GEOMETRY OF STATISTICAL F-CONNECTIONS ON THE ANTI-KÄHLER MANIFOLD

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Let M_{2k} be an anti-Kähler manifold with an almost complex structure F, a pseudo-Riemannian metric g and a totally symmetric $(0,3)$-tensor field C. We first introduce statistical F-connections which are a special class of $\alpha-$connections on M_{2k} and derive the conditions under which its curvature tensor field is holomorphic. Then, we present some local results concerning with curvature properties of the connection and the tensor C.

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1. Introduction

An anti-Kähler manifold means a triplet (M_{2k}, g, F) which consists of a $2k-$dimensional differentiable manifold M_{2k}, an almost complex structure F and a pseudo-Riemannian metric g such that

$$g(FX,Y) = g(X,FY)$$

and

$$\nabla^g F = 0 \tag{1}$$

for all vector fields X and Y on M_{2k}, where ∇^g is Levi-Civita conneciton of g. Such manifolds are also referred to as generalized B-manifolds [3] or as almost complex manifolds with Norden metric [1] or as almost complex manifolds with B-metric [2]. The condition (1) is equivalent to $\Phi_J g = 0$, where Φ_J is the Tachibana operator [8]: $(\Phi_J g)(X,Y,Z) = (L_{JX}g - L_X G)(Y,Z)$, where $G(Y,Z) = (g \circ J)(Y,Z) = g(JY,Z)$ is the twin anti-Hermitian metric. Also, $\Phi_J g = 0$ means that the pseudo-Riemannian metric g is holomorphic [4]. Denote by R^g the Riemannian curvature tensor on the anti-Kähler manifold (M_{2k}, g, F). The following properties are satisfied:

i) The Levi-Civita connection ∇^g is holomorphic, that is, $\nabla^g_{FX}Y = \nabla^g_X FY = F\nabla^g_X Y$ [5],

ii) The Riemannian curvature tensor R^g is holomorphic, that is,

$$(\phi_F R^g)(X, Y_1, Y_2, Y_3, Y_4) = 0,$$

where [4, 7]

$$(\phi_F R^g)(X, Y_1, Y_2, Y_3, Y_4) =(\nabla^g_{FX} R)(Y_1, Y_2, Y_3, Y_4) - (\nabla^g_X R)(FY_1, Y_2, Y_3, Y_4).$$

There are two equivalent ways of defining a statistical manifold. One of them is to define a totally symmetric tensor C on a (pseudo-)Riemannian manifold (M, g) and then

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define the statistical manifold by the triplet (M, g, C). The second one is a triplet (M, g, ∇), where ∇ is a torsion-free linear connection on M and $C : \nabla g$ as a $(0, 3)-$tensor field on M is symmetric in all arguments. The tensor C is called *cubic form*. In the paper, we aim to study the geometry of statistical F-connections on the anti-Kähler manifold (M_{2k}, g, F). Some global and local results are presented.

2. Statistical F-connections on the anti-Kähler manifold

In the following, let (M_{2k}, g, F) be an anti-Kähler manifold. An anti-Kähler manifold (M_{2k}, g, F) equipped with cubic form C will be called an anti-Kähler statistical manifold. For a real number α, we consider a linear connection on the quadruble (M_{2k}, g, F, C) given by

$$\overset{(\alpha)}{\nabla}_X Y = \nabla^g_X Y - \frac{\alpha}{2} C(X, Y).$$

The connection $\overset{(\alpha)}{\nabla}$ is the unique connection satisfying the properties [6]:

i) $\overset{(\alpha)}{\nabla}$ is a torsion-free linear connection,

ii) $(\overset{(\alpha)}{\nabla}_X g)(Y, Z) = \alpha C(X, Y, Z)$, where $C(X, Y, Z) = g(C(X, Y), Z)$.

