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OPTIMALITY CONDITIONS IN SUB-(b,m)-CONVEX PROGRAMMING

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By modulating the definitions of m-convex functions and sub-b-convex functions, the authors introduce a new class of generalized convex functions called sub-(b, m)-convex functions and derive some basic properties for the newly introduced functions. Furthermore, The sufficient conditions of optimality for both unconstrained and inequality constrained sub-(b, m)-convex programming are presented. Also some optimality conditions of nonlinear multi-objective sub-(b, m)-convex programming are established.

Keywords: generalized convex functions, sub-(b, m)-convex set, sub-(b, m)-convex function, optimality conditions, multi-objective programming.

MSC2010: 90C26, 26B25.

1. Introduction

Owing to the importance of the convexity and generalized convexity in the study of optimality to solve the mathematical programming, researchers worked a lot on the generalized convex functions. For example, in earlier papers, Toader(1984) [22] introduced a class of functions called m-convex functions. Bector and Singh(1991) [3] introduced a class b-vex functions. Yang *et al.*(2002) [24] established some properties of explicitly B-preinvex functions.

Recently, Long *et al.*(2006) [13] discussed a class of functions called semi-b-preinvex functions, which is a generalization of the semi preinvex functions and the b-vex functions. Mishra *et al.*(2011) [15] studied a class of E-b-vex functions, observed some of its basic properties, and discussed certain interrelations with other functions. Emam(2012) [9] researched a new class of functions called roughly b-invex functions, discussed their properties, and obtained sufficient optimality criteria for nonlinear programming involving these functions. Alimohammady *et al.*(2011) [1] have solved some basic notions of convex analysis and convex optimization via convex semi-closed functions. Wang *et al.*(2012) [23] introduced and investigated a certain subclass of meromorphic close-to-convex functions and discussed some results as coefficient inequalities, convolution property, distortion property and radius of meromorphic convexity. These scholars's research promoted the development of the generalized convex functions like b-vex functions. Meanwhile, these extensions of convexity such as sub-b-convexity and m-convexity sparking our research interest, so we turn our attention to this new research.

More recently, some significant results involving the properties of generalized convex function are optimality conditions for nonlinear generalized convex programming were created in, for instance, see the papers [5, 7, 14, 16, 25] and closely related references therein. Estimating a possible impact to applied sciences, Pitea *et al.* studied multiobjective optimization problems by means of several classes of generalized convexity in a geometric framework; please, see [2], and [16, 18, 19, 20].

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Inspired by the research works [9, 10, 21, 26] and based on our work [12], we present a new class of generalized convex functions which is called sub-(b,m)-convex functions and discuss some properties of the class of functions satisfying the sub-(b,m)-convexity. We also give the sufficient conditions of optimality for both unconstrained and inequality constrained sub-(b,m)-convex programming.

The remainder of this paper is organized as follows. In Section 2, we recall the definitions of b-convex, sub-b-convex and m-convex functions. Section 3 develops some properties of sub-(b,m)-convex function and sub-(b,m)-convex sets. In Section 4, we introduce a new sub-(b,m)-convex programming and establish the sufficient optimality conditions under the b-(E,m)-convexity. Some optimality conditions for the nonlinear multi-objective sub-(b,m)-convex programming by using weighting approach and ε-constraint approach are presented in Section 5. Finally conclusions are given in Section 6.

2. **Preliminaries**

From now on, let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and M be a nonempty convex subset in $\mathbb{R}^n$. In the following, several definitions about b-vex, m-convex and sub-b-convex functions, which will be needed in sequel, from Bector and Singh [3], Chao and Jian [6] and Toader [22] are summarized below.

Definition 2.1. *Let M be a nonempty convex subset in $\mathbb{R}^n$. The function $f : M \to \mathbb{R}$ is said to be:*

(1): *b-vex function on M with respect to mapping $b : M \times M \times [0,1] \to \mathbb{R}$, if*

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda b(x,y,\lambda) f(x) + \big(1 - \lambda b(x,y,\lambda)\big) f(y)$$

holds for all $x, y \in M$ and $\lambda \in [0,1]$;

(2): *b-linear function on M with respect to mapping $b : M \times M \times [0,1] \to \mathbb{R}$, if*

$$f\big(\lambda x + (1-\lambda)y\big) = \lambda b(x,y,\lambda) f(x) + \big(1 - \lambda b(x,y,\lambda)\big) f(y)$$

holds for all $x, y \in M$ and $\lambda \in [0,1]$.

Definition 2.2. *The function $f\colon [0,b] \to \mathbb{R}$ is said to be m-convex if*

$$f\big(\lambda x + m(1-\lambda)y\big) \le \lambda f(x) + m(1-\lambda) f(y)$$

holds for all $x, y \in [0,b]$, $\lambda \in [0,1]$ and fixed $m \in (0,1]$.

Definition 2.3. *The function $f : M \to \mathbb{R}$ is said to be a sub-b-convex function on M with respect to mapping $b : M \times M \times [0,1] \to \mathbb{R}$, if*

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y) + b(x,y,\lambda)$$

holds for all $x, y \in M$ and $\lambda \in [0,1]$.

