.1 and 2.2, consequences of the existence result of a local minimum theorem [3, Theorem 3.1] which is inspired by the Ricceri Variational Principle [21].

For a given non-empty set X, and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following functions

$$\beta(r_1, r_2) := \inf_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$

$$\rho_2(r_1, r_2) := \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u)}{\Phi(v) - r_1},$$

for all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, and

$$\rho(r) := \sup_{v \in \Phi^{-1}(]r, +\infty[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{\Phi(v) - r},$$

for all $r \in \mathbb{R}$.

Theorem 2.1 ([3, Theorem 5.1]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Put $I_\lambda := \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho_2(r_1, r_2). \tag{3}$$

Then, for each $\lambda \in \left]\frac{1}{\rho_2(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)}\right[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.2 ([3, Theorem 5.3]). *Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X; $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that*

$$\rho(r) > 0, \tag{4}$$

and for each $\lambda > \frac{1}{\rho(r)}$ *the function* $I_\lambda := \Phi - \lambda\Psi$ *is coercive. Then, for each* $\lambda > \frac{1}{\rho(r)}$ *there is* $u_{0,\lambda} \in \Phi^{-1}(]r, +\infty[)$ *such that* $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]r, +\infty[)$ *and* $I'_\lambda(u_{0,\lambda}) = 0$.

Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|g(t_1) - g(t_2)| \leq L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$, and $g(0) = 0$.

We recall that $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if

(a) $x \mapsto f(x, \xi)$ is measurable for every $\xi \in \mathbb{R}$;
(b) $\xi \mapsto f(x, \xi)$ is continuous for almost every $x \in [a, b]$;
(c) for every $\rho > 0$ there is a function $l_\rho \in L^1([a, b])$ such that

$$\sup_{|\xi| \leq \rho} |f(x, \xi)| \leq l_\rho(x)$$

for almost every $x \in [a, b]$.

Corresponding to f and g we introduce the functions $F : [a, b] \times \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$, respectively, as follows

$$F(x, t) := \int_0^t f(x, \xi) d\xi$$

and

$$G(t) := \int_0^t g(\xi) d\xi$$

for all $x \in [a, b]$ and $t \in \mathbb{R}$.

We define a convenient function space (for details see Brezis [10]):

$$X := \Big\{ u \in W^{1,2}([a, b]) : u(a) = 0 \Big\}.$$

Then, X is a subspace of Sobolev space $W^{1,2}([a, b])$. The usual norm in X is defined by

$$\|u\|_X := \left(\int_a^b (u(x))^2 dx + \int_a^b (u'(x))^2 dx \right)^{1/2}.$$

For every $u, v \in X$, we define

$$(u, v) := \int_a^b p(x) u'(x) v'(x) dx + \int_a^b q(x) u(x) v(x) dx. \tag{5}$$

Clearly, (5) defines an inner product on X whose corresponding norm is

$$\|u\| := \left(\int_a^b p(x)(u'(x))^2 dx + \int_a^b q(x)(u(x))^2 dx \right)^{1/2}.$$

Due to the positivity of p and non-negativity of q, it is easily seen that the norm $\|\cdot\|$ on X is equivalent to $\|\cdot\|_X$. In the following, we will use $\|\cdot\|$ instead of $\|\cdot\|_X$ on X. Note that X is a reflexive real Banach space.

Definition 2.1. We say that a function $u \in X$ is a *weak solution* of problem (2) if

$$(u, v) = \lambda \int_a^b f(x, u(x))v(x)dx + \int_a^b g(u(x))v(x)dx \tag{6}$$

for all $v \in X$.

Remark 2.1. Definition 2.1 is quite natural, as it agrees with an intuitive notion of "*classical solution*" provided the involved functions are sufficiently smooth. Indeed, let f be a continuous function, $p \in C^1([a, b])$, $q \in C^0([a, b])$, and $u \in X \cap C^2([a, b])$ be a weak solution of (2). By (6) and integration by parts, we have for all $v \in X$

$$\int_a^b \left[-(p(x)u'(x))'v(x) + (p(x)u'(x)v(x))' + q(x)u(x)v(x)\right] dx$$

$$= \lambda \int_a^b f(x, u(x))v(x)dx + \int_a^b g(u(x))v(x)dx.$$

Taking an arbitrary $v \in W_0^{1,2}([a, b])$ we see that

$$-(pu')' + qu = \lambda f(x, u) + g(u) \qquad \text{in } (a, b),$$

which in turn implies for any $v \in X$

$$0 = p(a)u'(a)v(a) = p(b)u'(b)v(b),$$

hence $u'(b) = 0$. Thus, u solves (2) in a pointwise sense.

It is well known that $(X, \|\cdot\|)$ is compactly embedded in $(C^0([a, b]), \|\cdot\|_\infty)$ and

$$\|u\|_\infty \le \sqrt{\frac{b-a}{p_0}}\|u\| \tag{7}$$

for all $u \in X$ (see, e.g., [23]).

Also, we use the following notations:

$$\|p\|_\infty := \operatorname{ess\,sup}_{x\in[a,b]} p(x), \qquad \|q\|_\infty := \operatorname{ess\,sup}_{x\in[a,b]} q(x).$$

For other basic notations and definitions, we refer the reader to [14, 15, 24].