In particular, if we choose the cubic form C satisfying the following condition

$$C(FX, Y, Z) = C(X, FY, Z) = C(X, Y, FZ) \tag{2}$$

that is, C is pure with respect to F (for pure tensors, see [7]), then the following holds:

$$(\overset{(\alpha)}{\nabla}_X F)Y = 0.$$

We shall call such a connection *a statistical F-connection.*

Recall the curvature tensor field R of a linear connection ∇ is the tensor field

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

When ∇ is torsion-free, $[X, Y] = \nabla_X Y - \nabla_Y X$. then the curvature tensor field transforms into

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y} Z + \nabla_{\nabla_Y X} Z.$$

The $(0, 4)-$curvature tensor field R is defined by $R(X, Y, Z, W) = g(R(X, Y)Z, W)$.

Proposition 2.1. *The $(0, 4)-$curvature tensor field $R^{(\alpha)}$ of the statistical F-connection on the anti-Kähler statistical manifold (M_{2k}, g, F, C) satisfies*

$$\begin{aligned} R^{(\alpha)}(X, Y, Z, W) \quad &= \quad R^g(X, Y, Z, W) + \frac{\alpha}{2}\{(\nabla^g_Y C)(X, Z, W) - (\nabla^g_X C)(Y, Z, W)\} \\ &\quad + \frac{\alpha^2}{4}\{C(X, W, C(Y, Z)) - C(Y, W, C(X, Z))\} \end{aligned}$$

for all vector fields X, Y, Z.

Proof. By definition of the statistical F-connection and of curvature tensor field, standard calculations give the result. □

Proposition 2.2. *The $(0, 4)-$curvature tensor field $R^{(\alpha)}$ of the statistical $F-$connection on the anti-Kähler statistical manifold (M_{2k}, g, F, C) is pure with respect to F if the cubic form C is holomorphic, that is, $(\nabla^g_{FX} C)(Y_1, Y_2, Y_3) = (\nabla^g_X C)(FY_1, Y_2, Y_3)$.*

Proof. As is known that on an anti-Kähler manifold, the Riemannian curvature tensor field is pure. Also, the contracted tensor product of pure tensors is pure. The cubic form C on the anti-Kähler manifold (M_{2k}, g, F) is holomorphic if and only if $\phi_F C = 0$, where ϕ_F is the Tachibana operator applied to the cubic form C given by

$$(\phi_F C)(X, Y_1, Y_2, Y_3) = (\nabla^g_{FX} C)(Y_1, Y_2, Y_3) - (\nabla^g_X C)(FY_1, Y_2, Y_3)$$

for all vector fields X, Y_1, Y_2, Y_3. This means that the covariant differentiation of the cubic form C with respect to the Levi-Civita connection ∇^g is pure. The set of all pure tensors of arbitrary type is an algebra over the real numbers $\mathbb{R}$ [7]. Hence, when we take into account of the expression of the curvature tensor field $R^{(\alpha)}$ of the statistical F-connection we can immediately say that the curvature tensor field $R^{(\alpha)}$ is pure if the cubic form C is holomorphic. This completes the proof. □

The purity of the curvature tensor field of a linear connection is a necessary and sufficient condition for the connection to be holomorphic [5, 7]. Thus, as a direct result of Proposition 2.2, we get:

Corollary 2.1. *On the anti-Kähler statistical manifold (M_{2k}, g, F, C), the statistical F-connection is holomorphic, that is, $\overset{(\alpha)}{\nabla}_{(FX)} Y = \overset{(\alpha)}{\nabla}_X (FY) = F \overset{(\alpha)}{\nabla}_X Y$.*

Theorem 2.1. *The $(0,4)-$curvature tensor field $R^{(\alpha)}$ of the statistical F-connection on the anti-Kähler statistical manifold (M_{2k}, g, F, C) is holomorphic.*

Proof. When applying the Tachibana operator to the curvature tensor field $R^{(\alpha)}$ of the statistical F-connection, we have

$$\begin{aligned}(\phi_F R^{(\alpha)})(X, Y_1, Y_2, Y_3, Y_4) &= (\nabla^g_{FX} R^{(\alpha)})(Y_1, Y_2, Y_3, Y_4) \\ &\quad -(\nabla^g_X R^{(\alpha)})(FY_1, Y_2, Y_3, Y_4)\end{aligned}$$