3. **Sub-(b,m)-convex functions**

Before we introduce the concept of sub-(b,m)-convex functions, we give the definition of m-convex set as follows.

Definition 3.1. *A set $S \subset \mathbb{R}^n$ is said to be m-convex set, if there exists a fixed constant $m \in (0,1]$ such that*

$$\lambda x + m(1-\lambda)y \in S \tag{3.1}$$

for every $x, y \in S$ and $\lambda \in [0,1]$.

Remark 3.1. *By Definition 3.1, we can easily check that $mx \in S$ for all $x \in S$ and fixed $m \in (0,1]$. In addition, every convex set $S \subset \mathbb{R}^n$ is an m-convex set by taking $m = 1$.*

The following result is obvious.

Proposition 3.1. *If $S_i (i \in I = \{1, 2, \cdots, n\})$ is a family of m-convex sets, then $\bigcap_{i\in I} S_i$ is an m-convex set.*

As one can see, the definitions of m-convex, b-vex and sub-b-convex functions have similar forms. This observation leads us to generalize these varieties of convexity. Now, we introduce the so-called 'sub-(b, m)-convex function' by combining Definition 2.2 and Definition 2.3.

Definition 3.2. *The function $f\colon S \to \mathbb{R}$ is said to be sub-(b, m)-convex function on S with respect to mapping b: $\mathrm{S} \times \mathrm{S} \times [0, 1] \to \mathbb{R}$, if there exists a fixed constant $m \in (0, 1]$ such that S is an m-convex set and*

$$f\big(\lambda x + m(1-\lambda)y\big) \le \lambda f(x) + m(1-\lambda)f(y) + b(x, y, \lambda) \tag{3.2}$$

holds for all $x, y \in S$, $\lambda \in [0, 1]$. On the other hand, if

$$f\big(\lambda x + m(1-\lambda)y\big) \ge \lambda f(x) + m(1-\lambda)f(y) + b(x, y, \lambda) \tag{3.3}$$

holds for all $x, y \in S$, $\lambda \in [0, 1]$ and for some fixed $m \in (0, 1]$, then the function f is said to be sub-(b, m)-concave function. If the inequality signs in the previous two inequalities are strict, then f is called strictly sub-(b, m)-convex or sub-(b, m)-concave function.

Proposition 3.2. *Every convex function f on the convex set S is a sub-(b, m)-convex function with respect to the mapping $b(x, y, \lambda) \equiv 0$ and $m = 1$, but the converse is not necessarily true.*

Remark 3.2. *If $m = 1$ in Definition 3.2, then the sub-(b, m)-convex function reduces to the sub-b-convex function. When $b(x, y, \lambda) \le 0$, the sub-(b, m)-convex function reduces to the m-convex function.*

In the following, we are going to point out, whether or not, the sub-(b, m)-convex function shares some similar properties with the sub-b-convex function. Some basic results of sub-(b, m)-convex functions are established without proof.

Proposition 3.3. *If f_i: $S \to \mathbb{R}$, $i \in I$ are sub-(b, m)-convex functions on m-convex set S with respect to b_i: $S \times S \times [0, 1] \to \mathbb{R}$, $i \in I$ for the same fixed $m \in (0, 1]$, respectively, then the function*

$$f = \sum_{i\in I} a_i f_i, a_i \ge 0, (i = 1, 2, \cdots, n)$$

is sub-(b, m)-convex on m-convex set S with respect to $b = \sum_{i\in I} a_i b_i$ for the same fixed $m \in (0, 1]$.

Proposition 3.4. *If f_i: $S \to \mathbb{R}$, $i \in I$ are sub-(b, m)-convex functions on m-convex set S with respect to b_i: $S \times S \times [0, 1] \to \mathbb{R}$, $i \in I$ for the same fixed $m \in (0, 1]$, respectively, then the function $f = \max\{f_i, i \in I\}$ is a sub-(b, m)-convex function on m-convex set S with respect to $b = \max\{b_i, i \in I\}$.*

Proposition 3.5. *Assume f: $S \to \mathbb{R}$ is a sub-(b, m)-convex function on m-convex set S with respect to b: $S \times S \times [0, 1] \to \mathbb{R}$ for the same fixed $m \in (0, 1]$ and g: $\mathbb{R} \to \mathbb{R}$ is an increasing linear function. Then $f' = g \circ f$ is a sub-(b, m)-convex function on m-convex set S with respect to $b' = g \circ b$ for fixed $m \in (0, 1]$.*

Next, we define the concept "sub-(b, m)-convex sets" and study some interrelationship involving the sub-(b, m)-convex function and the sub-(b, m)-convex sets.