3. Main results

In this section we present our main results. To be precise, we establish an existence result of at least one solution, Theorem 3.1, which is based on Theorem 2.1, and we point out some consequences, Theorems 3.2, 3.3 and 3.4. Finally, we present another existence result of at least one solution, Theorem 3.5, which is based in turn on Theorem 2.2.

Suppose that the Lipschitz constant $L > 0$ of the function g satisfies the condition $L(b-a)^2 < p_0$. Now, put

$$\begin{aligned} k_1 &:= \frac{3(b-a)}{6p_0 + 2(b-a)^2 q_0}, \\ k_2 &:= \frac{3(b-a)}{6\|p\|_\infty + 2(b-a)^2\|q\|_\infty}, \\ A &:= \frac{p_0 - L(b-a)^2}{2p_0}, \\ B &:= \frac{p_0 + L(b-a)^2}{2p_0}, \end{aligned}$$

and suppose that $\frac{B}{A} \le \frac{p_0}{b-a}$.

Given a nonnegative constant c_1 and two positive constants c_2 and d with $k_1 c_1^2 < d^2 < k_2 c_2^2$, put

$$a(c_2, k_2) := \frac{\int_a^b \max_{|t|\le c_2} F(x,t)dx - \int_{(a+b)/2}^b F(x,d)dx}{Bc_2^2 - Bd^2/k_2}$$

and

$$b(c_1, k_2) := \frac{\int_{(a+b)/2}^b F(x,d)dx - \int_a^b \max_{|t|\le c_1} F(x,t)dx}{Bd^2/k_2 - Ac_1^2}.$$

Theorem 3.1. *Assume that there exist a nonnegative constant c_1 and two positive constants c_2, d, with $k_1 c_1^2 < d^2 < k_2 c_2^2$, such that*

(A_1) $F(x,t) \ge 0$ *for all* $(x,t) \in [a, (a+b)/2] \times [0, d]$;

(A_2) $a(c_2, k_2) < b(c_1, k_2)$.

Then, for each $\lambda \in \left]\frac{1}{b(c_1,k_2)}, \frac{1}{a(c_2,k_2)}\right[$, *problem* (2) *admits at least one non-trivial weak solution* $\bar{u} \in X$, *such that*

$$\frac{A}{B}c_1^2 < \|\bar{u}\|^2 < \frac{B}{A}c_2^2.$$

Proof. Our aim is to apply Theorem 2.1 to our problem. To this end, for each $u \in X$, let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \frac{1}{2}\|u\|^2 - \int_a^b G(u(x))dx$$

and

$$\Psi(u) := \int_a^b F(x, u(x))dx,$$

and put

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u) \qquad \forall\ u \in X.$$

First, we show that Φ is a Gâteaux differentiable sequentially weakly lower semicontinuous functional on X. Indeed, put $\Phi(u) := \Upsilon(u) - \Theta(u)$, where

$$\Upsilon(u) := \frac{1}{2}\|u\|^2 = \frac{1}{2}\left(\int_a^b p(x)(u'(x))^2dx + \int_a^b q(x)(u(x))^2dx\right),$$

and

$$\Theta(u) := \int_a^b G(u(x))dx = \int_a^b \left(\int_0^{u(x)} g(\xi)d\xi \right) dx,$$

for every $u \in X$.

If $u_n \rightharpoonup u$ in X then compactness of embedding $X \hookrightarrow C^0([a,b])$ implies $u_n \to u$ in $C^0([a,b])$ i.e. $u_n \to u$ uniformly on $[a,b]$ (see Proposition 2.2.4 of [13]). Hence, for some constant $M > 0$ and any $n \in \mathbb{N}$ we have $\|u_n\|_\infty \leq M$, and so

$$|G(u_n(x)) - G(u(x))|dx \leq L \int_{u(x)}^{u_n(x)} |t|dt \leq \frac{L}{2}\left(|u_n(x)|^2 + |u(x)|^2\right) \leq \frac{L}{2}(M^2 + \|u\|_\infty^2)$$

for every $n \in \mathbb{N}$ and $x \in [a,b]$. Furthermore, $G(u_n(x)) \to G(u(x))$ at any $x \in [a,b]$ and therefore, the Lebesgue Convergence Theorem yields

$$\Theta(u_n) = \int_a^b G(u_n(x))dx \to \int_a^b G(u(x))dx = \Theta(u).$$

Therefore Θ is a sequentially weakly continuous functional on X. Since the norm $\|\cdot\|$ on X is a weakly sequentially lower semi-continuous functional in X (see Proposition 2.1.22(iii) in [13]), the functional Φ is sequentially weakly lower semicontinuous.