$$\begin{aligned}= \ & (\nabla^g_{FX} R^g)(Y_1, Y_2, Y_3, Y_4) + \frac{\alpha}{2}\{\nabla^g_{FX}(\nabla^g_{Y_2} C)(Y_1, Y_3, Y_4) \\ & -\nabla^g_{FX}(\nabla^g_{Y_1} C)(Y_2, Y_3, Y_4)\} + \frac{\alpha^2}{4}\{(\nabla^g_{FX} C)(Y_1, Y_4, C(Y_2, Y_3)) \\ & +C(Y_1, Y_4, (\nabla^g_{FX} C))(Y_2, Y_3) - (\nabla^g_{FX} C)(Y_2, Y_4, C(Y_1, Y_3)) \\ & -C(Y_2, Y_4, (\nabla^g_{FX} C))(Y_1, Y_3)\} - (\nabla^g_X R^g)(FY_1, Y_2, Y_3, Y_4) \\ & -\frac{\alpha}{2}\{\nabla^g_X(\nabla^g_{Y_2} C)(FY_1, Y_3, Y_4) + \nabla^g_X(\nabla^g_{FY_1} C)(Y_2, Y_3, Y_4))\} \\ & -\frac{\alpha^2}{4}\{(\nabla^g_X C)(FY_1, Y_4, C(Y_2, Y_3)) + C(FY_1, Y_4, (\nabla^g_X C))(Y_2, Y_3) \\ & -(\nabla^g_X C)(Y_2, Y_4, C(FY_1, Y_3)) - C(Y_2, Y_4, (\nabla^g_X C))(FY_1, Y_3)\}.\end{aligned}$$

In view of the holomorphy of the cubic form C and the Levi-Civita connection ∇^g, the relation above reduces to

$$\begin{aligned}(\phi_F R^{(\alpha)})(X, Y_1, Y_2, Y_3, Y_4) &= (\nabla^g_{FX} R^g)(Y_1, Y_2, Y_3, Y_4) - (\nabla^g_X R^g)(FY_1, Y_2, Y_3, Y_4) \\ &= (\phi_F R^g)(X, Y_1, Y_2, Y_3, Y_4).\end{aligned}$$

Since the Riemannian curvature tensor R^g on the anti-Kähler manifold is holomorhic, the result immediately follows. □

3. Some local results for a special case

Under the local coordinates, in the anti-Kähler statistical manifold (M_{2k}, g, F, C) we shall specialize the statistical F-connection by choosing the cubic form C as:

$$C_{ij}^{\ \ k} = p_i \delta_j^k + p_j \delta_i^k + p^k g_{ij} - p_t F_i^{\ t} F_j^{\ k} - p_t F_j^{\ t} F_i^{\ k} - p_t F^{kt} F_{ij} \tag{3}$$

where the $1-$form p is holomorphic, that is,

$$F_k^m(\nabla_m^g p_j) = F_j^m(\nabla_k^g p_m).$$

In the case, the cubic form C is holomorphic.

Theorem 3.1. *In the anti-Kähler statistical manifold (M_{2k}, g, F), the cubic form C is recurrent, that is, $\nabla_k^g C_{ij}^{\ l} = \omega_k C_{ij}^{\ l}$ if and only if the $1-$form p is recurrent, where ω_k is the recurrence $1-$form.*

Proof. Assume that the cubic form C be recurrent, that is,

$$\nabla_k^g C_{ij}^{\ l} = \omega_k C_{ij}^{\ l}.$$

By contracting the last relation with respect to i and l, in view of (3) we obtain

$$\begin{aligned} \nabla_k^g C_{lj}^{\ l} &= \omega_k C_{lj}^{\ l} \\ \nabla_k^g[(n+4)p_j] &= \omega_k[(n+4)p_j] \\ \nabla_k^g p_j &= \omega_k p_j. \end{aligned}$$