Definition 3.3. *Let $\mathrm{X} \subseteq \mathbb{R}^{n+1}$ be a nonempty set. X is said to be sub-(b, m)-convex set with respect to b: $\mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \to \mathbb{R}$ for some fixed $m \in (0, 1]$, if*

$$\Big(\lambda x + m(1-\lambda)y, \lambda\alpha + m(1-\lambda)\beta + b(x, y, \lambda)\Big) \in \mathrm{X} \tag{3.4}$$

holds for $\lambda \in [0, 1]$, where $(x, \alpha), (y, \beta) \in \mathrm{X}$ and $x, y \in \mathbb{R}^n$.

Here, we give a characterization of sub-(b,m)-convex function $f\colon S\to\mathbb{R}$ in terms of their epigraph $E(f)$, which is given by

$$E(f)=\{(x,\alpha)|x\in S,\alpha\in\mathbb{R},f(x)\le\alpha\}. \tag{3.5}$$

Theorem 3.1. *A function $f\colon S\to\mathbb{R}$ is a sub-(b,m)-convex function on m-convex set S with respect to $b\colon S\times S\times[0,1]\to\mathbb{R}$ for some fixed $m\in(0,1]$, if and only if $E(f)$ is a sub-(b,m)-convex set with respect to b.*

Proof Suppose that f is a sub-(b,m)-convex function with respect to b. Let (x_1,α_1), $(x_2,\alpha_2)\in E(f)$. Then, $f(x_1)\le\alpha_1$, $f(x_2)\le\alpha_2$. It follows that

$$\begin{aligned}f\big(\lambda x_1+m(1-\lambda)x_2\big)&\le\lambda f(x_1)+m(1-\lambda)f(x_2)+b(x_1,x_2,\lambda)\\&\le\lambda\alpha_1+m(1-\lambda)\alpha_2+b(x_1,x_2,\lambda)\end{aligned}$$

holds for all $x_1,x_2\in S$, $\lambda\in[0,1]$. Hence, we have that

$$\Big(\lambda x_1+m(1-\lambda)x_2,\lambda\alpha_1+m(1-\lambda)\alpha_2+b(x_1,x_2,\lambda)\Big)\in E(f).$$

Thus, by Definition 3.3, $E(f)$ is a sub-(b,m)-convex set with respect to b.

Conversely, let's assume that $E(f)$ is a sub-(b,m)-convex set with respect to b. Let $x_1,x_2\in S$, then $\big(x_1,f(x_1)\big),\big(x_2,f(x_2)\big)\in E(f)$. Thus, for $\lambda\in[0,1]$ and some fixed $m\in(0,1]$, it yields that

$$\Big(\lambda x_1+m(1-\lambda)x_2,\lambda f(x_1)+m(1-\lambda)f(x_2)+b(x_1,x_2,\lambda)\Big)\in E(f).$$

This implies that

$$f\big(\lambda x_1+m(1-\lambda)x_2\big)\le\lambda f(x_1)+m(1-\lambda)f(x_2)+b(x_1,x_2,\lambda).$$

That is, f is a sub-(b,m)-convex function with respect to b and the proof of Theorem 3.1 is completed.

Theorem 3.2. *If X_i, $i\in I$ is a family of sub-(b,m)-convex sets with respect to the same $b(x,y,\lambda)$ for the same fixed $m\in(0,1]$, then $\bigcap_{i\in I}X_i$ is a sub-(b,m)-convex set with respect to $b(x,y,\lambda)$.*

Theorem 3.3. *If $\{f_i|i\in I\}$ is a family of numerical functions, and each f_i is a sub-(b,m)-convex function with respect to the same $b(x,y,\lambda)$ for the same fixed $m\in(0,1]$, then the numerical function $f=\sup_{i\in I}f_i(x)$ is a sub-(b,m)-convex function with respect to $b(x,y,\lambda)$.*

The proofs of Theorem 3.2 and Theorem 3.3 are not particularly difficult, so no proofs will be given here.

4. Sub-(b,m)-convex Programming

Here the mapping $b:S\times S\times[0,1]\to\mathbb{R}$, where S is an m-convex set.

We make

Assumption 1: *The limit $\lim_{\lambda\to0+}\frac{b(x,y,\lambda)}{\lambda}$ exists for fixed $x,y\in S$.*

Assumption 2: *The function $f(x)$ satisfies that $f\big(my+\lambda(x-my)\big)\ge mf\big(y+\frac{\lambda}{m}(x-my)\big)$ holds for any $x,y\in S$, $\lambda\in(0,1]$ and some fixed $m\in(0,1]$. For fixed $x,y\in S$, when $\lambda\to0+$, $f\big(my+\lambda(x-my)\big)\to mf(y)$.*

We present

Theorem 4.1. *Suppose that $f : S \to \mathbb{R}$ is a differentiable sub-(b,m)-convex function on m-convex set S with respect to mapping $b(x,y,\lambda)$ for some fixed $m \in (0,1]$, the mapping b satisfies Assumption 1 and the function f satisfies Assumption 2, then*

$$\nabla f(y)^T(x-my) \leq \big(f(x)-f(my)\big) + \lim_{\lambda\to 0+} \frac{b(x,y,\lambda)}{\lambda}.$$