At this point, we have

$$\begin{aligned}
\Upsilon'(u)(v) &= \frac{1}{2}\frac{d}{d\theta}\left(\int_a^b \left[p(x)\left((u+\theta v)'(x)\right)^2 + q(x)\left((u+\theta v)(x)\right)^2 \right] dx \right)\Bigg|_{\theta=0} \\
&= \left(\int_a^b \left[p(x)\left((u'(x)v'(x) + \theta(v'(x))^2\right) + q(x)\left((u(x)v(x) + \theta v(x)\right)\right] dx \right)\Bigg|_{\theta=0} \\
&= \int_a^b p(x)u'(x)v'(x)dx + \int_a^b q(x)u(x)v(x)dx.
\end{aligned}$$

For proving Gâteaux differentiability of Θ, suppose $u, v \in X$. Then for $t \neq 0$, by the Mean Value Theorem

$$\begin{aligned}
\left| \frac{\Theta(u+tv) - \Theta(u)}{t} - \int_a^b g(u(x))v(x)dx \right| &\leq \int_a^b \left| \frac{G(u+tv) - G(u)}{t} - g(u(x))v(x) \right| dx \\
&= \int_a^b \left| g\big(u(x) + t\zeta(x)v(x)\big) - g(u(x)) \right| |v(x)|dx \\
&\leq L\|v\|_\infty^2 |t|(b-a),
\end{aligned}$$

where $0 < \zeta(x) < 1$ for every $x \in [a,b]$. Therefore, $\Theta : X \to \mathbb{R}$ is a Gâteaux differentiable at every $u \in X$ with derivative

$$\Theta'(u)(v) = \int_a^b g(u(x))v(x)dx$$

for every $v \in X$. Thus, we have that Φ is Gâteaux differentiable and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_a^b p(x)u'(x)v'(x)dx + \int_a^b q(x)u(x)v(x)dx - \int_a^b g(u(x))v(x)dx \tag{8}$$

for every $v \in X$. Furthermore, the differential $\Phi' : X \to X^*$ is a Lipschitzian operator. Indeed, for any $u, v \in X$, there holds

$$\begin{aligned}
\|\Phi'(u) - \Phi'(v)\|_{X^*} &= \sup_{\|w\|\le 1} |(\Phi'(u) - \Phi'(v), w)| \\
&\le \sup_{\|w\|\le 1} |(u - v, w)| + \sup_{\|w\|\le 1} \int_a^b |g(u(x)) - g(v(x))|\, |w(x)| dx \\
&\le \sup_{\|w\|\le 1} \|u - v\|\, \|w\| \\
&\quad + \sup_{\|w\|\le 1} \left(\int_a^b |g(u(x)) - g(v(x))|^2 \right)^{1/2} \left(\int_a^b |w(x)|^2 \right)^{1/2}.
\end{aligned}$$

Recalling that g is Lipschitz continuous and the embedding $X \hookrightarrow L^2([a,b])$ is compact, the claim is true. In particular, we derive that Φ is continuously differentiable. Since $|g(t)| \le L|t|$ for every $t \in \mathbb{R}$, the inequality (7) yields for any $u, v \in X$ the estimate

$$\begin{aligned}
(\Phi'(u) - \Phi'(v), u - v) &= (u - v, u - v) - \int_a^b \big(g(u(x)) - g(v(x))\big)\big(u(x) - v(x)\big) dx \\
&\ge \|u - v\|^2 - L \int_a^b \big(u(x) - v(x)\big)^2 dx \\
&\ge \frac{p_0 - L(b-a)^2}{p_0} \|u - v\|^2.
\end{aligned}$$

By the assumption $L(b-a)^2 < p_0$, it turns out that Φ' is a strongly monotone operator. So, by applying Minty-Browder theorem (Theorem 26.A of [24]), $\Phi' : X \to X^*$ admits a Lipschitz continuous inverse.

For proving the Gâteaux differentiability of Ψ, let $u, v \in X$ with $\|u\| < \sqrt{\frac{p_0}{b-a}} M$ and $\|v\| < \sqrt{\frac{p_0}{b-a}} M$, where $M > 0$ is a constant. Then, for $t \ne 0$ by the Mean Value Theorem

$$\begin{aligned}
\left| \frac{\Psi(u + tv) - \Psi(u)}{t} - \int_a^b f(x, u(x)) v(x) dx \right| &\le \int_a^b |f(x, u(x) + t\zeta(x)v(x)) - f(x, u(x))|\, |v(x)| dx \\
&\le \|v\|_\infty \int_a^b \big|f(x, u(x) + t\zeta(x)v(x)) - f(x, u(x))\big| dx
\end{aligned}$$

where $0 < \zeta(x) < 1$ for every $x \in [a,b]$ for which $F(x,t)$ is differentiable with respect to t. Since the assumption (b) on $f(x,\xi)$ implies

$$\lim_{t\to 0} f(x, u(x) + t\zeta(x)v(x)) = f(x, u(x)) \quad \text{for almost every } x \in [a,b]$$

and by (7) we have $\|v\|_\infty \le M$ and $\|u\|_\infty \le M$, then by the assumption (c) on $f(x,\xi)$ we have

$$|f(x, u(x) + t\zeta(x)v(x)) - f(x, u(x))| \le l_{2M}(x) + l_M(x)$$

for any $|t| < 1$. Therefore, the Lebesgue Convergence Theorem implies

$$\lim_{t\to 0} \frac{\Psi(u + tv) - \Psi(u)}{t} = \int_a^b f(x, u(x)) v(x) dx.$$

Since for every $u, v \in X$, some constant $M > 0$ can be found so that both of inequalities $\|u\| < \sqrt{\frac{p_0}{b-a}}M$ and $\|v\| < \sqrt{\frac{p_0}{b-a}}M$ hold, thus Ψ is Gâteaux differentiable at every $u \in X$, whose Gâteaux derivative is given by

$$\Psi'(u)(v) = \int_a^b f(x, u(x))v(x)dx \tag{9}$$

for every $v \in X$.