In contrast, let us assume that the $1-$form p_i is recurrent. When we take the covariant derivative of (3) with respect to ∇^g, we have

$$\begin{aligned} \nabla_k^g C_{ij}^{\ l} &= (\nabla_k^g p_i)\,\delta_j^l + (\nabla_k^g p_j)\,\delta_i^l + \left(\nabla_k^g p^l\right) g_{ij} \\ &\quad - (\nabla_k^g p_t)\, F_i^{\ t} F_j^{\ l} - (\nabla_k^g p_t)\, F_j^{\ t} F_i^{\ l} - (\nabla_k^g p_t)\, F^{tl} F_{ij} \\ &= (\omega_k p_i)\,\delta_j^l + (\omega_k p_j)\,\delta_i^l + \left(\omega_k p^l\right) g_{ij} \\ &\quad - (\omega_k p_t)\, F_i^{\ t} F_j^{\ l} - (\omega_k p_t)\, F_j^{\ t} F_i^{\ l} - (\omega_k p_t)\, F^{tl} F_{ij} \\ &= \omega_k C_{ij}^{\ l}, \end{aligned}$$

which completes the proof. □

The local expression of the $(0,4)-$curvature tensor $R^{(\alpha)}$ of the statistical F-connection is as follows:

$$\begin{aligned} R_{ijkl}^{(\alpha)} &= R_{ijkl}^g - g_{jl}\mathcal{A}_{ik} + g_{il}\mathcal{A}_{jk} - F_{il}F_k^{\ t}\mathcal{A}_{jt} + F_{jl}F_k^{\ t}\mathcal{A}_{it} \\ &\quad - g_{jk}B_{il} + g_{ik}B_{jl} + F_{jk}F_l^{\ t}B_{it} - F_{ik}F_l^{\ t}B_{jt} \\ &\quad + F_{kl}F_j^{\ t}[\mathcal{A}_{it} - \mathcal{A}_{ti}] - g_{kl}[\mathcal{A}_{ij} - \mathcal{A}_{ji}], \end{aligned}$$

where

$$\mathcal{A}_{jk} = \frac{\alpha}{2}\nabla_j^g p_k + \frac{\alpha^2}{4}p_j p_k - \frac{\alpha^2}{4}p_t p_m F_j^{\ t} F_k^{\ m} + \frac{\alpha^2}{8}p^m p_m g_{jk} - \frac{\alpha^2}{8}p_m p^t F_t^{\ m} F_{jk} \tag{4}$$

and

$$B_{jk} = \frac{\alpha}{2}\nabla_j^g p_k - \frac{\alpha^2}{4}p_j p_k + \frac{\alpha^2}{4}p_t p_m F_j^{\ t} F_k^{\ m} - \frac{\alpha^2}{8}p^m p_m g_{jk} + \frac{\alpha^2}{8}p_m p^t F_t^{\ m} F_{jk}. \tag{5}$$

It immediately follows from (4) and (5) that

$$\mathcal{A}_{jk} - \mathcal{A}_{kj} = B_{jk} - B_{kj} = \frac{\alpha}{2}\left(\nabla_j^g p_k - \nabla_j^g p_k\right). \tag{6}$$

The Ricci tensor of the statistical F-connection has the components

$$\begin{aligned} R_{jk}^{(\alpha)} &= R_{jk}^g + (n-2)\mathcal{A}_{jk} + 2B_{jk} - g_{jk}traceB \\ &\quad + F_{jk}F_t^{\ l}B_l^t + 2(\mathcal{A}_{jk} - \mathcal{A}_{kj}). \end{aligned} \tag{7}$$

Proposition 3.1. *In the anti-Kähler statistical manifold (M_{2k}, g, F, C), the Ricci tensor of the statistical F-connection is symmetric if and only if the $1-$form p is closed.*

Proof. By (6) and (7), we have

$$\begin{aligned} R^{(\alpha)}_{jk} - R^{(\alpha)}_{kj} &= (n+2)(\mathcal{A}_{jk} - \mathcal{A}_{kj}) + 2(B_{jk} - B_{kj}) \\ &= (n+4)(\mathcal{A}_{jk} - \mathcal{A}_{kj}) \\ &= \frac{\alpha(n+4)}{2}\left(\nabla^g_j p_k - \nabla^g_k p_j\right). \end{aligned}$$

It is well known that if $(dp)_{jk} = \frac{1}{2}\left(\nabla^g_j p_k - \nabla^g_k p_j\right) = 0$, then the $1-$form p is closed. Hence, the proof is complete. □

Theorem 3.2. *In the anti-Kähler statistical manifold (M_{2k}, g, F, C), the scalar curvature $\tau^{(\alpha)}$ of the statistical F-connection with respect to g coincides with the scalar curvature τ^g of the Levi-Civita connection of g if and only if the $1-$form p is isotropic.*