Proof By the Taylor expansion and the assumption of f, we have that

$$\begin{aligned} f\big(\lambda x + m(1-\lambda)y\big) &= f\big(my + \lambda(x-my)\big) \\ &\geq m\Big(f\big(y + \frac{\lambda}{m}(x-my)\big)\Big) \\ &= m\Big(f(y) + \frac{\lambda}{m}\nabla f(y)^T(x-my) + o\big(\frac{\lambda}{m}\big)\Big) \\ &= mf(y) + \lambda\nabla f(y)^T(x-my) + o(\lambda). \end{aligned} \tag{4.1}$$

Since f is a sub-(b,m)-convex function on m-convex set, it follows that

$$f\big(\lambda x + m(1-\lambda)y\big) \leq \lambda f(x) + m(1-\lambda)f(y) + b(x,y,\lambda). \tag{4.2}$$

Combining the inequality (4.1) and (4.2), it yields that

$$\lambda\nabla f(y)^T(x-my) + o(\lambda) \leq \lambda\big(f(x) - mf(y)\big) + b(x,y,\lambda). \tag{4.3}$$

From Assumption 1, dividing the inequality (4.3) by λ and taking $\lambda \to 0+$, it follows that

$$\begin{aligned} \nabla f(y)^T(x-my) &\leq \big(f(x)-mf(y)\big) + \lim_{\lambda\to 0+}\frac{b(x,y,\lambda)}{\lambda} \\ &= \big(f(x)-f(my)\big) + \lim_{\lambda\to 0+}\frac{b(x,y,\lambda)}{\lambda} \end{aligned}$$

holds for all $x, y \in X$ and fixed $m \in (0,1]$. The statement in Theorem 4.1 is completed.

We can easily get the following

Corollary 4.1. *Suppose that $f\colon S \to \mathbb{R}$ is a differentiable strictly sub-(b,m)-convex function on m-convex set S with respect to mapping $b(x,y,\lambda)$ for some fixed $m \in (0,1]$, the mapping b satisfies Assumption 1 and the function f satisfies Assumption 2, Then for any $x, y \in X$*

$$\nabla f(y)^T(x-my) < \big(f(x)-f(my)\big) + \lim_{\lambda\to 0+} \frac{b(x,y,\lambda)}{\lambda}.$$

Proof Using the similar way in Theorem 4.1, we have that

$$f\big(\lambda x + m(1-\lambda)y\big) \geq mf(y) + \lambda\nabla f(y)^T(x-my) + o(\lambda). \tag{4.4}$$

Since f is a strictly sub-(b,m)-convex function on m-convex set, it follows that

$$f\big(\lambda x + m(1-\lambda)y\big) < \lambda f(x) + m(1-\lambda)f(y) + b(x,y,\lambda). \tag{4.5}$$

Combining the inequality (4.4) and (4.5), it yields that

$$\lambda\nabla f(y)^T(x-my) + o(\lambda) < \lambda\big(f(x) - mf(y)\big) + b(x,y,\lambda). \tag{4.6}$$

From Assumption 1, dividing the inequality (4.6) by λ and taking $\lambda \to 0+$, it follows that

$$\begin{aligned} \nabla f(y)^T(x-my) &< \big(f(x)-mf(y)\big) + \lim_{\lambda\to 0+}\frac{b(x,y,\lambda)}{\lambda} \\ &= \big(f(x)-f(my)\big) + \lim_{\lambda\to 0+}\frac{b(x,y,\lambda)}{\lambda} \end{aligned}$$

holds for all $x, y \in X$ and fixed $m \in (0,1]$. This ends the proof.

By using the associated results above, we consider the nonlinear unconstraint problem (P).

$$(P) : \min\{f(x), x \in S\}$$

We prove the following result.

Theorem 4.2. *Let $f\colon S \to \mathbb{R}$ be a differentiable sub-(b, m)-convex function on m-convex set S with respect to b for some fixed $m \in (0, 1]$, the mapping b satisfies Assumption 1 and the function f satisfies Assumption 2. If $\bar{x} \in S$ and the inequality*

$$\nabla f(\bar{x})^T (x - m\bar{x}) \geq \lim_{\lambda \to 0_+} \frac{b(x, \bar{x}, \lambda)}{\lambda}$$

holds for each $x \in \mathrm{S}$, $\lambda \in [0, 1]$ and some fixed $m \in (0, 1]$, then $m\bar{x}$ is the optimal solution to the problem (P) with respect to f on m-convex set S.

Proof For any $x \in S$, since f is a differentiable sub-(b, m)-convex function, by Theorem 4.1, it follows that

$$\nabla f(\bar{x})^T (x - m\bar{x}) - \lim_{\lambda \to 0_+} \frac{b(x, \bar{x}, \lambda)}{\lambda} \leq \big(f(x) - f(m\bar{x})\big)$$

holds for $\lambda \in [0, 1]$, some fixed $m \in (0, 1]$. On the other hand, since

$$\nabla f(\bar{x})^T (x - m\bar{x}) \geq \lim_{\lambda \to 0_+} \frac{b(x, \bar{x}, \lambda)}{\lambda},$$

we have $f(x) - f(m\bar{x}) \geq 0$. Therefore, $m\bar{x}$ is the optimal solution to the problem (P). This completes the proof.