We deduce from (8), (9) and the above discussion that the weak solutions of (2) are exactly the critical points of I_λ.

Now, we show that $\Psi' : X \to X^*$ is compact. Indeed, if $u_n \rightharpoonup u$ in X then compactness of embedding $X \hookrightarrow C^0([a, b])$ implies $u_n \to u$ uniformly on $[a, b]$, and the assumption (b) on $f(x, \xi)$ yields $f(x, u_n(x)) \to f(x, u(x))$ for almost every $x \in [a, b]$. Also, for some constant $M > 0$ and any $n \in \mathbb{N}$ we have $\|u_n\|_\infty \le M$. By the assumption (c) on $f(x, \xi)$ we have $|f(x, u_n(x))| \le l_M(x)$ for any $n \in \mathbb{N}$ and for almost every $x \in [a, b]$. Therefore, the Lebesgue Convergence Theorem yields

$$\int_a^b f(x, u_n(x))dx \to \int_a^b f(x, u(x))dx,$$

and so, for every $w \in X$, we have

$$\begin{aligned}\left(\Psi'(u_n) - \Psi'(u)\right)(w) &= \int_a^b \left(f(x, u_n(x)) - f(x, u(x))\right) w(x)dx \\ &\le \sqrt{\frac{b-a}{p_0}}\|w\| \int_a^b \left(f(x, u_n(x)) - f(x, u(x))\right) dx.\end{aligned}$$

Thus,

$$\|\Psi'(u_n) - \Psi'(u)\|_{X^*} \le \sqrt{\frac{b-a}{p_0}} \int_a^b \left(f(x, u_n(x)) - f(x, u(x))\right) dx,$$

and so $\Psi'(u_n) \to \Psi'(u)$. Therefore, by Proposition 26.2 in [24], Ψ' is compact.

Since g is Lipschitz continuous and $g(0) = 0$, we have from (7) that

$$A\|u\|^2 \le \Phi(u) \le B\|u\|^2 \qquad \text{for all } u \in X. \tag{10}$$

Now, put

$$r_1 := Ac_1^2, \qquad r_2 := Bc_2^2$$

and

$$w(x) := \begin{cases} \frac{2d}{b-a}(x-a), & \text{if } x \in [a, (a+b)/2[, \\ d, & \text{if } x \in [(a+b)/2, b]. \end{cases}$$

It is easy to verify that $w \in X$ and, in particular, one has

$$\frac{d^2}{k_1} \le \|w\|^2 \le \frac{d^2}{k_2}.$$

So, from (10), we have

$$\frac{Ad^2}{k_1} \le \Phi(w) \le \frac{Bd^2}{k_2}.$$

From the condition $k_1c_1^2 < d^2 < k_2c_2^2$, we obtain $r_1 < \Phi(w) < r_2$. Since $\frac{B}{A} \leq \frac{p_0}{b-a}$, for all $u \in X$ such that $\Phi(u) < r_2$, taking (7) into account, one has $|u(x)| < c_2$ for all $x \in [a,b]$, from which it follows

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \int_a^b F(x,u(x))dx \\ &\leq \int_a^b \max_{|t|\leq c_2} F(x,t)dx.\end{aligned}$$

Arguing as before, since $A < B$, we obtain

$$\sup_{u\in\Phi^{-1}(]-\infty,r_1])} \Psi(u) \leq \int_a^b \max_{|t|\leq c_1} F(x,t)dx.$$

Since $0 \leq w(x) \leq d$ for each $x \in [a,b]$, the assumption (A_1) ensures that

$$\Psi(w) \geq \int_{(a+b)/2}^b F(x,d)dx.$$

Therefore, one has

$$\begin{aligned}\beta(r_1,r_2) &\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \\ &\leq \frac{\int_a^b \max_{|t|\leq c_2} F(x,t)dx - \int_{(a+b)/2}^b F(x,d)dx}{Bc_2^2 - Bd^2/k_2} = a(c_2,k_2).\end{aligned}$$

On the other hand, one has

$$\begin{aligned}\rho_2(r_1,r_2) &\geq \frac{\Psi(w) - \sup_{u\in\Phi^{-1}(]-\infty,r_1])} \Psi(u)}{\Phi(w) - r_1} \\ &\geq \frac{\int_{(a+b)/2}^b F(x,d)dx - \int_a^b \max_{|t|\leq c_1} F(x,t)dx}{Bd^2/k_2 - Ac_1^2} = b(c_1,k_2).\end{aligned}$$

Hence, from the assumption (A_2), one has $\beta(r_1,r_2) < \rho_2(r_1,r_2)$. Therefore, from Theorem 2.1, for each $\lambda \in \left]\frac{1}{b(c_1,d)}, \frac{1}{a(c_2,d)}\right[$, the functional I_λ admits at least one critical point $\bar{u}$ such that

$$r_1 < \Phi(\bar{u}) < r_2,$$

that is

$$\frac{A}{B}c_1^2 < \|\bar{u}\|^2 < \frac{B}{A}c_2^2,$$

and the conclusion is achieved. □

Now, we point out an immediate consequence of Theorem 3.1.