Proof. The scalar curvature $\tau^{(\alpha)} = R^{(\alpha)}_{jk} g^{jk}$ of the statistical F-connection is given by

$$\tau^{(\alpha)} = \tau^g + (n-2)(trace\mathcal{A} - traceB).$$

From (4) and (5), we find

$$trace\mathcal{A} = \frac{\alpha}{2}\nabla^g_l p^l + \frac{\alpha^2(n+4)}{8}\|p\|$$

and

$$traceB = \frac{\alpha}{2}\nabla^g_l p^l - \frac{\alpha^2(n+4)}{8}\|p\|,$$

where $\|p\| = g^{lm}p_l p_m = p_l p^l$. As a result, these give

$$\tau^{(\alpha)} = \tau^g + \frac{\alpha^2(n-2)(n+4)}{4}\|p\|.$$

□

An Einstein manifold is a pseudo-Riemannian manifold whose Ricci tensor satisfies

$$R^g_{jk} = \lambda g_{jk},$$

where λ is a scalar function. The anti-Kähler statistical manifold (M_{2k}, g, F, C) with statistical F-connection in which the Ricci tensor satisfies

$$R^{(\alpha)}_{(jk)} = \mu g_{jk}$$

may be called an Einstein manifold, where μ is a scalar function.

Theorem 3.3. *Let (M_{2k}, g, F, C) be anti-Kähler statistical manifold and M_{2k} be an Einstein manifold, that is, $R_{jk} = \lambda g_{jk}$. Then, M_{2k} will be an Einstein manifold with respect to the statistical F-connection, that is, $R^{(\alpha)}_{(jk)} = \mu g_{jk}$ if*

$$\mu - \lambda = \frac{\alpha^2(n-2)(n+4)}{4n}\|p\|.$$

Proof. From (7), we have

$$\begin{aligned} R^{(\alpha)}_{(jk)} &= \frac{1}{2}(R^{(\alpha)}_{jk} + R^{(\alpha)}_{jk}) \\ &= R^g_{jk} + \frac{(n-2)}{2}(\mathcal{A}_{jk} + \mathcal{A}_{kj}) + (B_{jk} + B_{jk}) \\ &\quad - g_{jk} traceB + F_{jk}F_t{}^l B^t_l \end{aligned}$$

Transvecting the above relation with g^{jk}, we obtain

$$\begin{aligned} R^{(\alpha)}_{(jk)}g^{jk} &= R^{g}_{jk}g^{jk} + g^{jk}\{\frac{(n-2)}{2}(\mathcal{A}_{jk}+\mathcal{A}_{kj}) + (B_{jk}+B_{jk}) \\ & \quad -g_{jk}traceB + F_{jk}F_{t}{}^{l}B^{t}_{l}\} \\ \mu g_{jk}g^{jk} &= \lambda g_{jk}g^{jk} + (n-2)(trace\mathcal{A} - traceB) \\ \mu - \lambda &= \frac{\alpha^2(n-2)(n+4)}{4n}\|p\|. \end{aligned}$$

Hence (M_{2k}, g, F, C) with respect to the statistical F-connection is Einstein if $\mu - \lambda = \frac{\alpha^2(n-2)(n+4)}{4n}\|p\|$. □

4. **Conclusions**

The theory of statistical manifolds has been developing in information theory and statistics. Beyond these fields, it is providing interesting topics for differential geometry of manifolds with additional structures. Statistical manifolds are considered as a generalization of Riemannian manifolds. Following this idea in anti-Kähler geometry, we define a statistical F-connection on the anti-Kähler statistical manifold (M_{2k}, g, F, C) and compute its curvature tensor. We show that if the cubic form C is holomorphic, then the curvature tensor of the statistical F-connection is holomorphic. By choosing the particular cubic form, we search the recurrence property of the cubic form C with respect to the Levi-Civita connection and give the necessary and sufficient conditions under which the scalar curvature of the statistical F-connection and the scalar curvature of the Levi-Civita connection coincide each other. Also, we obtain the condition for the anti-Kähler statistical manifold (M_{2k}, g, F, C) to be Einstein with respect to the statistical F-connection.

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