Similary, we get the claim below.

Corollary 4.2. *Let $f\colon S \to \mathbb{R}$ be a differentiable strictly sub-(b, m)-convex function on m-convex set S with respect to b for some fixed $m \in (0, 1]$, the mapping b satisfies Assumption 1 and the function f satisfies Assumption 2. If $\bar{x} \in S$ and the inequality*

$$\nabla f(\bar{x})^T (x - m\bar{x}) \geq \lim_{\lambda \to 0_+} \frac{b(x, \bar{x}, \lambda)}{\lambda}$$

holds for each $x \in \mathrm{S}$, $\lambda \in [0, 1]$ and some fixed $m \in (0, 1]$, then $m\bar{x}$ is the unique optimal solution to the problem (P).

Proof Using Corollary 4.1 and the strictly sub-(b, m)-convexity of f on m-convex set S with respect to b, we have for $\bar{x} \in S$ with each $x \in S$ the following inequality

$$\nabla f(\bar{x})^T (x - m\bar{x}) < \big(f(x) - f(m\bar{x})\big) + \lim_{\lambda \to 0_+} \frac{b(x, \bar{x}, \lambda)}{\lambda}.$$

So, when the inequality

$$\nabla f(\bar{x})^T (x - m\bar{x}) \geq \lim_{\lambda \to 0_+} \frac{b(x, \bar{x}, \lambda)}{\lambda}$$

holds for each $x \in S$, any $\lambda \in [0, 1]$ and $m \in (0, 1]$, it follows that $f(x) - f(m\bar{x}) > 0$ for every $x \in S$. Therefore, $m\bar{x} \in X$ is the unique optimal solution to the problem (P), which ends the proof.

Next, we are going to apply the associated results to the nonlinear programming with inequality constraints as follows:

$$(P_s) : \min\big\{f(x) | x \in \mathbb{R}^n, g_i(x) \leq 0, i \in I\big\}, I = \{1, 2, \cdots, n\}.$$

Denote the feasible set of (P_s) by $F = \{x \in \mathbb{R}^n | g_i(x) \leq 0, i \in I\}$. For the convenience of discussion, we assume that f and g_i are all differentiable and F is a nonempty set in $\mathbb{R}^n$.

We present

Theorem 4.3. (Karush-Kuhn-Tucker Sufficient Conditions) *Suppose that the function f: $\mathbb{R}^n \to \mathbb{R}$ is differentiable sub-(b,m)-convex function with respect to b for some fixed $m \in (0,1]$, g_i: $\mathbb{R}^n \to \mathbb{R}$, $i \in I$ are differentiable sub-(b,m)-convex functions with respect to b_i, $i \in I$ for some fixed $m \in (0,1]$, the mapping b satisfies Assumption 1 and f, g_i satisfy Assumption 2. Assume that $x^* \in F$ is a KKT point of (P_s), i.e., there exist multipliers $u_i \geq 0$, $i \in I$ such that*

$$\nabla f(x^*) + \sum_{i\in I} u_i \nabla g_i(x^*) = 0, u_i g_i(x^*) = 0. \tag{4.7}$$

If

$$\lim_{\lambda\to 0_+} \frac{b(x,x^*,\lambda)}{\lambda} \leq -\sum_{i\in I} u_i \lim_{\lambda\to 0_+} \frac{b_i(x,x^*,\lambda)}{\lambda}, \tag{4.8}$$

then mx^* is an optimal solution to the problem (P_s).

Proof For any $x \in F$, we have that

$$g_i(x) \leq 0 = g_i(x^*), i \in I(x^*) = \{i \in I | g_i(x^*) = 0\}.$$

Therefore, by the sub-(b,m)-convexity of g_i and Theorem 4.1, for $i \in I(x^*)$, we obtain

$$\nabla g_i(x^*)^T(x - mx^*) - \lim_{\lambda\to 0_+} \frac{b_i(x,x^*,\lambda)}{\lambda} \leq \big(g_i(x) - mg_i(x^*)\big) \leq 0. \tag{4.9}$$

From (4.7), it follows that

$$\nabla f(x^*)^T(x - mx^*) = -\sum_{i\in I} u_i \nabla g_i(x^*)^T(x - mx^*).$$

Using the condition (4.8), it yields that

$$\begin{aligned} &\nabla f(x^*)^T(x - mx^*) - \lim_{\lambda\to 0_+} \frac{b(x,x^*,\lambda)}{\lambda} \\ &\geq -\sum_{i\in I} u_i \nabla g_i(x^*)^T(x - mx^*) + \sum_{i\in I} u_i \lim_{\lambda\to 0_+} \frac{b_i(x,x^*,\lambda)}{\lambda} \\ &= -\sum_{i\in I(x^*)} u_i \Big(\nabla g_i(x^*)^T(x - mx^*) - \lim_{\lambda\to 0_+} \frac{b_i(x,x^*,\lambda)}{\lambda}\Big). \end{aligned} \tag{4.10}$$

Combining the inequality (4.9) and (4.10), we can deduce that

$$\nabla f(x^*)^T(x - mx^*) - \lim_{\lambda\to 0_+} \frac{b(x,x^*,\lambda)}{\lambda} \geq 0.$$

From Theorem 4.2, we can get that $f(x) - f(mx^*) \geq 0$ for each $x \in F$. Therefore mx^* is an optimal solution to the problem (P_s). This ends the proof.