Theorem 3.2. *Assume that there exist two positive constants c, d, with $d^2 < k_2c^2$, such that the assumption (A_1) in Theorem 3.1 holds. Furthermore, suppose that*

(A_3) $\frac{\int_a^b \max_{|t|\leq c} F(x,t)dx}{c^2k_2} < \frac{\int_{(a+b)/2}^b F(x,d)dx}{d^2}$.

Then, for each

$$\lambda \in \left] \frac{Bd^2}{k_2 \int_{(a+b)/2}^{b} F(x,d)dx}, \frac{Bc^2}{\int_a^b \max_{|t|\le c} F(x,t)dx} \right[,$$

problem (2) *admits at least one non-trivial weak solution* $\bar{u} \in X$*, such that* $|\bar{u}(x)| < c$ *for all* $x \in [a,b]$*.*

Proof. The conclusion follows from Theorem 3.1, by taking $c_1 = 0$ and $c_2 = c$. Indeed, owing to the assumption (A_3), one has

$$\begin{aligned} a(c,k_2) &= \frac{\int_a^b \max_{|t|\le c} F(x,t)dx - \int_{(a+b)/2}^b F(x,d)dx}{Bc^2 - Bd^2/k_2} \\ &< \frac{\left(1 - \frac{d^2}{c^2k_2}\right)\int_a^b \max_{|t|\le c} F(x,t)dx}{B(c^2 - d^2/k_2)} \\ &= \frac{1}{Bc^2}\int_a^b \max_{|t|\le c} F(x,t)dx. \end{aligned}$$

On the other hand, one has

$$b(0,k_2) = \frac{\int_{(a+b)/2}^b F(x,d)dx}{Bd^2/k_2}.$$

Hence, taking the assumption (A_3) and (7) into account, Theorem 3.1 ensures the conclusion. □

Now, we point out a special situation of our main result when the nonlinear term has separable variables. To be precise, let $\alpha \in L^1([a,b])$ such that $\alpha(x) \ge 0$ almost every $x \in [a,b]$, $\alpha \not\equiv 0$, and let $h : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Consider the following mixed boundary value problem

$$\begin{cases} -(pu')' + qu = \lambda\alpha(x)h(u) + g(u) & \text{in } (a,b), \\ u(a) = u'(b) = 0. \end{cases} \tag{11}$$

Put $H(t) := \int_0^t h(\xi)d\xi$ for all $t \in \mathbb{R}$, and set $\|\alpha\|_1 := \int_a^b \alpha(x)dx$.

Theorem 3.3. *Assume that there exist two positive constants* c, d*, with* $d^2 < k_2c^2$*, such that*

(A_6) $\frac{H(c)}{c^2} < \left(\frac{k_2\int_{(a+b)/2}^b \alpha(x)dx}{\|\alpha\|_1}\right)\frac{H(d)}{d^2}$.

Then, for each

$$\lambda \in \left] \frac{B}{k_2\int_{(a+b)/2}^b \alpha(x)dx}\frac{d^2}{H(d)}, \frac{B}{\|\alpha\|_1}\frac{c^2}{H(c)} \right[,$$

problem (11) *admits at least one positive weak solution* $\bar{u} \in X$*, such that* $\bar{u}(x) < c$ *for all* $x \in [a,b]$*.*

Proof. Put $f(x,\xi) = \alpha(x)h(\xi)$ for all $(x,\xi) \in [a,b] \times \mathbb{R}$. Clearly, one has $F(x,t) = \alpha(x)H(t)$ for all $(x,t) \in [a,b] \times \mathbb{R}$. Therefore, taking into account that H is a non-decreasing function, Theorem 3.2 ensures the existence of a non-zero weak solution $\bar{u}$. We claim that it is nonnegative. In fact, arguing by contradiction and setting

$C := \{x \in [a,b] : \bar{u}(x) < 0\}$, one has $C \neq \varnothing$. Put $\bar{v} := \min\{\bar{u}, 0\}$, one has $\bar{v} \in X$. So, taking into account that $\bar{u}$ is a weak solution and by choosing $v = \bar{v}$, from our sign assumptions on the data, one has

$$\int_C p(x)|\bar{u}'(x)|^2 dx + \int_C q(x)|\bar{u}(x)|^2 dx - \int_C g(\bar{u}(x))\bar{u}(x)dx = \lambda \int_C \alpha(x)h(\bar{u}(x))\bar{u}(x)dx \leq 0.$$

On the other hand,

$$\frac{p_0 - L(m(C))^2}{p_0}\|\bar{u}\|^2_{W^{1,2}(C)} \leq \int_C p(x)|\bar{u}'(x)|^2 dx + \int_C q(x)|\bar{u}(x)|^2 dx - \int_C g(\bar{u}(x))\bar{u}(x)dx,$$

where $m(C)$ is the Lebesgue measure of the set C. So, $\|\bar{u}\|_{W^{1,2}(C)} = 0$ which is absurd. Hence, our claim is proved. Now, owing to the strong maximum principle (see, e.g., [18, Theorem 11.1]) the weak solution $\bar{u}$, being non-zero, is positive and the conclusion is achieved. □

We now give a special case of our main result as follows.