5. Multi-Objective Sub-(b,m)-convex Programming

Consider the following nonlinear multi-objective sub-(b,m)-convex programming (MP):

$$\begin{aligned} (MP) \qquad &\min \qquad f(x) = \big(f_1(x), f_2(x), \cdots, f_p(x)\big) \\ &s.t. \qquad x \in M = \{x \in \mathbb{R}^n | g_j(x) \leq 0,\ j = 1,2,\cdots,q\}, \end{aligned}$$

where $f_i(x) : \mathbb{R}^n \to \mathbb{R}$, $i \in P = \{1,2,\cdots,p\}$ and $g_j(x) : \mathbb{R}^n \to \mathbb{R}$, $j \in Q = \{1,2,\cdots,q\}$ are sub-(b,m)-convex functions.

Let us recall three necessary definitions which are used for further discussion.

Definition 5.1. *A feasible point $x^* \in M$ of problem (MP) is said to be an efficient solution if and only if there does not exist another $x \in M$ such that $f_i(x) \leq f_i(x^*)$ for every $i \in P$ with strict inequality holding for at least one $i_0 \in P$.*

Definition 5.2. *A feasible point $x^* \in M$ of problem (MP) is said to be a weakly efficient solution if and only if there does not exist another $x \in M$ such that $f_i(x) < f_i(x^*)$ for every $i \in P$.*

Definition 5.3. *A feasible point $x^* \in M$ of problem (MP) is said to be a properly efficient solution if there exists a scalar $\mu > 0$ such that for each $i \in P$ and each $x \in M$ satisfying $f_i(x) < f_i(x^*)$, there exists at least one $j \neq i$ with $f_i(x) > f_i(x^*)$ and*

$$\frac{f_i(x) - f_i(x^*)}{f_j(x^*) - f_j(x)} \leq \mu.$$

We prove the following

Theorem 5.1. *Let $b : \mathrm{S} \times \mathrm{S} \times [0,1] \to (-\infty, 0]$, If $f(x)$ is sub-(b,m)-convex function with respect to the mapping $b(x,y,\lambda)$ for some fixed $m \in (0,1]$, then the set*

$$A = \bigcup_{x \in M} A(x)$$

is a convex set, where $A(x) = \{z | z \in \mathbb{R}^P, z > f(x) - f(x^*), x \in M\}$.

Proof Let $z_1, z_2 \in A(x)$, and $b : S \times S \times [0,1] \to (-\infty, 0]$, then for any $\lambda \in [0,1]$, fixed $m \in (0,1]$, $x_1, x_2 \in M$ and $f(x)$ is sub-(b,m)-convex function, we have that

$$\begin{aligned} \lambda z_1 + (1-\lambda) z_2 &> \lambda[f(x_1) - f(x^*)] + (1-\lambda)[f(x_2) - f(x^*)] \\ &= \lambda f(x_1) + (1-\lambda) f(x_2) - f(x^*) \\ &\geq \lambda f(x_1) + m(1-\lambda) f(x_2) + b(x,y,\lambda) - f(x^*) \\ &\geq f(\lambda x_1 + m(1-\lambda)x_2) - f(x^*), \end{aligned}$$

which implies $\lambda z_1 + (1-\lambda) z_2 \in A$. Hence A is a convex set.

5.1. **Weighting approach**

To characterize an efficient solution for problem (MP) by weighting approach [8], let us scalarize problem (MP) to become in the following form.

$$\begin{aligned} (MP_\omega) \qquad & \min \qquad \sum_i^p \omega_i f_i(x) \\ & s.t. \qquad x \in M = \{x \in \mathbb{R}^n | g_j(x) \leq 0,\ j = 1, 2, \cdots, q\}, \end{aligned}$$

We prove the following result.

Theorem 5.2. *If $x^* \in M$ is an efficient solution for problem (MP), then there exists $\omega_i > 0, i \in P$, $\sum_i^p \omega_i = 1$, such that x^* is an optimal solution for problem (MP_ω).*

Proof Let $x^* \in M$ be an efficient solution for problem (MP), then the system $f_i(x) - f_i(x^*) < 0, i \in P$ has no solution $x \in M$, combining Theorem 4.1 and applying the generalized Gordan theorem [4], there exists $\nu_i, i \in P$ such that

$$\nu_i[f_i(x) - f_i(x^*)] > 0$$

and

$$\frac{\nu_i}{\sum_{i \in P} \nu_i}[f_i(x) - f_i(x^*)] > 0.$$

We denote

$$\frac{\nu_i}{\sum_{i \in P} \nu_i} = \omega_i,$$

then $\omega_i > 0, i \in P$ and $\sum_i^p \omega_i = 1$. Hence

$$\omega_i f_i(x^*) < \omega_i f_i(x).$$

That is, x^* is an optimal solution for problem (MP_ω).