Theorem 3.4. *Assume that*

(A_7) $\lim_{t\to 0^+} \frac{h(t)}{t} = +\infty$,

and put $\lambda^\star = \frac{B}{\|\alpha\|_1} \sup_{c>0} \frac{c^2}{H(c)}$. *Then, for each* $\lambda \in\,]0, \lambda^\star[$, *problem* (11) *admits at least one positive weak solution.*

Proof. For fixed $\lambda \in\,]0, \lambda^\star[$, there exists $c > 0$ such that

$$\lambda < \frac{B}{\|\alpha\|_1}\frac{c^2}{H(c)}.$$

Moreover, the assumption (A_7) follows that $\lim_{t\to 0^+} \frac{t^2}{H(t)} = 0$. Therefore, we can choose positive constant d satisfying $d < \sqrt{k_2}c$ such that

$$\frac{k_2 \int_{(a+b)/2}^b \alpha(x)dx}{B}\frac{H(d)}{d^2} > \frac{1}{\lambda}.$$

Hence, by applying Theorem 3.3 we arrive at the desired conclusion. □

Remark 3.1. Taking into account (A_7), fix $\gamma > 0$ such that $h(t) > 0$ for all $t \in]0, \gamma[$. Then, put $\lambda_\gamma := \frac{B}{\|\alpha\|_1} \sup_{c\in]0,\gamma[} \frac{c^2}{H(c)}$. Clearly, $\lambda_\gamma \leq \lambda^\star$. Now, fixed $\lambda \in]0, \lambda_\gamma[$ and arguing as in the proof of Theorem 3.4, there are $c \in]0, \gamma[$ and $d < \sqrt{k_2}c$ such that

$$\frac{B}{k_2 \int_{(a+b)/2}^b \alpha(x)dx}\frac{d^2}{H(d)} < \lambda < \frac{B}{\|\alpha\|_1}\frac{c^2}{H(c)}.$$

Hence, Theorem 3.3 ensures that, for each $\lambda \in]0, \lambda_\gamma[$, problem (11) admits at least one positive weak solution $\bar{u}_\lambda$ such that $\bar{u}_\lambda(x) < \gamma$ for all $x \in [a,b]$.

Finally, we present an application of Theorem 2.2 which we will use in next section to obtain multiple solutions.

Theorem 3.5. *Assume that there exist two constants* $\bar{c}, \bar{d}$, *with* $0 < k_1\bar{c}^2 < \bar{d}^2$, *such that*

(A_8) $\int_a^b \max_{|t|\leq\bar{c}} F(x,t)dx < \int_{(a+b)/2}^b F(x,\bar{d})dx$;

(A_9) $\limsup_{|t|\to+\infty} \frac{F(x,t)}{t^2} \le 0$ *uniformly in* x.

Then, for each $\lambda > \tilde{\lambda}$, *where*

$$\tilde{\lambda} := \frac{B\bar{d}^2/k_2 - A\bar{c}^2}{\int_{(a+b)/2}^{b} F(x,\bar{d})dx - \int_a^b \max_{|t|\le\bar{c}} F(x,t)dx},$$

problem (2) *admits at least one non-trivial weak solution* $\tilde{u}$ *such that* $\|\tilde{u}\|^2 > \frac{A}{B}\bar{c}^2$.

Proof. The functionals Φ and Ψ defined in the proof of Theorem 3.1 satisfy all regularity assumptions requested in Theorem 2.2. Moreover, the assumption (A_9) implies that I_λ, $\lambda > 0$, is coercive. So, our aim is to verify condition (4) of Theorem 2.2. To this end, put $r = A\bar{c}^2$ and

$$w(x) = \begin{cases} \frac{2\bar{d}}{b-a}(x-a), & \text{if } x \in [a,(a+b)/2[, \\ \bar{d}, & \text{if } x \in [(a+b)/2, b]. \end{cases}$$

Arguing as in the proof of Theorem 3.1 we obtain that

$$\rho(r) \ge \frac{\int_{(a+b)/2}^{b} F(x,\bar{d})dx - \int_a^b \max_{|t|\le\bar{c}} F(x,t)dx}{B\bar{d}^2/k_2 - A\bar{c}^2}.$$

So, from our assumption it follows that $\rho(r) > 0$.

Hence, from Theorem 2.2 for each $\lambda > \tilde{\lambda}$, the functional I_λ admits at least one local minimum $\tilde{u}$ such that $\|\tilde{u}\|^2 > \frac{A}{B}\bar{c}^2$ and the conclusion is achieved. □

4. **Multiplicity results**

The main aim of this section is to present multiplicity results. First, as consequence of Theorem 3.1, taking into account the classical theorem of Ambrosetti and Rabinowitz, we have the following multiplicity result.

Definition 4.1. We say that a functional $I : X \to \mathbb{R}$ satisfies the *Palais-Smale condition* if any sequence $\{u_n\} \subset X$ satisfying

$$\sup_n I(u_n) < +\infty, \qquad I'(u_n) \to 0,$$

contains a convergent subsequence.

Theorem 4.1. *Let the assumptions of Theorem 3.1 be satisfied. Assume also that* $f(\cdot,0) \ne 0$ *in* (a,b), *and*

(A_{10}) *there exist constants* $\nu > 2$ *and* $R > 0$ *such that, for all* $|t| \ge R$ *and for all* $x \in [a,b]$, *one has*

$$0 < \nu F(x,t) \le t f(x,t).$$

Then, for each $\lambda \in \left]\frac{1}{b(c_1,k_2)}, \frac{1}{a(c_2,k_2)}\right[$, *problem* (2) *admits at least two non-trivial weak solutions* $\bar{u}_1, \bar{u}_2$, *such that*

$$\frac{A}{B}c_1^2 < \|\bar{u}_1\|^2 < \frac{B}{A}c_2^2. \tag{12}$$

Proof. Fix λ as in the conclusion. So, Theorem 3.1 ensures that problem (2) admits at least one non-trivial weak solution $\bar{u}_1$ satisfying the condition (12) which is a local minimum of the functional I_λ.

Now, we prove the existence of the second local minimum distinct from the first one. To this end, we must show that the functional I_λ satisfies the hypotheses of the mountain pass theorem.

Clearly, the functional I_λ is of class C^1 and $I_\lambda(0) = 0$.

We can assume that $\bar{u}_1$ is a strict local minimum for I_λ in X. Therefore, there is $\rho > 0$ such that $\inf_{\|u-\bar{u}_1\|=\rho} I_\lambda(u) > I_\lambda(\bar{u}_1)$, so condition (I_1) in Theorem 2.2 in [19] is verified.

From (A_{10}), there is a positive constant C such that

$$F(x,t) \geq C|t|^\nu \tag{13}$$

for all $x \in [a,b]$ and $|t| > R$. In fact, setting $a(x) := \min_{|\xi|=R} F(x,\xi)$ and

$$\varphi_t(s) := F(x,st), \quad \forall s > 0, \tag{14}$$

by (A_{10}), for every $x \in [a,b]$ and $|t| > R$ one has

$$0 < \nu\varphi_t(s) = \nu F(x,st) \leq st \cdot f(x,st) = s\varphi_t'(s), \quad \forall s > \frac{R}{|t|}.$$

Therefore,

$$\int_{R/|t|}^1 \frac{\varphi_t'(s)}{\varphi_t(s)} ds \geq \int_{R/|t|}^1 \frac{\nu}{s} ds.$$

Then

$$\varphi_t(1) \geq \varphi_t\Big(\frac{R}{|t|}\Big)\frac{|t|^\nu}{R^\nu}.$$

Taking into account of (14), we obtain

$$F(x,t) \geq F\Big(x, \frac{R}{|t|}t\Big)\frac{|t|^\nu}{R^\nu} \geq a(x)\frac{|t|^\nu}{R^\nu} \geq C|t|^\nu,$$

where $C > 0$ is a constant. Thus, (13) is proved.

Now, choosing any $u \in X \setminus \{0\}$, one has

$$\begin{aligned} I_\lambda(tu) &= (\Phi - \lambda\Psi)(tu) \\ &= \frac{1}{2}\|tu\|^2 + \int_a^b G(tu(x))dx - \lambda \int_a^b F(x,tu(x))dx \\ &\leq \frac{Bt^2}{2}\|u\|^2 - \lambda t^\nu C \int_a^b |u(x)|^\nu dx \to -\infty \end{aligned}$$

as $t \to +\infty$, so condition (I_2) in Theorem 2.2 in [19] is verified. So, the functional I_λ satisfies the geometry of mountain pass.

Now, to verify the Palais-Smale condition it is sufficient to prove that any sequence of Palais-Smale is bounded. To this end, taking into account (A_{10}) one has

$$\begin{aligned}\nu I_\lambda(u_n) - \|I'_\lambda(u_n)\|_{X^*}\|u_n\| &\geq \nu I_\lambda(u_n) - I'_\lambda(u_n)(u_n)\\ &= \nu\Phi(u_n) - \lambda\nu\Psi(u_n) - \Phi'(u_n)(u_n) + \lambda\Psi'(u_n)(u_n)\\ &\geq (\nu A - 2A)\|u_n\|^2 - \lambda\int_a^b [\nu F(x,u_n(x)) - f(x,u_n(x))u_n(x)]\,dx\\ &\geq (\nu A - 2A)\|u_n\|^2. \qquad (15)\end{aligned}$$

If $\{u_n\}$ is not bounded, from (15) we have a contradiction. Thus, I_λ satisfies the Palais-Smale condition.

Hence, the classical theorem of Ambrosetti and Rabinowitz ensures a critical point $\bar{u}_2$ of I_λ such that $I_\lambda(\bar{u}_2) > I_\lambda(\bar{u}_1)$. So, $\bar{u}_1$ and $\bar{u}_2$ are two distinct weak solutions of (2) and the proof is complete. □

Corollary 4.1. *Assume that there exist two positive constants c, d, with $d^2 < k_2c^2$, such that* (A_6) *holds. Assume also that*

(A_{11}) *there exist constants $\nu > 2$ and $R > 0$ such that, for all $|t| \geq R$, one has*

$$0 < \nu H(t) \leq th(t).$$

Then, for each

$$\lambda \in \left]\frac{B}{k_2\int_{(a+b)/2}^b \alpha(x)dx}\frac{d^2}{H(d)}, \frac{B}{\|\alpha\|_1}\frac{c^2}{H(c)}\right[,$$

problem (11) *admits at least two nonnegative weak solutions $\bar{u}_1, \bar{u}_2$, such that $\bar{u}_1(x) < c$ for all $x \in [a,b]$.*

Corollary 4.2. *Assume that* (A_7) *and* (A_{11}) *are satisfied.*

Then, for each $\lambda \in]0, \lambda^\star[$, problem (11) *admits at least two nonnegative weak solutions.*

Next, as a consequence of Theorems 3.5 and 3.2, the following theorem of the existence of three solutions is obtained and its consequence for the nonlinearity with separable variables is presented.