We also have

Theorem 5.3. *If $\bar{x} \in M$ is an optimal solution for problem $(MP_{\bar{\omega}})$ with $\bar{\omega}_i, i \in P$, then $\bar{x}$ is an efficient solution for problem (MP), if one of the following two conditions holds:*

(1), $\bar{\omega}_i > 0$ for any $i \in P$;

(2), $\bar{x}$ is the unique solution to the problem $(MP_{\bar{\omega}})$.

The proof of Theorem 5.3 is similar to the Theorem 5.2.

5.2. ε-constraint approach

The ε-constraint approach is one of the common approaches for characterizing efficient solutions to multi-objective programming. In the following we shall characterize an efficient solution for the multi-objective programming (MP) in terms of an optimal solution to the following scalar problem:

$$\begin{array}{lll} (MP_\varepsilon) & \min & f_k(x) \\ & s.t. & x \in M = \{x \in \mathbb{R}^n | g_j(x) \leq 0,\ j = 1, 2, \cdots, q\}, \\ & & f_i(x) \leq \varepsilon_i, i \in P, i \neq k. \end{array}$$

We prove the following

Theorem 5.4. *If $x^* \in M$ is an efficient solution for problem (MP), then x^* is an optimal solution for problem (MP_ε) with $\varepsilon_i = f_i(x^*)$.*

Proof Let $x^* \in M$ be not an optimal solution for (MP_ε) with $\varepsilon_i = f_i(x^*), i \in P, i \neq k$, so there exists $x \in M$ such that

$$f_k(x) < f_k(x^*)$$

and

$$f_i(x) \leq \varepsilon_i = f_i(x^*), i \in P, i \neq k.$$

Therefore, $x^* \in M$ is not an efficient solution for problem (MP) which is a contradiction. Hence, x^* is an optimal solution for problem (MP_ε) with $\varepsilon_i = f_i(x^*)$.

We continue with

Theorem 5.5. *If $\bar{x} \in M$ is an optimal solution to the problem (MP_ε) with $\bar{\varepsilon}_i = f_i(\bar{x})$ for all $i = 1, 2, \cdots, p$, then $\bar{x}$ is an efficient solution for problem (MP).*

Proof Since $\bar{x} \in M$ is an optimal solution to the problem (MP_ε) for all $k = 1, 2, \cdots, p$, for each $x \in M$, we have that

$$\begin{aligned} & f_k(\bar{x}) \leq f_k(x), \\ & f_i(x) \leq \bar{\varepsilon}_i = f_i(\bar{x}), i = 1, 2, \cdots, p, i \neq k \end{aligned}$$

This implies that the system $f_i(x) - f_i(\bar{x}) < 0, i \in P$ has no solution $x \in M$. Hence, $\bar{x}$ is an efficient solution for problem (MP).

Next we discuss the sufficient conditions for a feasible solution x^* to be efficient or properly efficient for problem (MP) in the following theorems.

Theorem 5.6. *Let $f_i(x), i \in P$ and $g_j(x), j \in Q$ be differentiable sub-(b,m)-convex functions with respect to $b_i, i \in P$ and $b_j, j \in Q$, respectively. Suppose that there exist a feasible $x^* \in M$ to the problem (MP) and multipliers $\bar{\lambda}_i > 0, i \in P$, $u_j \geq 0, j \in Q$ such that*

$$\sum_{i \in P} \bar{\lambda}_i \nabla f_i(x^*) + \sum_{j \in Q} u_j \nabla g_j(x^*) = 0, \tag{5.1}$$

$$u_i g_i(x^*) = 0. \tag{5.2}$$

If

$$\sum_{i\in P}\bar{\lambda}_i \lim_{\lambda\to 0_+}\frac{b_i(x,x^*,\lambda)}{\lambda} \le -\sum_{j\in Q}u_j \lim_{\lambda\to 0_+}\frac{b_j(x,x^*,\lambda)}{\lambda}, \tag{5.3}$$

then x^* is a properly efficient solution to the problem (MP).

Proof For any $x \in M$, we have that

$$g_j(x) \le 0 = g_j(x^*), j \in Q(x^*) = \{j \in Q : g_j(x^*) = 0\}.$$

Through the equality (5.2), we deduce that $u_j = 0$ for $j \notin Q(x^*)$. Then, it follows that

$$\sum_{j\in Q}u_j\nabla g_j(x^*) = \sum_{j\in Q(x^*)}u_j\nabla g_j(x^*). \tag{5.4}$$

According to Theorem 4.1 and by the sub-(b,m)-convexity of g_j $(j \in Q(x^*))$, it is easy to show that

$$\nabla g_j(x^*)^T(x-mx^*) - \lim_{\lambda\to 0+}\frac{b_j(x,x^*,\lambda)}{\lambda} \le \big(g_j(x) - mg_j(x^*)\big) \le 0. \tag{5.5}$$

On account of $\bar{\lambda}_i > 0, i \in P$, $u_j \ge 0, j \in Q$ and combining the equality (5.4), inequality (5.3), (5.5), and the sub-(b,m)-convexity of $f_i(x), i \in P$, it yields

$$\begin{aligned}\sum_{i\in P}\bar{\lambda}_i(f_i(x)-f_i(mx^*)) &\ge \sum_{i\in P}\bar{\lambda}_i\nabla f_i(x^*)^T(x-mx^*) - \sum_{i\in P}\bar{\lambda}_i\lim_{\lambda\to 0+}\frac{b_i(x,y,\lambda)}{\lambda}\\
&= -\sum_{j\in Q(x^*)}u_j\nabla g_j(x^*)^T(x-mx^*) - \sum_{i\in P}\bar{\lambda}_i\lim_{\lambda\to 0+}\frac{b_i(x,y,\lambda)}{\lambda}\\
&\ge -\sum_{j\in Q(x^*)}u_j\nabla g_j(x^*)^T(x-mx^*) + \sum_{j\in Q(x^*)}u_j\lim_{\lambda\to 0_+}\frac{b_j(x,x^*,\lambda)}{\lambda}\\
&\ge -\sum_{j\in Q(x^*)}u_j(g_j(x)-mg_j(x^*))\\
&\ge 0.\end{aligned}$$

That is

$$\sum_{i\in P}\bar{\lambda}_i f_i(x) - \sum_{i\in P}\bar{\lambda}_i f_i(mx^*) \ge 0$$

holds for all $x \in M$. It follows that mx^* minimizes $\sum_{i\in P}\bar{\lambda}_i f_i(x)$ subject to $g_j(x) \le 0,\ j \in Q$. Therefore, from Theorem 1 in [11], mx^* is a proper efficient solution to the problem (MP) which ends the proof.

Finally, we prove the following

Theorem 5.7. *Let $f_i(x), i \in P$ and $g_j(x), j \in Q$ be differentiable sub-(b,m)-convex functions with respect to $b_i, i \in P$ and $b_j, j \in Q$, respectively. Suppose that there exist a feasible $x^* \in M$ to the problem (MP) and scalars $\omega_i \ge 0, i \in P$, $\sum_{i\in P}\omega_i = 1$ such that the triplet (x^*, ω_i, u_j) satisfies*

$$\sum_{i\in P}\omega_i\nabla f_i(x^*) + \sum_{j\in Q}u_j\nabla g_j(x^*) = 0, \tag{5.6}$$

$$u_i g_i(x^*) = 0. \tag{5.7}$$

If

$$\sum_{i\in P}\omega_i \lim_{\lambda\to 0_+}\frac{b_i(x,x^*,\lambda)}{\lambda} \le -\sum_{j\in Q}u_j \lim_{\lambda\to 0_+}\frac{b_j(x,x^*,\lambda)}{\lambda}, \tag{5.8}$$

and $\sum_{i\in P}\omega_i f_i(x)$ is a strictly sub-(b,m)-convex functions. Then x^* is an efficient solution to the problem (MP).

Proof Suppose that x^* is not an efficient solution for (MP), then there exist a feasible $x \in M$ and an index κ such that

$$f_\kappa(x) < f_\kappa(x^*)$$

and

$$f_i(x) \leq f_i(x^*), for\ i \neq \kappa.$$

Since $\sum_{i\in P} \omega_i f_i(x)$ is a strictly sub-(b,m)-convex functions, we have that

$$\sum_{i\in P} \omega_i f_i(x) - \sum_{i\in P} \omega_i f_i(x^*) < 0.$$

Combining Theorem 4.1, it yields that

$$\sum_{i\in P} \omega_i \nabla f_i(x^*)^T (x - mx^*) - \sum_{i\in P} \omega_i \lim_{\lambda\to 0+} \frac{b_i(x, x^*, \lambda)}{\lambda} < 0, \tag{5.9}$$

and for $u_j \geq 0$, $j \in Q(x^*)$, combining the equality (5.7), it follows that

$$\sum_{j\in Q} u_j \nabla g_j(x^*)^T (x - mx^*) - \sum_{j\in Q} u_j \lim_{\lambda\to 0+} \frac{b_j(x, x^*, \lambda)}{\lambda} \leq 0. \tag{5.10}$$

According to the inequality (5.8), Adding (5.9) and (5.10) that contradicts (5.6). Then, we conclude that x^* is an efficient solution to the problem (MP).

6. **Conclusion**

In this paper, we have introduced sub-(b,m)-convex sets and sub-(b,m)-convex functions. It is observed that sub-(b,m)-convex functions can be simplified into m-convex function on the conditions that $b(x, y, \lambda) \leq 0$ and can be simplified into sub-b-convex function on the conditions that $m = 1$. Therefore, the sub-(b,m)-convex function is a generalization of m-convex and sub-b-convex function. Also we have studied optimality conditions for obtaining an optimal solution to sub-(b,m)-convex programming.

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