Theorem 4.2. *Assume that* (A_9) *holds. Moreover, assume that there exist four positive constants $c, d, \bar{c}, \bar{d}$, with $d^2 < k_2c^2 \leq k_1\bar{c}^2 < \bar{d}^2$, such that* ($A_3$), ($A_8$) *and*

(A_{12}) $\frac{\int_a^b \max_{|t|\leq c} F(x,t)dx}{Bc^2} < \frac{\int_{(a+b)/2}^b F(x,\bar{d})dx - \int_a^b \max_{|t|\leq \bar{c}} F(x,t)dx}{B\bar{d}^2/k_2 - A\bar{c}^2}$

are satisfied. Then, for each

$$\lambda \in \Lambda := \left]\max\left\{\tilde{\lambda}, \frac{Bd^2}{k_2\int_{(a+b)/2}^b F(x,d)dx}\right\}, \frac{Bc^2}{\int_a^b \max_{|t|\leq c} F(x,t)dx}\right[,$$

problem (2) *admits at least three weak solutions.*

Proof. First, we observe that $\Lambda \neq \varnothing$ owing to (A_{12}). Next, fix $\lambda \in \Lambda$. Theorem 3.2 ensures a non-trivial weak solution $\bar{u}$ such that $\|\bar{u}\|^2 < \frac{B}{A}c^2$ which is a local minimum for the associated functional I_λ, as well as Theorem 3.5 guarantees a non-trivial weak solution $\tilde{u}$ such that $\|\tilde{u}\|^2 > \frac{A}{B}\bar{c}^2$ which is a local minimum for I_λ.

Hence, the mountain pass theorem as given by Pucci and Serrin (see [17]) ensures the conclusion. □

Theorem 4.3. *Assume that*

(A_{13}) $\limsup_{t\to 0^+} \frac{H(t)}{t^2} = +\infty$;

(A_{14}) $\limsup_{t\to +\infty} \frac{H(t)}{t^2} = 0$.

Further, assume that there exist two positive constants $\bar{c}, \bar{d}$, *with* $k_1\bar{c}^2 < \bar{d}^2$, *such that*

(A_{15}) $\frac{H(\bar{c})}{\bar{c}^2} < \left(\frac{k_2\int_{(a+b)/2}^{b}\alpha(x)dx}{\|\alpha\|_1}\right)\frac{H(\bar{d})}{\bar{d}^2}$.

Then, for each

$$\lambda \in \left]\frac{B}{k_2\int_{(a+b)/2}^{b}\alpha(x)dx}\frac{\bar{d}^2}{H(\bar{d})}, \frac{B}{\|\alpha\|_1}\frac{\bar{c}^2}{H(\bar{c})}\right[,$$

problem (11) *admits at least three nonnegative weak solutions.*

Proof. Clearly, (A_{14}) implies (A_9). Moreover, by choosing d small enough and $c = \bar{c}$, simple computations show that (A_{13}) implies (A_3). Finally, from (A_{15}) we get (A_8) and also (A_{12}). Hence, Theorem 4.2 ensures the conclusion. □

Remark 4.1. If $h(0) \neq 0$, Corollaries 4.1 and 4.2 ensure two positive weak solutions while Theorem 4.3 ensures three positive weak solutions (see proof of Theorem 3.3).

Finally, we present the following example to illustrate our results.

Example 4.1. Consider the problem

$$\begin{cases} -(2e^x u')' + e^x u = \lambda e^x(\frac{1}{6} + |u|^2 u) + \frac{1}{2}u & \text{in } (0,1), \\ u(0) = u'(1) = 0. \end{cases} \tag{16}$$

Let $p(x) = 2e^x$ and $q(x) = \alpha(x) = e^x$ for every $x \in [0,1]$. Also, let $h(t) = \frac{1}{6} + |t|^2 t$ and $g(t) = \frac{1}{2}t$ for all $t \in \mathbb{R}$. Clearly, $h(0) \neq 0$. Since

$$\lim_{t\to 0^+}\frac{h(t)}{t} = \lim_{t\to 0^+}\left(\frac{1}{6t} + |t|^2\right) = +\infty,$$

condition (A_7) holds true. Choose $\nu = 3$ and $R = 1$, we have

$$0 < 3H(t) \leq th(t),$$

for all $|t| \geq 1$. Moreover, one has

$$\lambda^\star = \frac{B}{\|\alpha\|_1}\sup_{c>0}\frac{c^2}{H(c)} \geq \frac{15}{8(e-1)}.$$

Then, owing to Corollary 4.2 and Remark 4.1, for each $\lambda \in \left]0, \frac{15}{8(e-1)}\right[$, problem (16) admits at least two positive classical solutions. In particular, the problem

$$\begin{cases} -(2e^x u')' + e^x u = e^x(\frac{1}{6} + |u|^2 u) + \frac{1}{2}u & \text{in } (0,1), \\ u(0) = u'(1) = 0, \end{cases}$$

admits at least two positive classical solutions.

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